Complete solutions to Exercise 12(e)

1. (i) Using (12.24) to find $\mathbf{r}\times\mathbf{s}$ gives

$$(5\mathbf{i}+\mathbf{j}+3\mathbf{k})\times(2\mathbf{i}-3\mathbf{j}-4\mathbf{k})=\det\begin{pmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\5&1&3\\2&-3&-4\end{pmatrix}$$

$$=\mathbf{i}\left[\det\begin{pmatrix}1&3\\-3&-4\end{pmatrix}\right]-\mathbf{j}\left[\det\begin{pmatrix}5&3\\2&-4\end{pmatrix}\right]+\mathbf{k}\left[\det\begin{pmatrix}5&1\\2&-3\end{pmatrix}\right]$$

$$=\mathbf{i}(-4+9)-\mathbf{j}(-20-6)+\mathbf{k}(-15-2)=5\mathbf{i}+26\mathbf{j}-17\mathbf{k}$$

(ii) Remember $\mathbf{r}\times\mathbf{s}=-(\mathbf{s}\times\mathbf{r})=-(5\mathbf{i}+26\mathbf{j}-17\mathbf{k})=-5\mathbf{i}-26\mathbf{j}+17\mathbf{k}$

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2. We need to find $(\mathbf{r}_1\times\mathbf{F}_1)+(\mathbf{r}_2\times\mathbf{F}_2)+(\mathbf{r}_3\times\mathbf{F}_3)$ (*)

Applying (12.24) to find $\mathbf{r}_1\times\mathbf{F}_1$ gives

$$(\mathbf{i}+3\mathbf{j}+\mathbf{k})\times(2\mathbf{i}+\mathbf{j}-3\mathbf{k})=\det\begin{pmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&3&1\\2&1&-3\end{pmatrix}$$

$$=\mathbf{i}\left[\det\begin{pmatrix}3&1\\1&-3\end{pmatrix}\right]-\mathbf{j}\left[\det\begin{pmatrix}1&1\\2&-3\end{pmatrix}\right]+\mathbf{k}\left[\det\begin{pmatrix}1&3\\2&1\end{pmatrix}\right]$$

$$=\mathbf{i}(-9-1)-\mathbf{j}(-3-2)+\mathbf{k}(1-6)=-10\mathbf{i}+5\mathbf{j}-5\mathbf{k}$$

Similarly we find $\mathbf{r}_2\times\mathbf{F}_2$

$$(\mathbf{i}-\mathbf{j}+2\mathbf{k})\times(3\mathbf{i}-4\mathbf{j}+6\mathbf{k})=\det\begin{pmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&2\\3&-4&6\end{pmatrix}$$

$$=\mathbf{i}\left[\det\begin{pmatrix}-1&2\\-4&6\end{pmatrix}\right]-\mathbf{j}\left[\det\begin{pmatrix}1&2\\3&6\end{pmatrix}\right]+\mathbf{k}\left[\det\begin{pmatrix}1&-1\\3&-4\end{pmatrix}\right]$$

$$=\mathbf{i}(-6+8)-\mathbf{j}(6-6)+\mathbf{k}(-4+3)=2\mathbf{i}-0\mathbf{j}-\mathbf{k}$$

Also $\mathbf{r}_3\times\mathbf{F}_3$ is

$$(\mathbf{i}+4\mathbf{j}+2\mathbf{k})\times(\mathbf{i}+10\mathbf{j}+7\mathbf{k})=\det\begin{pmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&4&2\\1&10&7\end{pmatrix}=\mathbf{i}\left[\det\begin{pmatrix}4&2\\10&7\end{pmatrix}\right]-\mathbf{j}\left[\det\begin{pmatrix}1&2\\1&7\end{pmatrix}\right]+\mathbf{k}\left[\det\begin{pmatrix}1&4\\1&10\end{pmatrix}\right]$$

$$=\mathbf{i}(28-20)-\mathbf{j}(7-2)+\mathbf{k}(10-4)=8\mathbf{i}-5\mathbf{j}+6\mathbf{k}$$

Substituting these into (*) gives

$$(-10\mathbf{i}+5\mathbf{j}-5\mathbf{k})+(2\mathbf{i}-\mathbf{k})+(8\mathbf{i}-5\mathbf{j}+6\mathbf{k})=(-10+2+8)\mathbf{i}+(5+0-5)\mathbf{j}+(-5-1+6)\mathbf{k}$$ T

$$=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}$$

hus the body is in equilibrium.

(12.24) $$(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\times(d\,\mathbf{i}+e\mathbf{j}+f\,\mathbf{k})=\det\begin{pmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a&b&c\\d&e&f\end{pmatrix}$$

3. Similar to solution 2. We need to find the vector products in each case.

$$(3\mathbf{i}+\mathbf{j}+5\mathbf{k})\times(2\mathbf{i}-\mathbf{j}+\mathbf{k})=\det\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 1 & 5\\ 2 & -1 & 1\end{pmatrix}=6\mathbf{i}+7\mathbf{j}-5\mathbf{k}$$

$$(5\mathbf{i}-2\mathbf{j}-\mathbf{k})\times(8\mathbf{i}-6\mathbf{j}+6\mathbf{k})=\det\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5 & -2 & -1\\ 8 & -6 & 6\end{pmatrix}=-18\mathbf{i}-38\mathbf{j}-14\mathbf{k}$$

$$(\mathbf{i}-6\mathbf{j}+7\mathbf{k})\times(4\mathbf{i}+3\mathbf{k})=\det\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -6 & 7\\ 4 & 0 & 3\end{pmatrix}=-18\mathbf{i}+25\mathbf{j}+24\mathbf{k}$$

$$(i+6k)\times(i-5j)=\det\begin{pmatrix}i & j & k\\ 1 & 0 & 6\\ 1 & -5 & 0\end{pmatrix}=30i+6j-5k$$

Thus the body is in equilibrium because

$$(6\mathbf{i}+7\mathbf{j}-5\mathbf{k})+(-18\mathbf{i}-38\mathbf{j}-14\mathbf{k})+(-18\mathbf{i}+25\mathbf{j}+24\mathbf{k})+(30\mathbf{i}+6\mathbf{j}-5\mathbf{k})$$
$$=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}$$

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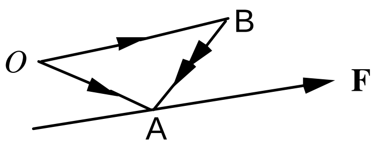
4. (a) We have



$$\vec{BA}=\vec{OA}-\vec{OB}$$

The position vectors of A and B are given by

$$\vec{OA}=\mathbf{i}+2\mathbf{j}+3\mathbf{k},\ \vec{OB}=5\mathbf{i}+7\mathbf{j}+9\mathbf{k}$$

Thus

$$\vec{BA}=(1-5)\mathbf{i}+(2-7)\mathbf{j}+(3-9)\mathbf{k}=-4\mathbf{i}-5\mathbf{j}-6\mathbf{k}$$

Using (12.25)

$$\mathbf{M}=(-4\mathbf{i}-5\mathbf{j}-6\mathbf{k})\times(2\mathbf{i}-\mathbf{j}+\mathbf{k})=\det\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -4 & -5 & -6\\ 2 & -1 & 1\end{pmatrix}=-11\mathbf{i}-8\mathbf{j}+14\mathbf{k}$$

(b) Similarly we have $\vec{OA}=\mathbf{i}-\mathbf{j},\ \vec{OB}=3\mathbf{i}-\mathbf{j}-\mathbf{k}$,

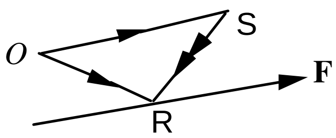
$$\vec{BA}=\vec{OA}-\vec{OB}=(\mathbf{i}-\mathbf{j})-(3\mathbf{i}-\mathbf{j}-\mathbf{k})=-2\mathbf{i}+\mathbf{k}$$

Applying (12.25)

$$\mathbf{M}=(-2\mathbf{i}+\mathbf{k})\times(4\mathbf{i}-3\mathbf{j}-6\mathbf{k})=\det\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -2 & 0 & 1\\ 4 & -3 & -6\end{pmatrix}=3\mathbf{i}-8\mathbf{j}+6\mathbf{k}$$

(12.25) $\mathbf{M}=\mathbf{r}\times\mathbf{F}$

5. We have



$$\vec{SR} = \vec{OR} - \vec{OS}$$
$$= (\mathbf{i} - \mathbf{j} - \mathbf{k}) - (7\mathbf{i} + \mathbf{j} - 9\mathbf{k}) = -6\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$$

Moment about S is

$$\vec{SR} \times \mathbf{F} = (-6\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}) \times (14\mathbf{i} + 2\mathbf{j} - 18\mathbf{k})$$
$$= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & -2 & 8 \\ 14 & 2 & -18 \end{pmatrix}$$
$$= \mathbf{i}\left[\det\begin{pmatrix} -2 & 8 \\ 2 & -18 \end{pmatrix}\right] - \mathbf{j}\left[\det\begin{pmatrix} -6 & 8 \\ 14 & -18 \end{pmatrix}\right] + \mathbf{k}\left[\det\begin{pmatrix} -6 & -2 \\ 14 & 2 \end{pmatrix}\right]$$
$$= \mathbf{i}(20) - \mathbf{j}(-4) + \mathbf{k}(16) = 20\mathbf{i} + 4\mathbf{j} + 16\mathbf{k}$$

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6. Applying (12.24) we have

$$(a\mathbf{i} + b\mathbf{j}) \times (c\mathbf{i} + d\,\mathbf{j}) = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & 0 \\ c & d & 0 \end{pmatrix} = \mathbf{i}\left[\det\begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix}\right] - \mathbf{j}\left[\det\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}\right] + \mathbf{k}\left[\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \mathrm{T}$$
$$= \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(ad - cb)$$

hus we have the required result:

$$(a\mathbf{i} + b\mathbf{j}) \times (c\mathbf{i} + d\,\mathbf{j}) = \mathbf{k}(ad - cb)$$

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7. We have

$$\mathbf{r} \times \mathbf{s} = (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (d\,\mathbf{i} + e\mathbf{j} + f\,\mathbf{k}) = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{pmatrix}$$
$$= \mathbf{i}\left[\det\begin{pmatrix} b & c \\ e & f \end{pmatrix}\right] - \mathbf{j}\left[\det\begin{pmatrix} a & c \\ d & f \end{pmatrix}\right] + \mathbf{k}\left[\det\begin{pmatrix} a & b \\ d & e \end{pmatrix}\right]$$
$$= \mathbf{i}(bf - ec) - \mathbf{j}(af - dc) + \mathbf{k}(ae - db)$$

Examining $\mathbf{s} \times \mathbf{r}$ gives

$$\mathbf{s} \times \mathbf{r} = (d\,\mathbf{i} + e\mathbf{j} + f\,\mathbf{k}) \times (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ d & e & f \\ a & b & c \end{pmatrix}$$
$$= \mathbf{i}\left[\det\begin{pmatrix} e & f \\ b & c \end{pmatrix}\right] - \mathbf{j}\left[\det\begin{pmatrix} d & f \\ a & c \end{pmatrix}\right] + \mathbf{k}\left[\det\begin{pmatrix} d & e \\ a & b \end{pmatrix}\right]$$
$$= \mathbf{i}(ec - bf) - \mathbf{j}(dc - af) + \mathbf{k}(db - ae)$$

(12.24)
$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (d\,\mathbf{i} + e\mathbf{j} + f\,\mathbf{k}) = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{pmatrix}$$

We can rewrite $\mathbf{s}\times\mathbf{r}$ as

$$\begin{aligned}\mathbf{s}\times\mathbf{r} &= \mathbf{i}(ec-bf)-\mathbf{j}(dc-af)+\mathbf{k}(db-ae)\\ &= -\mathbf{i}(bf-ec)+\mathbf{j}(af-dc)-\mathbf{k}(ae-db)\\ &= -\left[\mathbf{i}(bf-ec)-\mathbf{j}(af-dc)+\mathbf{k}(ae-db)\right] = -(\mathbf{r}\times\mathbf{s})\end{aligned}$$

Hence we have shown the required result, $\mathbf{r}\times\mathbf{s} = -(\mathbf{s}\times\mathbf{r})$.