Complete solutions to Exercise 13(c)

1. (i) By applying Kirchhoff's voltage law we have

$$iR + L\frac{di}{dt} = 0, \quad \frac{di}{dt} = -\frac{iR}{L}$$

(ii) Separating variables $\frac{di}{i} = -\frac{R}{L}dt$. Integrating

$$\int \frac{di}{i} = -\int \frac{R}{L}dt$$

$$\underbrace{\ln(i)}_{\text{by (8.2)}} = -\frac{R}{L}t + C$$

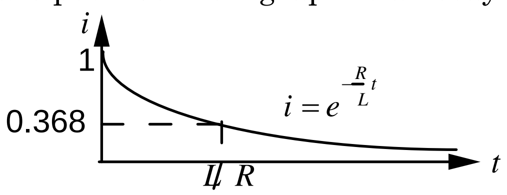
Substituting $t = 0, \; i = 1$

$$\ln(1) = 0 + C \text{ which gives } C = 0 \text{ because } \ln(1) = 0$$

$$\ln(i) = -\frac{R}{L}t$$

Taking exponentials gives $i = e^{-\frac{R}{L}t}$

(iii) Since R and L are positive the i graph is a decaying graph:



(iv) Substituting $t = \tau = L/R$ into i gives $i = e^{-\frac{R}{L}\frac{L}{R}} = e^{-1} = 0.368\,\text{A}$ (A = amp)

2. (i) Applying Kirchhoff's law we have

(voltage across resistor) + (voltage across capacitor)=0

There is no applied voltage so the right hand side is zero. Let v be the voltage across the capacitor then by using (13.6) we have

$$iR + v = 0 \qquad (*)$$

By (13.8) we have $i = C\frac{dv}{dt}$, substituting this into (*) gives a first order differential equation

$$CR\frac{dv}{dt} + v = 0 \text{ and so } \frac{dv}{dt} = -\frac{v}{RC}$$

(ii) Separating variables

$$\frac{dv}{v} = -\frac{dt}{RC}$$

Integrating $\int \frac{dv}{v} = -\int \frac{dt}{RC}$ gives

$$\underbrace{\ln(v)}_{\text{by (8.2)}} = -\frac{t}{RC} + D \qquad (\dagger)$$

Substituting the initial condition $t = 0, \; v = E$ gives $\ln(E) = D$. Substituting this into $(\dagger)$ yields

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(8.2) $\int du/u = \ln|u|$

(13.6) $v = iR$

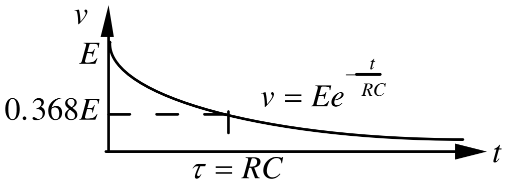
$$\ln(v) = -\frac{t}{RC} + \ln(E), \text{ rearranging } \ln(v) - \ln(E) = -\frac{t}{RC}$$

$$\underbrace{\ln\left(\frac{v}{E}\right)}_{\text{by (5.12)}} = -\frac{t}{RC}$$

Taking exponentials gives

$$\frac{v}{E} = e^{-\frac{t}{RC}} \text{ which gives } v = Ee^{-\frac{t}{RC}}$$

(iii) Similar to solution 1.



(iv) We have $v = Ee^{-RC/RC} = Ee^{-1} = 0.368E$

3. Similar to solution 2. Applying Kirchhoff's voltage law we have

$E = iR + v$ where v is the voltage across capacitor

Substituting $i = C\frac{dv}{dt}$ from (13.8), gives $E = C\frac{dv}{dt}R + v = RC\frac{dv}{dt} + v$. Rearranging

$$\frac{dv}{dt} = \frac{E - v}{RC}$$

Separating variables

$$\frac{dv}{E - v} = \frac{dt}{RC}$$

Integrating both sides $\int \frac{dv}{E - v} = \int \frac{dt}{RC}$ gives

$$\underbrace{-\ln(E - v)}_{\text{by (8.42)}} = \frac{t}{RC} + D \qquad (*)$$

Substituting the initial condition $t = 0, \ v = 0$ yields $-\ln(E) = D$. The remaining solution is very similar to 2. Thus we have

$$v = E\left(1 - e^{-t/RC}\right)$$

4. (i) By Kirchhoff's law we have (v is the voltage across the capacitor)

$$15 = iR + v = \left(10 \times 10^{3}\right)i + v \qquad (*)$$

We have $C = 10 \times 10^{-6}$ (because $\mu = 10^{-6}$), by (13.8) $i = C\frac{dv}{dt} = \left(10 \times 10^{-6}\right)\frac{dv}{dt}$.

Substituting for i into (*)

$$\left(10 \times 10^{3}\right)\left(10 \times 10^{-6}\right)\frac{dv}{dt} + v = 15$$

$$0.1\frac{dv}{dt} + v = 15, \text{ rearranging } 0.1\frac{dv}{dt} = 15 - v$$

(5.12) $\ln(A) - \ln(B) = \ln(A/B)$

(8.42) $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|$

(ii) Separating variables

$$\frac{0.1dv}{15-v}=dt, \quad \int \frac{0.1dv}{15-v}=\int dt$$

$$-0.1\ln(15-v)=t+C$$

Using the initial condition $t=0,\ v=0$ gives

$$-0.1\ln(15)=C$$

Thus by rearranging and using properties of logs we have

$$t=0.1\ln(15)-0.1\ln(15-v)=0.1\left[\ln(15)-\ln(15-v)\right]\underbrace{=}_{\text{by (5.12)}}0.1\ln\left(\frac{15}{15-v}\right)$$

Multiplying by 10:

$$10t=\ln\left(\frac{15}{15-v}\right)$$

Taking exponentials

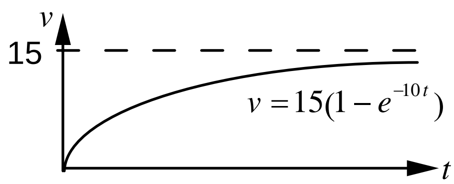
$$e^{10t}=e^{\ln\left(\frac{15}{15-v}\right)}\underbrace{=}_{\text{by (5.16)}}\frac{15}{15-v}$$

Rearranging yields

$$15-v=\frac{15}{e^{10t}}=15e^{-10t}$$

$$v=15-15e^{-10t}=15\left(1-e^{-10t}\right)$$

(iii) As $t\to\infty,\ v\to 15$ and when $t=0$ the voltage $v=15(1-1)=0$. Thus



(iv) By substituting the given values of t into $v=15\left(1-e^{-10t}\right)$ we have

$t=0.1,\ v=15\times\left(1-e^{-1}\right)=15\times 0.632,$ 63.2% of 15 volts

$t=0.2,\ v=15\times\left(1-e^{-2}\right)=15\times 0.865,$ 86.5% of 15 volts

$t=0.3,\ v=15\times\left(1-e^{-3}\right)=15\times 0.950,$ 95.0% of 15 volts

5. (i) Let v be the voltage across the capacitor then with $C=\left(3\times 10^{-6}\right)$:

$$i=C\frac{dv}{dt}=\left(3\times 10^{-6}\right)\frac{dv}{dt}$$

By Ohm's law the voltage across resistor is $iR=\left(15\times 10^{3}\right)i$. Substituting for i gives $iR=\left(15\times 10^{3}\right)i=\left(15\times 10^{3}\right)\left(3\times 10^{-6}\right)\frac{dv}{dt}=0.045\frac{dv}{dt}$

Using Kirchhoff's law yields

(5.12) $\ln(A)-\ln(B)=\ln(A/B)$

(5.16)
$$e^{\ln(x)} = x$$
$$0.045\frac{dv}{dt} + v = 15t^2$$
$$0.045\frac{dv}{dt} = 15t^2 - v$$

5. (ii) Maple output is:

```
> de_5:=0.045*diff(v(t),t)=15*t^2-v(t);
```

$$de\_5 := 0.045\left(\frac{d}{dt}\mathrm{v}(t)\right) = 15\,t^2 - \mathrm{v}(t)$$

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> soln:=dsolve({de_5,v(0)=1},v(t));
```

$$soln := \mathrm{v}(t) = 15\,t^2 - \frac{27\,t}{20} + \frac{243}{4000} + \frac{3757}{4000}\,\mathbf{e}^{\left(-\frac{200\,t}{9}\right)}$$

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> plot(rhs(soln),t=0..0.5);
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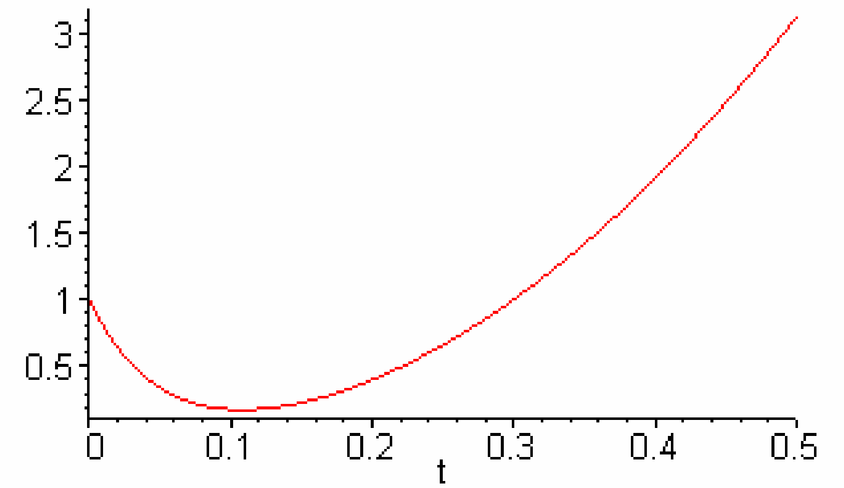

6. (i) Applying Kirchhoff's law gives $L\frac{di}{dt} + iR = 9\sin(t)$. Substituting the given values in the circuit, $L = 0.1\times10^{-3}$, $R = 3\times10^{3}$:

$$\left(0.1\times10^{-3}\right)\frac{di}{dt} + \left(3\times10^{3}\right)i = 9\sin(t)$$
$$\left(0.1\times10^{-3}\right)\frac{di}{dt} = 9\sin(t) - \left(3\times10^{3}\right)i$$

Dividing both sides by 0.1×10^{-3} gives

$$\frac{di}{dt} = \left(90\times10^{3}\right)\sin(t) - \left(30\times10^{6}\right)i$$

(ii) Maple output is:

```
> de_6:=diff(i(t),t)=(90*10^3)*sin(t)-(30*10^6)*i(t);
```

$$de\_6 := \frac{d}{dt}\mathrm{i}(t) = 90000\,\sin(t) - 30000000\,\mathrm{i}(t)$$

```
> soln:=evalf(dsolve({de_6,i(0)=1*10^(-3)},i(t)));
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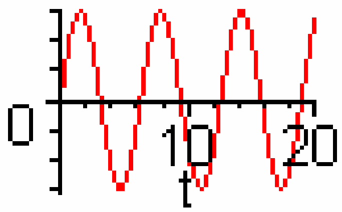
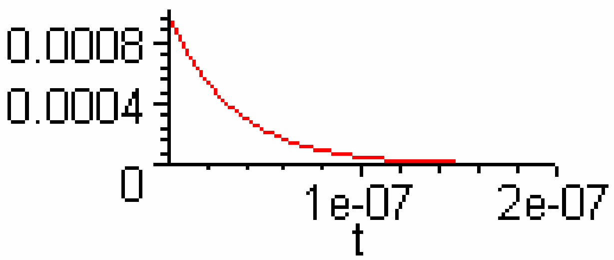
$$soln := \mathrm{i}(t) = -0.1000000000\;10^{-9}\cos(t) + 0.003000000000\sin(t)$$
$$+\;0.001000000100\,\mathbf{e}^{\left(-0.3000000000\,10^{8}\,t\right)}$$

```
> plot(rhs(soln),t=0..20);plot(rhs(soln),t=0..2*10^(-7));
```



(a)



(b)

7. (i) Applying Kirchhoff's law gives $L\frac{di}{dt}+iR=5\cos(100\pi t)$. Substituting the given values

$$\left(0.1\times10^{-3}\right)\frac{di}{dt}+\left(9\times10^{3}\right)i=5\cos\left(100\pi t\right)$$

$$\left(0.1\times10^{-3}\right)\frac{di}{dt}=5\cos\left(100\pi t\right)-\left(9\times10^{3}\right)i$$

$$\frac{di}{dt}=\left(50\times10^{3}\right)\cos\left(100\pi t\right)-\left(90\times10^{6}\right)i$$

Maple output is:

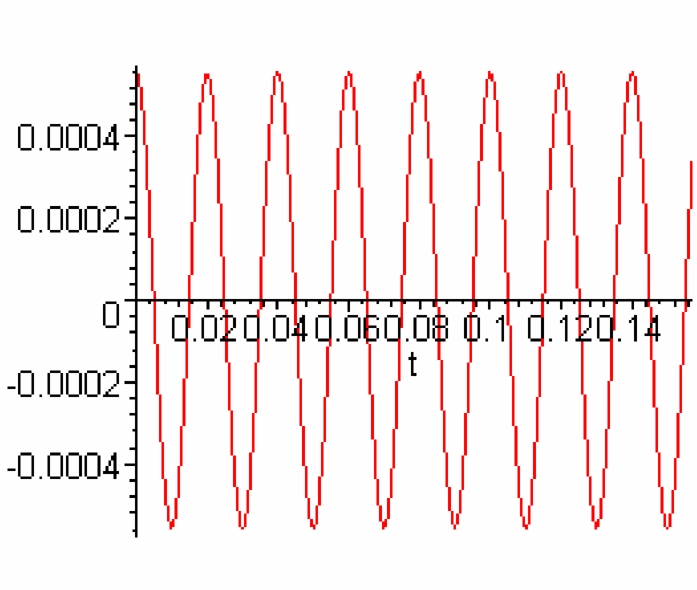
```
>de_7:=diff(i(t),t)=(50*10^3)*cos(100*Pi*t)-
(90*10^6)*i(t);
```

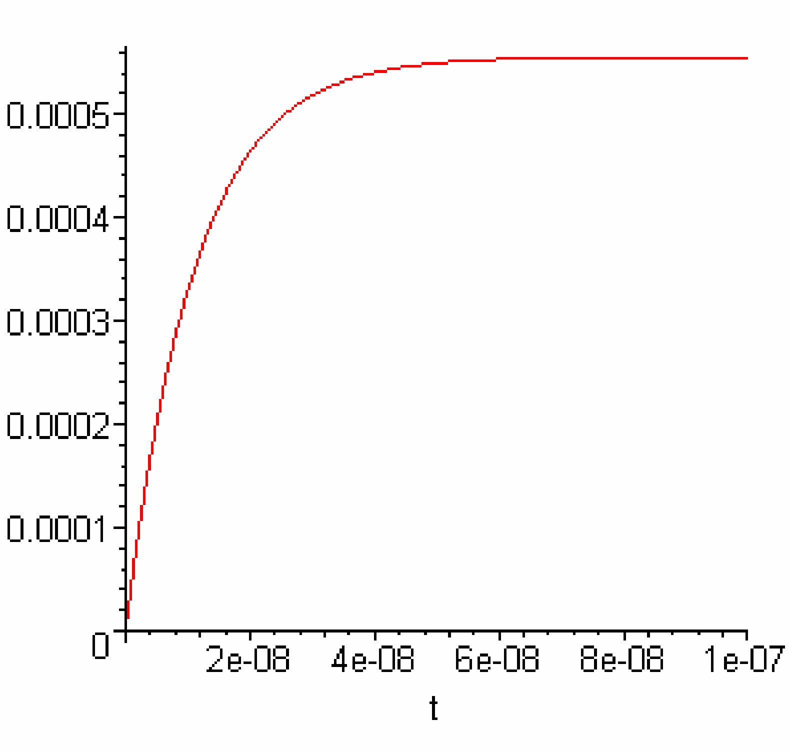
$$de\_7 := \frac{d}{dt}\,\mathrm{i}(t) = 50000\cos(100\,\pi\,t) - 90000000\,\mathrm{i}(t)$$

```
> soln:=evalf(dsolve({de_7,i(0)=0},i(t)));
```

$$soln := \mathrm{i}(t) = -0.0005555555554\,\mathrm{e}^{(-0.9000000000\,10^{8}\,t)} + 0.0005555555555\cos(314.1592654\,t)$$
$$+\,0.1939254724\,10^{-8}\sin(314.1592654\,t)$$

```
> plot(rhs(soln),t=0..Pi/20);plot(rhs(soln),t=0..10^(-7));
```





8. (i) Using Kirchhoff's law we have

$$\left(3\times10^{-3}\right)\frac{di}{dt}+\left(9\times10^{3}\right)i=240\sin\left(\omega t\right)$$

Dividing through by 3×10^{-3} gives

$$\frac{di}{dt}+\left(3\times10^{6}\right)i=\frac{240}{3\times10^{-3}}\sin\left(\omega t\right)$$

$$\frac{di}{dt}=\left(80\times10^{3}\right)\sin\left(\omega t\right)-\left(3\times10^{6}\right)i$$

(ii) Maple output is:

```
> de_8:=diff(i(t),t)=(80*10^3)*sin(omega*t)-(3*10^6)*i(t);
```

$$de\_8 := \frac{d}{dt}\,\mathrm{i}(t) = 80000\sin(\omega\, t) - 3000000\,\mathrm{i}(t)$$

```
> soln:=evalf(dsolve({de_8,i(0)=0},i(t)));
```

$soln :=$

$$\mathrm{i}(t) = \frac{80000.\, \mathbf{e}^{(-0.3000000\, 10^7\, t)}\, \omega}{0.9000000000\, 10^{13} + \omega^2} + \frac{80000.\, (-1.\, \cos(\omega\, t)\, \omega + 0.3000000\, 10^7 \sin(\omega\, t))}{0.9000000000\, 10^{13} + \omega^2}$$

```
> subs(omega=10000,rhs(soln));
```

$$0.00008888790128\, \mathbf{e}^{(-0.3000000\, 10^7\, t)} - 0.00008888790128 \cos(10000\, t) + 0.02666637038 \sin(10000\, t)$$

```
> plot(%,t=0..0.005);
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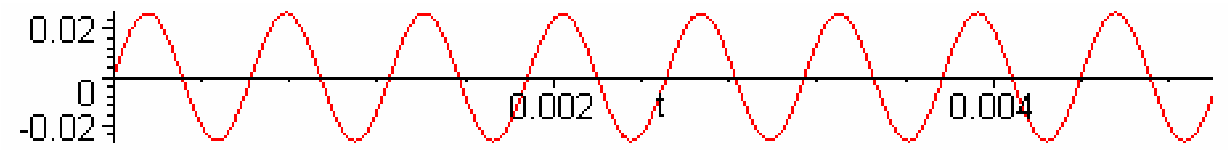


```
> subs(omega=100000000,rhs(soln)):plot(%,t=0..1e-6);
```