Complete solutions to Exercise 11(f)

1. We substitute the given matrix, **A**, into $\det(\mathbf{A}-\lambda\mathbf{I})=\mathbf{0}$:

(a) We have

$$\det\begin{pmatrix}7-\lambda & 3\\ 0 & -4-\lambda\end{pmatrix} \underbrace{=}_{\text{by (11.1)}} \left[(7-\lambda)(-4-\lambda)-0\right]$$

$$=-(7-\lambda)(4+\lambda)=0$$

Thus the eigenvalues are $\lambda=-4,\ 7$.

(b)

$$\det\begin{pmatrix}5-\lambda & -2\\ 4 & -1-\lambda\end{pmatrix}=(5-\lambda)(-1-\lambda)+8$$

$$=-(5-\lambda)(1+\lambda)+8=-\left[5+4\lambda-\lambda^2\right]+8=\lambda^2-4\lambda-5+8$$

Putting this quadratic to zero and solving

$$\lambda^2-4\lambda-5+8=\lambda^2-4\lambda+3=0$$

$$(\lambda-3)(\lambda-1)=0 \text{ gives } \lambda_1=1,\ \lambda_2=3$$

(c) Substituting the given matrix into $\det(\mathbf{A}-\lambda\mathbf{I})$ yields

$$\det\begin{pmatrix}-1-\lambda & 4\\ 2 & 1-\lambda\end{pmatrix} \underbrace{=}_{\text{by (11.1)}} (-1-\lambda)(1-\lambda)-(2\times 4)$$

$$=-(1+\lambda)(1-\lambda)-8$$

$$=-\left(1-\lambda^2\right)-8=-1+\lambda^2-8=\lambda^2-9$$

Solving the equation $\lambda^2-9=0$ gives

$$\lambda^2=9$$

$$\lambda=\sqrt{9}=-3,\ 3$$

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2. (a) We have 2 eigenvalues $\lambda=-4,\ 7$. Let $\mathbf{u}=\begin{pmatrix}x\\ y\end{pmatrix}$ be the eigenvector for $\lambda=7$. We have

$$\begin{pmatrix}7-7 & 3\\ 0 & -4-7\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}0 & 3\\ 0 & -11\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

Multiplying out the matrices gives

$$3y=0$$

$$-11y=0$$

Thus y=0 and x is any real number apart from zero. A particular value of x can be 1. So a particular eigenvector for $\lambda=7$ is $\mathbf{u}=\begin{pmatrix}1\\ 0\end{pmatrix}$.

Similarly let **v** be an eigenvector for $\lambda=-4$:

(11.1) $$\det\begin{pmatrix}a & b\\ c & d\end{pmatrix}=ad-cb$$

$$\begin{pmatrix} 7-(-4) & 3 \\ 0 & -4-(-4) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 11 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Multiplying out the first row yields

$$11x+3y=0$$

$$x=-\frac{3}{11}y$$

If y=1 then $x=-3/11$, thus $\mathbf{v}=\begin{pmatrix} -3/11 \\ 1 \end{pmatrix}$ or using smallest integers gives $\begin{pmatrix} -3 \\ 11 \end{pmatrix}$

(b) Let $\mathbf{u}=\begin{pmatrix} x \\ y \end{pmatrix}$ be the eigenvector for $\lambda=1$:

$$\begin{pmatrix} 5-1 & -2 \\ 4 & -1-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Multiplying out the matrix

$$4x-2y=0$$
$$4x-2y=0$$

Solving these gives $x=1,\ y=2$. Thus $\mathbf{u}=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector for $\lambda=1$.

Let $\mathbf{v}=\begin{pmatrix} x \\ y \end{pmatrix}$ be the eigenvector for $\lambda=3$:

$$\begin{pmatrix} 5-3 & -2 \\ 4 & -1-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Multiplying gives

$$2x-2y=0$$
$$4x-4y=0$$

Solving these gives $x=y=1$. An eigenvector corresponding to $\lambda=3$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(c) Let $\mathbf{u}=\begin{pmatrix} x \\ y \end{pmatrix}$ be the eigenvector for $\lambda=-3$.

$$\begin{pmatrix} -1-(-3) & 4 \\ 2 & 1-(-3) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving gives $x=-2,\ y=1$. An eigenvector is $\begin{pmatrix}-2\\1\end{pmatrix}$ corresponding to $\lambda=-3$.

Let $\mathbf{v}=\begin{pmatrix}x\\y\end{pmatrix}$ be an eigenvector for $\lambda=3$:

$$\begin{pmatrix}-1-3 & 4\\ 2 & 1-3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}-4 & 4\\ 2 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Hence $x=y=1$. The eigenvector $\begin{pmatrix}1\\1\end{pmatrix}$ corresponds to $\lambda=3$.

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3. (a) Using $\det(\mathbf{A}-\lambda\mathbf{I})=0$:

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{pmatrix}-1-\lambda & 1\\ -2 & 1-\lambda\end{pmatrix}$$

$$=(-1-\lambda)(1-\lambda)-(-2\times 1)$$

$$=-(1+\lambda)(1-\lambda)+2=-(1-\lambda^2)+2=-1+\lambda^2+2=\lambda^2+1$$

Putting this quadratic to zero yields

$$\lambda^2+1=0$$

$$\lambda^2=-1 \text{ which gives } \lambda=\pm\sqrt{-1}=\pm j$$

The system poles are $\lambda_1=j,\ \lambda_2=-j$

(b) Substituting the given matrix into $\det(\mathbf{A}-\lambda\mathbf{I})$ gives

$$\det\begin{pmatrix}1-\lambda & 2\\ -4 & -3-\lambda\end{pmatrix}=(1-\lambda)(-3-\lambda)-(-4\times 2)$$

$$=-(1-\lambda)(3+\lambda)+8$$

$$=-(3+\lambda-3\lambda-\lambda^2)+8$$

$$=-(3-2\lambda-\lambda^2)+8=-3+2\lambda+\lambda^2+8=\lambda^2+2\lambda+5$$

Putting the resulting quadratic to zero $\lambda^2+2\lambda+5=0$. How do we solve this quadratic?

Using the quadratic formula (1.16) with $a=1,\ b=2$ and $c=5$ gives

$$\lambda=\frac{-2\pm\sqrt{4-(4\times1\times5)}}{2}=\frac{-2\pm\sqrt{-16}}{2}=\frac{-2\pm j4}{2}=-1\pm j2$$

The system poles are $\lambda_1=-1+j2,\ \lambda_2=-1-j2$.

(c) The system poles are given by the eigenvalues of the matrix.

(1.16) $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\det\begin{pmatrix}5-\lambda & -5\\ 5 & -2-\lambda\end{pmatrix}=(5-\lambda)(-2-\lambda)-(5\times(-5))$$
$$=-(5-\lambda)(2+\lambda)+25$$
$$=-(10+5\lambda-2\lambda-\lambda^2)+25$$
$$=-(10+3\lambda-\lambda^2)+25=-10-3\lambda+\lambda^2+25=\lambda^2-3\lambda+15$$

Putting the quadratic to zero, $\lambda^2-3\lambda+15=0$ and solving by using (1.16):

$$\lambda=\frac{3\pm\sqrt{9-(4\times1\times15)}}{2}=\frac{3\pm\sqrt{-51}}{2}=1.50\pm j3.57$$

System poles are $\lambda_1=1.50+j3.57,\ \lambda_2=1.50-j3.57$

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4. Let $\mathbf{w}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ be the eigenvector for the eigenvalue $\lambda=-5$ of the matrix in

EXAMPLE 29. Substituting this, $\lambda=-5$, into $(\mathbf{A}-\lambda\mathbf{I})\mathbf{w}=\mathbf{0}$ gives

$$\begin{pmatrix}1-(-5) & 0 & 4\\ 0 & 4-(-5) & 0\\ 3 & 5 & -3-(-5)\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$
$$\begin{pmatrix}6 & 0 & 4\\ 0 & 9 & 0\\ 3 & 5 & 2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Multiplying the matrices

$$6x \qquad +4z=0 \qquad (\dagger)$$
$$9y \qquad =0 \qquad (\dagger\dagger)$$
$$3x+5y+2z=0 \qquad (\dagger\dagger\dagger)$$

From $(\dagger\dagger)$ we have $y=0$. Substituting this into $(\dagger\dagger\dagger)$

$$3x+2z=0$$
$$3x=-2z$$
$$x=-\frac{2}{3}z$$

Let $z=a$ where $a\neq0$, thus the general eigenvector is $a\begin{pmatrix}-2/3\\0\\1\end{pmatrix}$.

Let $\mathbf{v}$ be the eigenvector for $\lambda=4$:

(1.16) $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\begin{pmatrix} 1-4 & 0 & 4 \\ 0 & 4-4 & 0 \\ 3 & 5 & -3-4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 0 & 4 \\ 0 & 0 & 0 \\ 3 & 5 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The general eigenvector is $\mathbf{v} = a\begin{pmatrix} 20 \\ 9 \\ 15 \end{pmatrix}$.

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5. We first find $\det(\mathbf{A} - \lambda\mathbf{I})$:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{pmatrix} 5-\lambda & 0 & 0 \\ -9 & 4-\lambda & -1 \\ -6 & 2 & 1-\lambda \end{pmatrix}$$

$$= (5-\lambda)\left[(4-\lambda)(1-\lambda) - (2\times(-1))\right]$$

$$= (5-\lambda)\left[4-5\lambda+\lambda^2+2\right] = (5-\lambda)\left[\lambda^2-5\lambda+6\right] = (5-\lambda)(\lambda-3)(\lambda-2)$$

Putting this to zero and solving gives

$$\lambda_1 = 2,\ \lambda_2 = 3 \text{ and } \lambda_3 = 5$$

Let $\mathbf{u}$ be the eigenvector for $\lambda_1 = 2$, then we have

$$\begin{pmatrix} 5-2 & 0 & 0 \\ -9 & 4-2 & -1 \\ -6 & 2 & 1-2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 & 0 \\ -9 & 2 & -1 \\ -6 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Multiplying out

$$3x = 0 \qquad (*)$$

$$-9x + 2y - z = 0 \qquad (**)$$

$$-6x + 2y - z = 0 \qquad (***)$$

From (*) we have x=0. By substituting x=0 into (**) or (***) we have

$$2y - z = 0 \text{ gives } 2y = z$$

If $y = 1$ then $z = 2$. Thus the eigenvector $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ corresponds to $\lambda_1 = 2$.

Let $\mathbf{v}$ be the eigenvector for $\lambda_2 = 3$:

$$\begin{pmatrix} 5-3 & 0 & 0 \\ -9 & 4-3 & -1 \\ -6 & 2 & 1-3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ -9 & 1 & -1 \\ -6 & 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Multiplying the matrices gives the equations.

$$2x = 0 \quad (\dagger)$$

$$-9x + y - z = 0 \quad (\dagger\dagger)$$

$$-6x + 2y - 2z = 0 \quad (\dagger\dagger\dagger)$$

From $(\dagger)$ we have $x = 0$. From $(\dagger\dagger)$ or $(\dagger\dagger\dagger)$ we have $y = z$. If $y = 1$ then $z = 1$

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ is an eigenvector for } \lambda_2 = 3$$

Let **w** be an eigenvector for $\lambda_3 = 5$:

$$\begin{pmatrix} 5-5 & 0 & 0 \\ -9 & 4-5 & -1 \\ -6 & 2 & 1-5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ -9 & -1 & -1 \\ -6 & 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

A solution for this matrix is $x = 1,\ y = -5,\ z = -4$. Thus $\begin{pmatrix} 1 \\ -5 \\ -4 \end{pmatrix}$ is an eigenvector for $\lambda_3 = 5$.

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6. Very similar to **EXAMPLE 28**.

$$\det\begin{pmatrix} -15-\lambda & 5 \\ 10 & -10-\lambda \end{pmatrix} = (-15-\lambda)(-10-\lambda) - 50$$

$$= (15+\lambda)(10+\lambda) - 50$$

$$= 150 + 25\lambda + \lambda^2 - 50 = \lambda^2 + 25\lambda + 100$$

Factorizing this, $\lambda^2 + 25\lambda + 100$, and putting the quadratic to zero gives

$$(\lambda + 20)(\lambda + 5) = 0 \text{ which gives } \lambda_1 = -20,\ \lambda_2 = -5$$

Substituting these into T gives the period

$$T_1 = \frac{2\pi}{\sqrt{20}} = 1.40s,\ T_2 = \frac{2\pi}{\sqrt{5}} = 2.81s$$
