Complete solutions to Exercise 12(d)

1. Using (12.13) in each case

(a) $(3\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(5\mathbf{i}+\mathbf{j}+7\mathbf{k})=(3\times 5)+(1\times 1)+(1\times 7)=23$

(b) $(-\mathbf{i}+\mathbf{j}-\mathbf{k})\cdot(3\mathbf{i}+2\mathbf{j}-\mathbf{k})=(-1\times 3)+(1\times 2)+(-1\times(-1))=0$

(c) $(10\mathbf{i}-15\mathbf{j}+\mathbf{k})\cdot(-21\mathbf{i}-\mathbf{k})=(10\times(-21))+(-15\times 0)+(1\times(-1))=-211$

2. Applying (12.15)

$$w=(100\mathbf{i}+220\mathbf{j}-250\mathbf{k})\cdot(150\mathbf{i}+200\mathbf{j})=(100\times 150)+(220\times 200)=59000=59\text{ kJ}$$

3. The displacement vector, $\mathbf{s}$, from $A=\mathbf{i}+\mathbf{j}+\mathbf{k}$ to $B=5\mathbf{i}-\mathbf{j}+2\mathbf{k}$ is given by

$$\mathbf{s}=(5\mathbf{i}-\mathbf{j}+2\mathbf{k})-(\mathbf{i}+\mathbf{j}+\mathbf{k})$$

$$=(5-1)\mathbf{i}+(-1-1)\mathbf{j}+(2-1)\mathbf{k}=4\mathbf{i}-2\mathbf{j}+\mathbf{k}$$

The work done, w, is found by using (12.15):

$$w=(5\mathbf{i}+3\mathbf{j}-2\mathbf{k})\cdot(4\mathbf{i}-2\mathbf{j}+\mathbf{k})=20-6-2=12\text{ J}$$

4. We first obtain the unit vector in the direction of $3\mathbf{i}+\mathbf{j}+\mathbf{k}$. The magnitude of this vector is

$$|3\mathbf{i}+\mathbf{j}+\mathbf{k}|=\sqrt{3^2+1^2+1^2}=\sqrt{11}$$

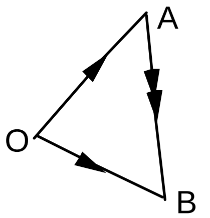
By (12.4) the unit vector, $\mathbf{u}$, is

$$\mathbf{u}=\frac{1}{\sqrt{11}}(3\mathbf{i}+\mathbf{j}+\mathbf{k})$$

The force, $\mathbf{F}$, along this direction is given by

$$\mathbf{F}=20\times\frac{1}{\sqrt{11}}(3\mathbf{i}+\mathbf{j}+\mathbf{k})=\frac{20}{\sqrt{11}}(3\mathbf{i}+\mathbf{j}+\mathbf{k})$$

Next we determine $\vec{AB}$ in $(\mathbf{i},\mathbf{j},\mathbf{k})$ components.



$$\vec{AB}=\vec{OB}-\vec{OA} \qquad (*)$$

We have $A=(3,-1,5)$ and $B=(-1,7,10)$, so

$$\vec{OB}=-\mathbf{i}+7\mathbf{j}+10\mathbf{k},\ \vec{OA}=3\mathbf{i}-\mathbf{j}+5\mathbf{k}$$

Substituting these into (*) gives

$$\vec{AB}=(-\mathbf{i}+7\mathbf{j}+10\mathbf{k})-(3\mathbf{i}-\mathbf{j}+5\mathbf{k})=-4\mathbf{i}+8\mathbf{j}+5\mathbf{k}$$

By (12.15) we have

$$\text{work done}=\frac{20}{\sqrt{11}}(3\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(-4\mathbf{i}+8\mathbf{j}+5\mathbf{k})$$

$$=\frac{20}{\sqrt{11}}\left((3\times(-4))+(1\times 8)+(1\times 5)\right)=6.03\text{ J}$$

(12.4) $\quad \mathbf{u}=\dfrac{1}{\sqrt{a^2+b^2+c^2}}(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})$

(12.13) $\quad (a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\cdot(d\,\mathbf{i}+e\mathbf{j}+f\,\mathbf{k})=ad+be+cf$

(12.15) $\quad$ Work done $=\mathbf{F}\cdot\mathbf{r}$

5. (i) We have

$\mathbf{r}\cdot\mathbf{r} = (x\,\mathbf{i} + y\mathbf{j} + z\mathbf{k})\cdot(x\,\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = x^2 + y^2 + z^2$

(ii) The magnitude of the vector $\mathbf{r}$ is given by

$$|\mathbf{r}| = |x\,\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = \sqrt{x^2 + y^2 + z^2}$$

Squaring both sides

$$|\mathbf{r}|^2 = |x\,\mathbf{i} + y\mathbf{j} + z\mathbf{k}|^2 = x^2 + y^2 + z^2$$

Notice that $|\mathbf{r}|^2 = \mathbf{r}\cdot\mathbf{r}$.

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6. We have

$$\cos(\theta) = \frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}||\mathbf{a}|} \qquad (*)$$

Finding the scalar product and magnitude gives

$$\mathbf{v}\cdot\mathbf{a} = (2\mathbf{i} + 3\mathbf{j})\cdot(2\mathbf{i} + 2\mathbf{j}) = 4 + 6 = 10$$

$$|\mathbf{v}| = |2\mathbf{i} + 3\mathbf{j}| = \sqrt{2^2 + 3^2} = \sqrt{13}$$

$$|\mathbf{a}| = |2\mathbf{i} + 2\mathbf{j}| = \sqrt{2^2 + 2^2} = \sqrt{8}$$

Substituting these into (*)

$$\cos(\theta) = \frac{10}{\sqrt{13}\sqrt{8}} \text{ which gives } \theta = \cos^{-1}\left(\frac{10}{\sqrt{13}\sqrt{8}}\right) = 11.31°$$

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7. (i) Differentiating $\mathbf{r}$ with respect to t gives the velocity and acceleration:

$$\mathbf{v} = \dot{\mathbf{r}} = (3t^2 - 2t)\mathbf{i} + 4t^3\mathbf{j}$$

$$\mathbf{a} = \ddot{\mathbf{r}} = (6t - 2)\mathbf{i} + 12t^2\mathbf{j}$$

(ii) Substituting $t = 1$ into $\mathbf{v}$ and $\mathbf{a}$ gives

$$\mathbf{v} = (3 - 2)\mathbf{i} + 4\mathbf{j} = \mathbf{i} + 4\mathbf{j}$$

$$\mathbf{a} = (6 - 2)\mathbf{i} + 12\mathbf{j} = 4\mathbf{i} + 12\mathbf{j}$$

To find the angle, θ, between the velocity and acceleration we use:

$$\cos(\theta) = \frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}||\mathbf{a}|} \qquad (\dagger)$$

We find $\mathbf{v}\cdot\mathbf{a}$ and $|\mathbf{v}||\mathbf{a}|$ and then substitute into $(\dagger)$:

$$\mathbf{v}\cdot\mathbf{a} = (\mathbf{i} + 4\mathbf{j})\cdot(4\mathbf{i} + 12\mathbf{j}) = (1\times 4) + (4\times 12) = 52$$

$$|\mathbf{v}| = |\mathbf{i} + 4\mathbf{j}| = \sqrt{1^2 + 4^2} = \sqrt{17}$$

$$|\mathbf{a}| = |4\mathbf{i} + 12\mathbf{j}| = \sqrt{4^2 + 12^2} = \sqrt{160}$$

Thus using $(\dagger)$

$$\cos(\theta) = \frac{52}{\sqrt{17}\sqrt{160}} \text{ which gives } \theta = \cos^{-1}\left(\frac{52}{\sqrt{17}\sqrt{160}}\right) = 4.399°$$

The angle between velocity and acceleration is $4.4°$.