Complete solutions to Exercise 13(b)

1. Putting $P=2$ into (13.4) yields

$$\text{I.F.}=e^{\int 2dx}=e^{2x}$$

$$ye^{2x}=\int\left(e^{2x}e^{-x}\right)dx=\int\left(e^{2x-x}\right)dx=\int e^{x}dx=e^{x}+C$$

Dividing through by e^{2x} and using the rules of indices gives

$$y=\frac{e^{x}}{e^{2x}}+\frac{C}{e^{2x}}=e^{x-2x}+Ce^{-2x}=e^{-x}+Ce^{-2x}$$

Substituting the initial condition $x=0,\ y=0$, into $y=e^{-x}+Ce^{-2x}$ yields

$$0=1+C \text{ which gives } C=-1$$

Thus $y=e^{-x}-e^{-2x}$

2. Substituting $P=-1$ into (13.4) gives

$$\text{I.F.}=e^{\int -1dx}=e^{-x}$$

$$ye^{-x}=\int\left(e^{2x}e^{-x}\right)dx=\int\left(e^{2x-x}\right)dx=\int e^{x}dx=e^{x}+C$$

Divide through by e^{-x} into $ye^{-x}=e^{x}+C$ gives

$$y=\frac{e^{x}}{e^{-x}}+\frac{C}{e^{-x}}=e^{x-(-x)}+Ce^{-(-x)}$$

$$y=e^{2x}+Ce^{x}$$

3. *Can we use the integrating factor method?*

Yes but we need to put the given differential equation into the correct format. Divide through by $1-x^2$

$$\frac{dy}{dx}-\frac{2x}{1-x^2}y=\frac{1}{1-x^2}$$

This time our P for the integrating factor is **not** a constant but $P=-\dfrac{2x}{1-x^2}$

$$\text{I.F.}=\exp\left(\int\left(\frac{-2x}{1-x^2}\right)dx\right)$$

where exp is the exponential function. Thus the integration is

$$\int\left(\frac{-2x}{1-x^2}\right)dx \underbrace{=}_{\text{by (8.42)}} \ln\left(1-x^2\right) \quad\text{thus}\quad \text{I.F.}=e^{\ln\left(1-x^2\right)} \underbrace{=}_{\text{by (5.16)}} 1-x^2$$

Substituting this, I.F $=1-x^2$, into (13.4) gives

$$y\left(1-x^2\right)=\int\left(1-x^2\right)\frac{1}{\left(1-x^2\right)}dx=\int dx=x+C$$

Using the initial condition $x=0,\ y=0$ gives $C=0$ and we have

$y\left(1-x^2\right)=x$

Dividing through by $1-x^2$ gives $y=\dfrac{x}{1-x^2}$.

(5.16)	$e^{\ln(x)}=x$
(8.42)	$\int\frac{f'(x)}{f(x)}dx=\ln\lvert f(x)\rvert$
(13.4)	$y(\text{I.F.})=\int(\text{I.F.})Q(x)dx$

4. Dividing the given differential equation by m yields

$$\frac{dv}{dt}=g-\frac{k}{m}v,\ \text{ rearranging }\ \frac{dv}{dt}+\frac{k}{m}v=g$$

We find the integrating factor (I.F.) with $P=\dfrac{k}{m}$ and integrate this with respect to t:

$$\text{I.F.}=e^{\int \frac{k}{m}dt}=e^{\frac{k}{m}t}$$

$$ve^{\frac{k}{m}t}=\int\left(ge^{\frac{k}{m}t}\right)dt \underbrace{=}_{\text{by (8.41)}} \frac{ge^{\frac{k}{m}t}}{k/m}+C=\frac{mge^{\frac{k}{m}t}}{k}+C$$

Dividing through by $e^{\frac{k}{m}t}$ gives

$$v=\frac{mge^{\frac{k}{m}t}}{ke^{\frac{k}{m}t}}+\frac{C}{e^{\frac{k}{m}t}}=\frac{mg}{k}+Ce^{-\frac{k}{m}t}$$

Using the initial condition $t=0,\ v=0$ gives $0=\dfrac{mg}{k}+C$ yields $C=-\dfrac{mg}{k}$

Substituting this, $C=-\dfrac{mg}{k}$, into v gives the required result: $v=\dfrac{mg}{k}\left(1-e^{-\frac{k}{m}t}\right)$

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5. By rearranging we have $\dfrac{dv}{dt}+\dfrac{v}{t}=\dfrac{1}{t}$

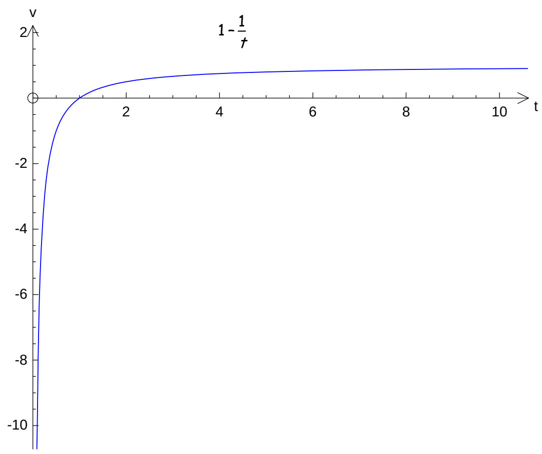
With $P=\dfrac{1}{t}$ the I.F.$=e^{\int dt/t}=e^{\ln(t)}=t$. Applying (13.4) with I.F.$=t$

$$vt=\int t\frac{1}{t}dt=\int dt=t+C$$

Dividing through by t gives $v=\dfrac{t}{t}+\dfrac{C}{t}=1+\dfrac{C}{t}$. Using the initial condition $t=1,\ v=0$ gives $C=-1$. Substituting this

$$v=1+\frac{C}{t}=1-\frac{1}{t} \qquad (*)$$

The terminal velocity is the velocity as $t\to\infty$. As $t\to\infty$ in (*), $v\to 1$.



(8.41) $\int e^{kx}dx=e^{kx}/k$

(13.4) $y(\text{I.F.})=\int(\text{I.F.})Q(x)dx$

6. (i) Dividing the given differential equation through by L:

$$\frac{di}{dt}+\frac{R}{L}i=\frac{1}{L}E(t) \qquad (*)$$

The integrating factor, $\text{I.F.}=e^{\frac{R}{L}t}$. Multiplying (*) by $\text{I.F.}=e^{\frac{R}{L}t}$ gives

$$e^{\frac{R}{L}t}\frac{di}{dt}+\frac{R}{L}e^{\frac{R}{L}t}i=\frac{e^{\frac{R}{L}t}}{L}E(t)$$

$$\frac{d}{dt}\left[e^{\frac{R}{L}t}i\right]=\frac{E(t)}{L}e^{\frac{R}{L}t}$$

Integrating

$$ie^{\frac{R}{L}t}=\int\left(\frac{E(t)}{L}e^{\frac{R}{L}t}\right)dt,\ \text{ rearranging gives }\ i=\frac{e^{-\frac{R}{L}t}}{L}\int\left(E(t)e^{\frac{R}{L}t}\right)dt$$

(ii) Substituting $E(t)=E$ into result (i) gives

$$i=\frac{e^{-\frac{R}{L}t}}{L}\int\left(Ee^{\frac{R}{L}t}\right)dt=\frac{e^{-\frac{R}{L}t}}{L}\left[\frac{Ee^{\frac{R}{L}t}}{(R/L)}+C\right]=\frac{E}{R}+\frac{Ce^{-\frac{R}{L}t}}{L}=\frac{E}{R}+Ae^{-\frac{R}{L}t}=i$$

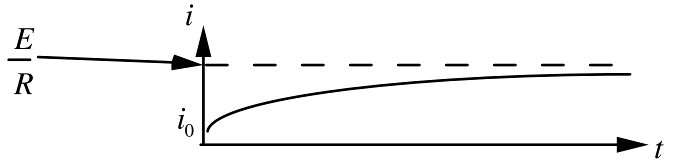
where $A=\dfrac{C}{L}$ (a constant). Substituting the initial condition $i(0)=i_0$ which means $t=0,\ i=i_0$:

$$i_0=\frac{E}{R}+A \ \text{ which gives }\ A=i_0-\frac{E}{R}$$

Thus we have the required result:

$$i=\frac{E}{R}+\left(i_0-\frac{E}{R}\right)e^{-\frac{Rt}{L}} \qquad (**)$$

(iii) From the initial condition $i(0)=i_0$ we know at $t=0,\ i=i_0$. Also as $t\to\infty$ in (**), $i\to\dfrac{E}{R}$ because the exponential in (**) goes to zero. Thus



7. Dividing through by RC gives the differential equation

$$\frac{dv}{dt}+\frac{v}{RC}=\frac{E}{RC}$$

The $\text{I.F.}=e^{t/(RC)}$ because $P=1/(RC)$. Thus

$$ve^{t/(RC)}=\int\left(e^{t/(RC)}\frac{E}{RC}\right)dt=\frac{E}{RC}\int\left(e^{t/(RC)}\right)dt\underbrace{=}_{\text{by (8.41)}}\frac{E}{RC}\left(\frac{e^{t/(RC)}}{1/(RC)}\right)+D=Ee^{t/(RC)}+D$$

We can find D by substituting the initial conditions $t=0,\ v=0$:

$$0=Ee^0+D \ \text{ thus }\ D=-E$$

We have

(8.41) $$\int e^{kx}dx=e^{kx}/k$$

$$ve^{t/(RC)} = Ee^{t/(RC)} - E = E\left(e^{t/(RC)} - 1\right)$$

To find v we divide through by $e^{t/(RC)}$

$$v = \frac{E}{e^{t/(RC)}}\left(e^{t/(RC)} - 1\right)$$

$$= E\left(\frac{e^{t/(RC)}}{e^{t/(RC)}} - \frac{1}{e^{t/(RC)}}\right)$$

$$= E\left(1 - e^{-t/(RC)}\right)$$

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8. Dividing through by $\left(1 \times 10^{-3}\right)$ gives the differential equation

$$\frac{di}{dt} + \left(3\times10^{6}\right)i = \left(10\times10^{3}\right)e^{t}$$

The integrating factor is

$$\text{I.F.} = e^{\int\left(3\times10^{6}\right)dt} = e^{\left(3\times10^{6}\right)t}$$

Applying (13.4)

$$ie^{\left(3\times10^{6}\right)t} = \int\left(\left(10\times10^{3}\right)e^{\left(3\times10^{6}\right)t}e^{t}\right)dt = \left(10\times10^{3}\right)\int\left(e^{\left[\left(3\times10^{6}\right)+1\right]t}\right)dt$$

$$= \left(10\times10^{3}\right)\underbrace{\frac{e^{\left[\left(3\times10^{6}\right)+1\right]t}}{\left(3\times10^{6}\right)+1}}_{\text{by (8.41)}} + C$$

$$= \left(3.33\times10^{-3}\right)e^{\left[\left(3\times10^{6}\right)+1\right]t} + C$$

To find i on it's own we divide through by $e^{\left(3\times10^{6}\right)t}$ and then simplify by using the rules of indices:

$$i = \frac{\left(3.33\times10^{-3}\right)e^{\left[\left(3\times10^{6}\right)+1\right]t}}{e^{\left(3\times10^{6}\right)t}} + \frac{C}{e^{\left(3\times10^{6}\right)t}}$$

$$= \left(3.33\times10^{-3}\right)e^{\left[\left(3\times10^{6}\right)+1-\left(3\times10^{6}\right)\right]t} + Ce^{-\left(3\times10^{6}\right)t}$$

$$= \left(3.33\times10^{-3}\right)e^{t} + Ce^{-\left(3\times10^{6}\right)t} = i$$

Using the initial condition, $t = 0,\ i = 0$ gives $C = -3.33\times10^{-3}$

Hence using this value for $C = -3.33\times10^{-3}$ we have

$$i = \left(3.33\times10^{-3}\right)\left[e^{t} - e^{-\left(3\times10^{6}\right)t}\right]\text{A} \quad \text{(A=amp)}$$

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9. Dividing the original differential equation by R

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{V}{R}e^{-t}$$

For the integrating factor we have $P = \dfrac{1}{RC}$ so, $\text{I.F.} = e^{\int\frac{1}{RC}dt} = e^{\frac{t}{RC}}$

(8.41)	$\int e^{kx} = e^{kx}/k$
(13.4)	$y(\text{I.F.}) = \int(\text{I.F.})Q(x)\,dx$

Applying (13.4) with this integrating factor

$$qe^{\frac{t}{RC}} = \int\left(\frac{V}{R}e^{\frac{t}{RC}}e^{-t}\right)dt$$

$$= \frac{V}{R}\int\left(e^{\frac{t}{RC}-t}\right)dt$$

$$= \frac{V}{R}\int\left(e^{\left(\frac{1}{RC}-1\right)t}\right)dt \underbrace{=}_{\text{by (8.41)}} \frac{V}{R}\frac{e^{\left(\frac{1}{RC}-1\right)t}}{\left(\frac{1}{RC}-1\right)}+D \qquad (\dagger)$$

Simplify the denominator:

$$R\left(\frac{1}{RC}-1\right) = R\left(\frac{1}{RC}-\frac{RC}{RC}\right)$$

$$= R\left(\frac{1-RC}{RC}\right)$$

$$= \frac{1-RC}{C}$$

Dividing through by $e^{\frac{t}{RC}}$ and replacing the denominator in $(\dagger)$:

$$q = \frac{Ve^{\left(\frac{1}{RC}-1-\frac{1}{RC}\right)t}}{(1-RC)/C}+De^{-\frac{t}{RC}}$$

$$= \frac{VCe^{-t}}{1-RC}+De^{-\frac{t}{RC}}$$

Using $t=0,\ q=0$ gives $D=-\dfrac{VC}{1-RC}$. Thus $q=\dfrac{VC}{1-RC}\left(e^{-t}-e^{-t/(RC)}\right)$

(13.4) $$y(\text{I.F.}) = \int(\text{I.F.})Q(x)\,dx$$