Complete solutions to Exercise 13(a)

Throughout the solutions, C and D represent constant of integration.

1. We have

$$dp = -\rho g dz$$

Integrating

$$\int dp = \int -\rho g dz$$
$$p = -\rho g z + C$$

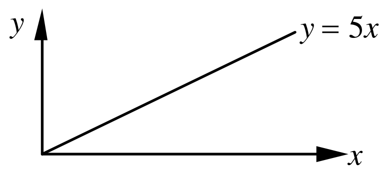
2. Similar to solution 1. We have $dp = -\rho\omega^2 r dr$, integrating this gives

$$p = -\frac{\rho\omega^2 r^2}{2} + C$$

3. By substituting $C = 5$ we have $\frac{dy}{dx} = 5$, rearranging this

$$dy = 5dx$$

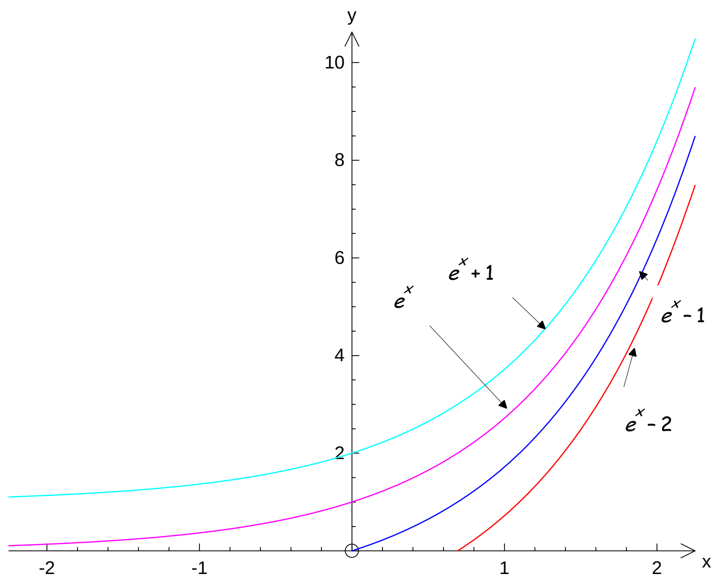
Integrating gives $y = 5x + D$. Since we want to plot the streamline which goes through the origin we have $x = 0,\ y = 0$ which gives $D = 0$. Thus we have a straight line $y = 5x$:



4. By rearranging the given differential equation we have

$$dy = e^x dx$$

Integrating yields $y = e^x + C$. We sketch the streamlines for $C = -2,\ -1,\ 0$ and 1:



5. We have

$$dv = adt$$

Integrating

$$\int dv = \int adt$$

$$v = at + C$$

Substituting $t = 0,\ v = u$ gives

$$u = 0 + C \text{ therefore } C = u$$

Thus $v = u + at$

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6. We have

$$a = \frac{dv}{dt} = 5 - 3t$$

$$dv = (5 - 3t)\,dt$$

Integrating both sides

$$\int dv = \int (5 - 3t)\,dt$$

$$v = 5t - \frac{3t^2}{2} + C$$

Substituting $t = 0,\ v = 8$ gives $C = 8$

$$v = 5t - \frac{3t^2}{2} + 8$$

By using hint, $v = \frac{ds}{dt}$, we have $ds = vdt$. Substituting for v gives

$$ds = \left(5t - \frac{3t^2}{2} + 8\right)dt$$

Integrating

$$\int ds = \int \left(5t - \frac{3t^2}{2} + 8\right)dt$$

$$s = \frac{5t^2}{2} - \frac{t^3}{2} + 8t + D$$

Substituting the initial conditions $t = 0,\ s = -2.1$ gives $D = -2.1$. Thus

$$s = 2.5t^2 - 0.5t^3 + 8t - 2.1$$

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7. We have $d\omega = \alpha dt$. Integrating this gives

$$\omega = \alpha t + C$$

Putting $t = 0,\ \omega = \omega_0$

$$\omega_0 = 0 + C \text{ which gives } C = \omega_0$$

Thus $\omega = \omega_0 + \alpha t$. We now have

$$\frac{d\theta}{dt} = \omega_0 + \alpha t \text{ therefore } d\theta = (\omega_0 + \alpha t)dt$$

Integrating gives

$$\int d\theta = \int (\omega_0 + \alpha t)\,dt$$

$$\theta = \omega_0 t + \frac{\alpha t^2}{2} + C$$

Substituting $t = 0,\ \theta = 0$ gives $C = 0$. Hence

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

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8. We have $ydy = -xdx$. Integrating

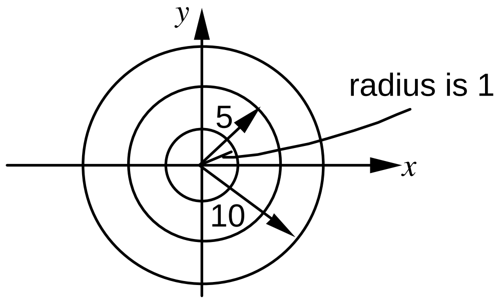
$$\int ydy = -\int xdx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C, \quad \frac{y^2}{2} + \frac{x^2}{2} = C$$

Multiplying both sides by 2:

$$y^2 + x^2 = 2C = A \text{ (constant)}$$

The equation $y^2 + x^2 = A$ are circles with centre origin and radius, $\sqrt{A}$.



For A=1, 25 and 100 we have circles of radius 1, 5 and 10 respectively.

9. Separating the variables gives

$$\frac{dy}{y} = -\frac{dx}{x}$$

Integrating both sides by using (8.2) we have

$$\ln(y) = -\ln(x) + C$$

$$\ln(y) + \ln(x) = C$$

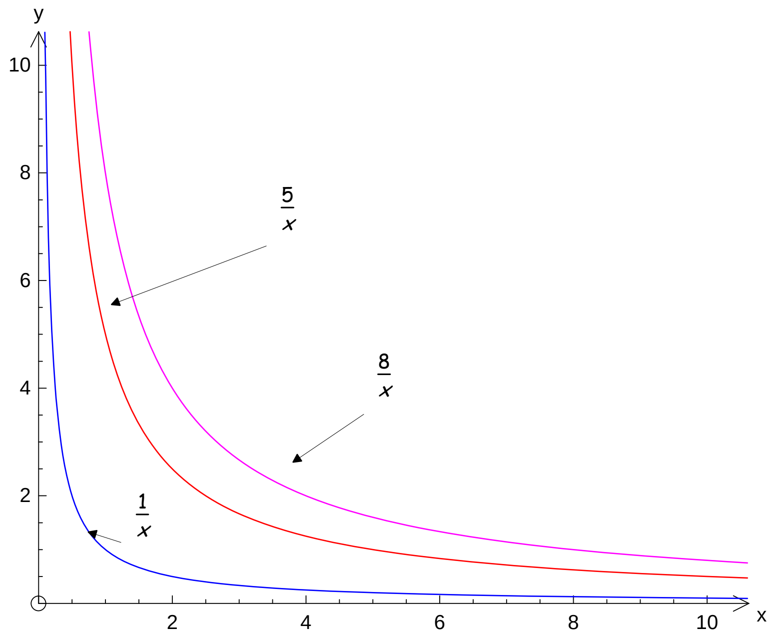
By applying (5.11) on the left hand side we have

$$\ln(xy) = C$$

Taking exponentials of both sides

$$xy = e^C = A \text{ (a constant)}$$

Thus rearranging yields $y = \frac{A}{x}$. Sketching the streamlines for $A = 1, 5$ and 8



(5.11) $$\ln(A) + \ln(B) = \ln(AB)$$

(8.2) $$\int \frac{du}{u} = \ln|u|$$

10. Separating the variables

$$\frac{dy}{y+1}=\frac{dx}{x+1}$$

Integrating gives

$$\ln(y+1)=\ln(x+1)+C$$

$$\ln(y+1)-\ln(x+1)=C$$

Applying (5.12) to the left hand side yields:

$$\ln\left(\frac{y+1}{x+1}\right)=C$$

Taking exponentials of both sides

$$\frac{y+1}{x+1}=e^C=A$$

Rearranging

$$y+1=A(x+1) \text{ which gives } y=Ax+A-1$$

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11. We have $d\theta = Cdx$. Integrating both sides yields

$$\theta = Cx + D$$

Substituting $x=0,\ \theta=\theta_1$ gives

$$\theta_1=(C\times 0)+D, \text{ hence } D=\theta_1$$

Substituting the other condition, $x=t,\ \theta=\theta_2$ gives

$$\theta_2=Ct+D=Ct+\theta_1$$

Rearranging gives $C=\dfrac{\theta_2-\theta_1}{t}$. Putting this into $\dfrac{d\theta}{dx}=C$ gives:

$$\frac{d\theta}{dx}=\frac{\theta_2-\theta_1}{t}$$

Substituting this into Fourier's law gives the required result:

$$Q=-kA\left(\frac{\theta_2-\theta_1}{t}\right)$$

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12. Separating the variables

$$\frac{dp}{p}=-\frac{mg}{RT}dz$$

$$\int\frac{dp}{p}=-\int\frac{mg}{RT}dz$$

$$\ln(p)=-\frac{mg}{RT}z+C$$

Taking exponentials and using the rules of indices

$$p=e^{-\frac{mg}{RT}z+C}=e^{-\frac{mg}{RT}z}e^C=e^{-\frac{mg}{RT}z}A \text{ where } A=e^C$$

$$p=Ae^{-\frac{mg}{RT}z}$$

(5.12) $$\ln(A)-\ln(B)=\ln(A/B)$$

13. Separating the variables

$$\frac{dp}{p^{1/\gamma}} = -kdz$$

$$\int p^{-1/\gamma} dp = -\int kdz$$

$$\frac{p^{-\frac{1}{\gamma}+1}}{-\frac{1}{\gamma}+1} = -kz + C$$

We can simplify: $-\frac{1}{\gamma}+1 = -\frac{1}{\gamma}+\frac{\gamma}{\gamma} = \frac{-1+\gamma}{\gamma} = \frac{\gamma-1}{\gamma}$. Thus we have

$$\frac{p^{\frac{\gamma-1}{\gamma}}}{\left(\frac{\gamma-1}{\gamma}\right)} = \frac{\gamma p^{\frac{\gamma-1}{\gamma}}}{(\gamma-1)} = -kz + C$$

Subtracting C and dividing by $-k$ gives $\frac{\gamma p^{\frac{\gamma-1}{\gamma}}}{-k(\gamma-1)} - \frac{C}{-k} = z$

Thus $z = \frac{\gamma p^{\frac{\gamma-1}{\gamma}}}{k(1-\gamma)} + A$ where $A = \frac{C}{k}$.