Complete solutions to Exercise 11(e)

1. Similar to **EXAMPLE 23**. Let **A** be the matrix of coefficients:

$$\mathbf{A}=\begin{pmatrix}1 & 3 & 5\\ 7 & -1 & 1\\ 2 & 3 & -8\end{pmatrix}$$

We need to find the determinant of this matrix:

$$\det\mathbf{A}=1\det\begin{pmatrix}-1 & 1\\ 3 & -8\end{pmatrix}-3\det\begin{pmatrix}7 & 1\\ 2 & -8\end{pmatrix}+5\det\begin{pmatrix}7 & -1\\ 2 & 3\end{pmatrix}$$

$$\underset{\text{by (11.1)}}{=}(8-3)-3(-56-2)+5(21+2)=294$$

Since $\det\mathbf{A}\neq 0$ so by (11.16) we only have the trivial solution

$$x=0,\ y=0,\ z=0$$

2. Solving the equations by the inverse matrix method gives $x=1,\ y=1$. To check if this solution is unique we test whether the following matrix has a non zero determinant:

$$\mathbf{A}=\begin{pmatrix}2 & 3\\ 4 & 5\end{pmatrix}$$

$$\det\begin{pmatrix}2 & 3\\ 4 & 5\end{pmatrix}=(2\times 5)-(4\times 3)=-2\neq 0$$

By (11.13) $x=1,\ y=1$ is the unique solution of the given equations.

3. We can write the given equations as $\mathbf{Au}=\mathbf{b}$ where

$$\mathbf{A}=\begin{pmatrix}2 & 1\\ 4 & 2\end{pmatrix},\ \mathbf{u}=\begin{pmatrix}x\\ y\end{pmatrix},\ \mathbf{b}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

We can find the determinant of **A** by using (11.1):

$$\det\begin{pmatrix}2 & 1\\ 4 & 2\end{pmatrix}=(2\times 2)-(4\times 1)=0$$

By (11.15) there are an infinite number of solutions for

$$2x+y=0 \qquad (*)$$

$$4x+2y=0 \qquad (**)$$

From (*) we have

$$y=-2x$$

The general solution is $x=a,\ y=-2a$ where a is any real number.

4. For the solution to be unique, $\det\mathbf{A}\neq 0$ where **A** is the matrix of coefficients.

For question 1(a), let $\mathbf{A}=\begin{pmatrix}1 & 2 & -3\\ 2 & -1 & -1\\ 3 & 2 & 1\end{pmatrix}$ and so the determinant is given by

$$\det\begin{pmatrix}1 & 2 & -3\\ 2 & -1 & -1\\ 3 & 2 & 1\end{pmatrix}=1\det\begin{pmatrix}-1 & -1\\ 2 & 1\end{pmatrix}-2\det\begin{pmatrix}2 & -1\\ 3 & 1\end{pmatrix}-3\det\begin{pmatrix}2 & -1\\ 3 & 2\end{pmatrix}=-30$$

Since $\det\mathbf{A}=-30\neq 0$ the solution to the given equations is unique.

Similarly for (b) $\det \mathbf{A} = 8$ (c) $\det \mathbf{A} = -8$
Therefore the solutions are unique.

For question 4; $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 1 \end{pmatrix}$, $\det \mathbf{A} = 3$. The solution is unique.

For question 5; $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & -2 & 1 \end{pmatrix}$, $\det \mathbf{A} = 5$. Solution is unique.

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5. (a) For a non trivial solution we need

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

Using the given **A** we have

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \det\left[\begin{pmatrix} -1 & 1 \\ 9 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$= \det\left[\begin{pmatrix} -1 & 1 \\ 9 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right]$$

$$= \det\left[\begin{pmatrix} -1-\lambda & 1 \\ 9 & -1-\lambda \end{pmatrix}\right]$$

$$= (-1-\lambda)(-1-\lambda) - 9 = (1+\lambda)(1+\lambda) - 9 = 1 + 2\lambda + \lambda^2 - 9 = \lambda^2 + 2\lambda - 8$$

Solving the quadratic equation by putting it to zero gives

$$\lambda^2 + 2\lambda - 8 = 0$$

$$(\lambda + 4)(\lambda - 2) = 0 \text{ which gives } \lambda = -4,\ \lambda = 2$$

The values of λ for a non trivial solution are $\lambda = -4,\ \lambda = 2$.

(b) Similar to (a):

$$\mathbf{A} - \lambda \mathbf{I} = \begin{pmatrix} 1 & 3 \\ -6 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 3 \\ -6 & 5-\lambda \end{pmatrix}$$

For a non trivial solution we need the determinant of this to be zero.

$$\det\begin{pmatrix} 1-\lambda & 3 \\ -6 & 5-\lambda \end{pmatrix} = (1-\lambda)(5-\lambda) + 18 = 5 - \lambda - 5\lambda + \lambda^2 + 18 = \lambda^2 - 6\lambda + 23$$

Putting the quadratic to zero and solving for λ.

$$\lambda^2 - 6\lambda + 23 = 0$$

Using the quadratic formula, (1.16), with $a = 1,\ b = -6$ and $c = 23$ gives

$$\lambda = \frac{6 \pm \sqrt{(-6)^2 - (4 \times 1 \times 23)}}{2} = \frac{6 \pm \sqrt{-56}}{2} = \frac{6 \pm \sqrt{-4 \times 14}}{2} = 3 \pm j\sqrt{14}$$

The values of λ are $3 + j\sqrt{14},\ 3 - j\sqrt{14}$.

(1.16) $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$