Complete solutions to Exercise 13(d)

1. Very similar to **EXAMPLE 13**.

(i) We have $A_0 = \pi \times 0.1^2$. Cross-section area of tank $A = 2^2 = 4$. Applying (13.10)

$$\frac{dh}{dt} = -\frac{\pi \times 0.1^2}{4}\sqrt{2gh} = -\left(34.79\times 10^{-3}\right)h^{1/2}$$

(ii) Separating variables

$$\frac{dh}{h^{1/2}} = -\left(34.79\times 10^{-3}\right)dt$$

Integrating gives

$$2h^{1/2} = -\left(34.79\times 10^{-3}\right)t + C$$

Substituting $t = 0,\ h = 1.3$ gives $2 \times 1.3^{1/2} = C$. Thus

$$h^{1/2} = \frac{-\left(34.79\times 10^{-3}\right)t + 2\left(1.3\right)^{1/2}}{2} = -\left(17.395\times 10^{-3}\right)t + 1.14$$

Squaring both sides

$$h = \left[-\left(17.395\times 10^{-3}\right)t + 1.14\right]^2$$

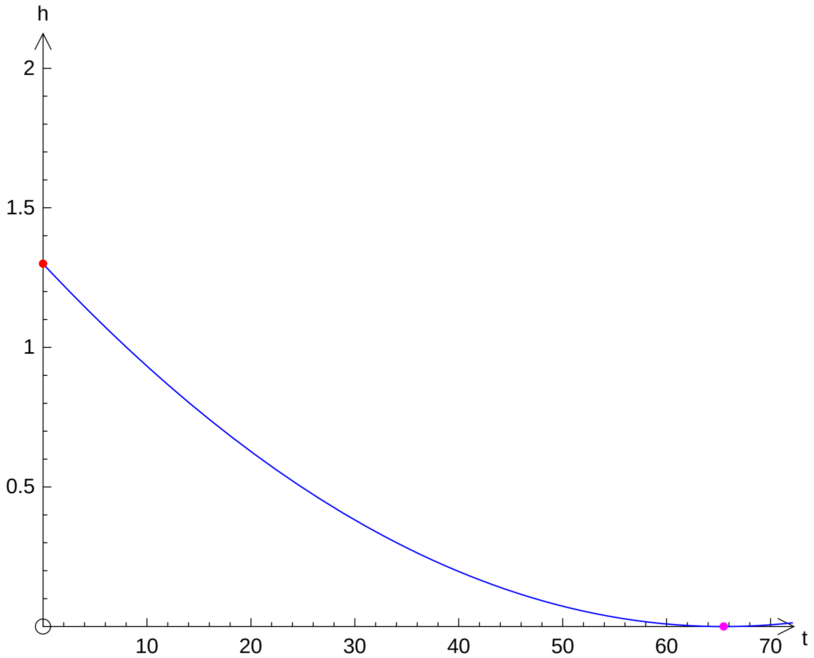
(iii) Need to find t for $h = 0$.

$$0 = -\left(17.395\times 10^{-3}\right)t + 1.14$$

$$t = \frac{1.14}{17.395\times 10^{-3}} = 65.5$$

In minutes, $t = \frac{65.5}{60} = 1.09\,\text{mins}$.

(iv) The MAPLE commands are given in section D. The graph has the same shape as Fig 21 and note the height, h, decreases with time, t, and $h = 0$ when $t \geq 65.5$.



(13.10) $$\frac{dh}{dt} = -\frac{A_0}{A}\sqrt{2gh}$$

2. (i) The area of the square outlet is $A_0 = 0.05^2$. Cross-sectional area of tank is $A = 0.75^2\pi$. By (13.10)

$$\frac{dh}{dt} = -\frac{0.05^2}{0.75^2\pi}\sqrt{2\times 9.81\times h} = -\left(6.266\times 10^{-3}\right)h^{1/2}$$

(ii) Separating variables

$$\frac{dh}{h^{1/2}} = -\left(6.266\times 10^{-3}\right)dt$$

$$h^{-1/2}dh = -\left(6.266\times 10^{-3}\right)dt$$

Integrating

$$2h^{1/2} = -\left(6.266\times 10^{-3}\right)t + C \qquad (\dagger)$$

Substituting $t = 0,\ h = 1.6$

$$2\times 1.6^{1/2} = 0 + C \text{ which gives } C = 2.53$$

Substituting for C into $(\dagger)$ and dividing both sides by 2

$$h^{1/2} = \frac{-\left(6.266\times 10^{-3}\right)t + 2.53}{2} = -\left(3.133\times 10^{-3}\right)t + 1.265$$

Squaring both sides

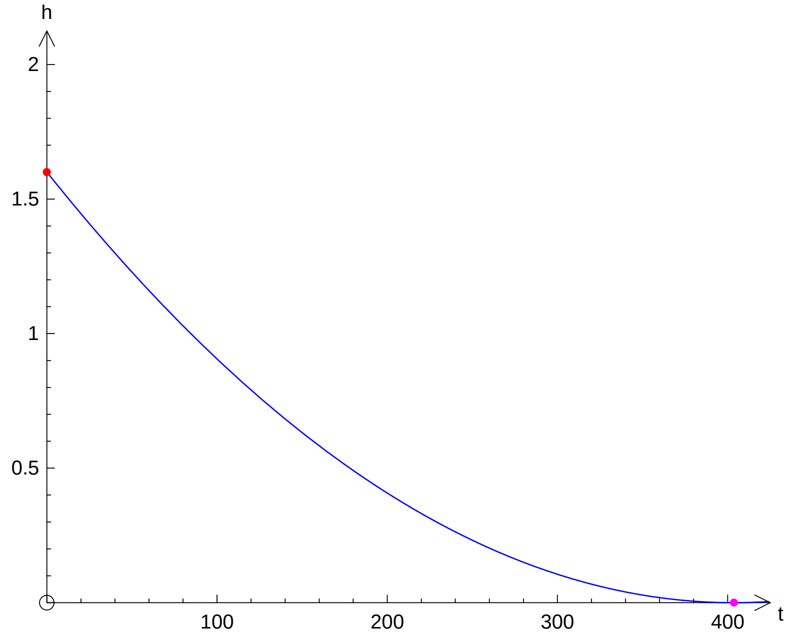
$$h = \left[-\left(3.133\times 10^{-3}\right)t + 1.265\right]^2$$

(iii) The tank is empty when $h = 0$. Hence

$$h = \left[-\left(3.133\times 10^{-3}\right)t + 1.265\right]^2 = 0 \text{ gives } t = \frac{1.265}{3.133\times 10^{-3}} = 403.8$$

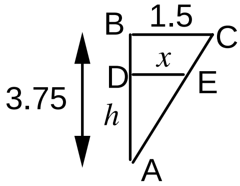
The time taken in minutes to empty the tank is $t = \dfrac{403.8}{60} = 6.73\text{mins}$.

(iv) The graph is the same shape as 1(iv) but cuts the axis at 1.6 and the t axis at 403.8. When $t \ge 403.8$ the tank is empty, $h = 0$.



(13.10) $$\frac{dh}{dt} = -\frac{A_0}{A}\sqrt{2gh}$$

3. (i) We have to use similarity of triangles to obtain a relationship between height, h, and x as shown.



Triangles ABC and ADE are similar so using hint we have

$$\frac{x}{h}=\frac{1.5}{3.75}=0.4 \quad \text{transposing gives} \qquad x=0.4h$$

The cross-sectional area, A, is a circle with radius $x=0.4h$

$$A=\pi(0.4h)^2=0.16\pi h^2$$

Outlet area, A_0, is a circle

$$A_0=\pi\left(\frac{0.05}{2}\right)^2=\left(6.25\times10^{-4}\right)\pi$$

Substituting these into (13.10) and using $g=9.81$ we have

$$\frac{dh}{dt}=-\frac{\left(6.25\times10^{-4}\right)\pi}{0.16\pi h^2}\sqrt{2gh}=-\frac{\left(6.25\times10^{-4}\right)}{0.16}\sqrt{2g}\,\frac{\sqrt{h}}{h^2}=-\left(17.3\times10^{-3}\right)h^{-3/2}$$

(ii) Separating variables

$$h^{3/2}dh=-\left(17.3\times10^{-3}\right)dt$$

$$\frac{2h^{5/2}}{5}=-\left(17.3\times10^{-3}\right)t+C$$

Substituting the initial condition $t=0,\ h=3$ gives $C=\left(2\times3^{5/2}\right)/5=6.235$

$$\frac{2h^{5/2}}{5}=-\left(17.3\times10^{-3}\right)t+6.235$$

Transposing to make h the subject gives

$$h^{5/2}=\frac{5}{2}\left[-\left(17.3\times10^{-3}\right)t+6.235\right]=-\left(43.25\times10^{-3}\right)t+15.59$$

$$h=\left[-\left(43.25\times10^{-3}\right)t+15.59\right]^{2/5}$$

(iii) We need to find t for $h=0$

$$0=-\left(17.3\times10^{-3}\right)t+6.235 \quad \text{transposing} \quad t=\frac{6.235}{17.3\times10^{-3}}=360$$

In minutes $t=6$ mins. Takes 6 mins to empty the tank which is 3m full.
(iv) By using a graphical calculator or a symbolic manipulator we have

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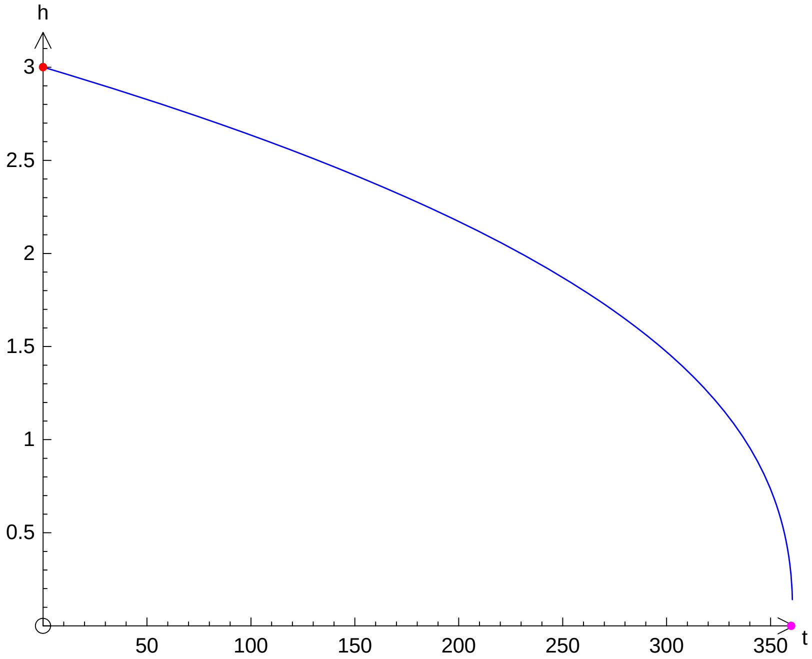
(13.10) $$\frac{dh}{dt}=-\frac{A_0}{A}\sqrt{2gh}$$



The graph indicates that the height of water in the tank decreases slowly at the start and decreases more rapidly towards the end as you would expect from a conical tank.

(v) Impossible because the tank only has a height of 3.75m.

4. We have

$$\frac{dh}{dt} = -k\sqrt{2gh} = -k\sqrt{2g}\sqrt{h}$$

Separating variables

$$\frac{dh}{h^{1/2}} = -\left(k\sqrt{2g}\right)dt$$

$$\int h^{-1/2}dh = -\int\left(k\sqrt{2g}\right)dt$$

$$2h^{1/2} = -\left(k\sqrt{2g}\right)t + C$$

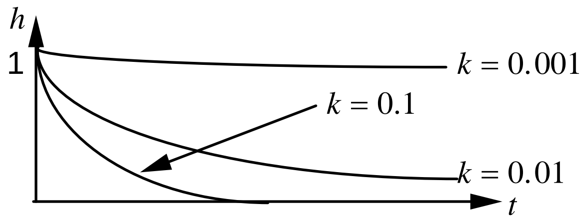
Substituting the initial condition $t = 0, \ h = 1$ gives $C = 2$. Thus we have

$$h^{1/2} = -\frac{1}{2}\left(k\sqrt{2g}\right)t + \frac{2}{2} = -\left(k\sqrt{\frac{2g}{4}}\right)t + 1$$

Squaring both sides

$$h = \left[-\left(k\sqrt{\frac{g}{2}}\right)t + 1\right]^2$$

The graph of height, h, against time, t, for $k = 0.1, \ 0.01$ and $\ 0.001$ is



The graphs show that the height, h, of water in the tank decreases more rapidly for large k.

5. Similar to **EXAMPLE 14**. Substituting $T = 300$ into (13.11) we have

$$\frac{d\theta}{dt} = k(\theta - 300)$$

Separating variables

$$\frac{d\theta}{\theta - 300} = kdt$$

Integrating both sides $\int \frac{d\theta}{\theta - 300} = \int kdt$ gives

$$\ln(\theta - 300) = kt + C \qquad (\dagger)$$

Substituting $t = 0,\ \theta = 373$ into $(\dagger)$

$$\ln(373 - 300) = C \text{ gives } C = \ln(73)$$

Putting this into $(\dagger)$

$$\ln(\theta - 300) = kt + \ln(73)$$

$$kt = \ln(\theta - 300) - \ln(73) = \ln\left(\frac{\theta - 300}{73}\right) \qquad (\dagger\dagger)$$

When $t = 5 \times 60 = 300,\ \theta = 330$, substituting these into $(\dagger\dagger)$

$$300k = \ln\left(\frac{330 - 300}{73}\right)$$

$$k = \frac{\ln(30/73)}{300} = -2.96 \times 10^{-3}$$

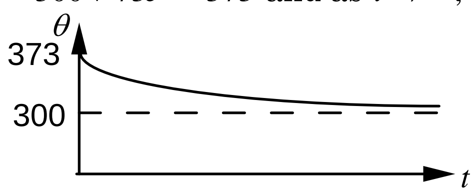
Putting $k = -2.96 \times 10^{-3}$ into $(\dagger\dagger)$ gives

$$\ln\left(\frac{\theta - 300}{73}\right) = -\left(2.96 \times 10^{-3}\right)t$$

Taking exponentials

$$\frac{\theta - 300}{73} = e^{-\left(2.96\times10^{-3}\right)t} \quad \text{rearranging} \quad \theta = 300 + 73e^{-\left(2.96\times10^{-3}\right)t}$$

For graph: At $t = 0,\ \theta = 300 + 73e^{0} = 373$ and as $t \to \infty,\ \theta \to 300$



6. From (13.11) we have $\frac{d\theta}{dt} = k(\theta - 300)$

because the surrounding temperature is $300K$. Separating variables

$$\frac{d\theta}{\theta - 300} = kdt \quad \text{and integrating} \quad \ln(\theta - 300) = kt + C$$

Using $t = 0,\ \theta = 400$ gives $C = \ln(100)$. We have

$$\ln(\theta - 300) - \ln(100) = kt \quad \text{which gives} \quad \ln\left(\frac{\theta - 300}{100}\right) = kt$$

(13.11) $$\frac{d\theta}{dt} = k(\theta - T)$$

Taking exponentials and rearranging gives $\theta = 300 + 100e^{kt}$. We cannot find the value of k in this case because we do not have enough information.

7. Separating variables $\dfrac{d\theta}{\theta - T} = kdt$. Integrating both sides

$$\int \frac{d\theta}{\theta - T} = \int kdt \quad \text{gives} \quad \ln(\theta - T) = kt + C \qquad (*)$$

Substituting the initial condition $t = 0,\ \theta = T_0$ gives

$$\ln(T_0 - T) = C$$

We have

$$\ln(\theta - T) = kt + \ln(T_0 - T)$$
$$\ln(\theta - T) - \ln(T_0 - T) = kt$$

By using the properties of logs

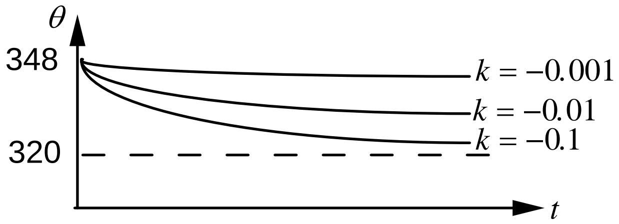
$$\ln\left(\frac{\theta - T}{T_0 - T}\right) = kt$$

Taking exponentials of both sides

$$\frac{\theta - T}{T_0 - T} = e^{kt}, \quad \text{rearranging} \quad \theta - T = (T_0 - T)e^{kt}$$

Thus $\theta = T + (T_0 - T)e^{kt}$

8. Using the result of question 7, $\theta = T + (T_0 - T)e^{kt}$, with $T_0 = 348,\ T = 320$ we have $\theta = 320 + 28e^{kt}$. Plotting graphs for $k = -0.001,\ -0.01$ and -0.1 gives



The graph shows the larger the absolute value of k the more rapidly the temperature drops to the surrounding temperature of $320K$.

9. Separating variables $\dfrac{d\theta}{\theta^4 - T^4} = kdt$. To find θ we have to integrate the left hand side by using partial fractions.

$$\theta^4 - T^4 = (\theta^2 - T^2)(\theta^2 + T^2)$$
$$= (\theta - T)(\theta + T)(\theta^2 + T^2)$$

We have

$$\frac{1}{\theta^4 - T^4} = \frac{1}{(\theta - T)(\theta + T)(\theta^2 + T^2)} = \frac{A}{\theta - T} + \frac{B}{\theta + T} + \frac{C\theta + D}{\theta^2 + T^2} \qquad (*)$$

Thus

$$1 = A(\theta + T)(\theta^2 + T^2) + B(\theta - T)(\theta^2 + T^2) + (C\theta + D)(\theta - T)(\theta + T) \qquad (**)$$

Substituting $\theta = -T$ into (**) gives

$$1 = 0 + B(-T - T)\left((-T)^2 + T^2\right) + 0$$

$$= B(-2T)\left(2T^2\right) \text{ which gives } B = -\frac{1}{4T^3}$$

How can we find A?

Substitute $\theta = T$ into (**)

$$1 = A(T + T)\left(T^2 + T^2\right) + 0 + 0$$

$$1 = 4T^3 A \text{ which gives } A = \frac{1}{4T^3}$$

To find C we equate coefficients of θ^3 in (**):

$$0 = A + B + C$$

$$0 = \frac{1}{4T^3} - \frac{1}{4T^3} + C \text{ gives } C = 0$$

To find D we equate coefficients of θ^2 in (**):

$$0 = AT - BT + D$$

$$= \frac{T}{4T^3} + \frac{T}{4T^3} + D = \frac{1}{4T^2} + \frac{1}{4T^2} + D = \frac{1}{2T^2} + D \text{ thus } D = -\frac{1}{2T^2}$$

Substituting $A = \frac{1}{4T^3}$, $B = -\frac{1}{4T^3}$, $C = 0$ and $D = -\frac{1}{2T^2}$ into (*)

$$\frac{1}{\theta^4 - T^4} = \frac{1}{4T^3(\theta - T)} - \frac{1}{4T^3(\theta + T)} - \frac{1}{2T^2\left(\theta^2 + T^2\right)}$$

$$= \frac{1}{4T^3}\left[\frac{1}{\theta - T} - \frac{1}{\theta + T} - \frac{2T}{\theta^2 + T^2}\right]$$

$$\int \frac{d\theta}{\theta^4 - T^4} = \frac{1}{4T^3}\left[\int \frac{d\theta}{\theta - T} - \int \frac{d\theta}{\theta + T} - 2T\int \frac{d\theta}{\theta^2 + T^2}\right]$$

$$= \frac{1}{4T^3}\left[\ln(\theta - T) - \ln(\theta + T) - 2T\underbrace{\frac{1}{T}\tan^{-1}\left(\frac{\theta}{T}\right)}_{\text{by (8.26)}}\right]$$

$$= \frac{1}{4T^3}\left[\ln\left(\frac{\theta - T}{\theta + T}\right) - 2\tan^{-1}\left(\frac{\theta}{T}\right)\right]$$

The constant of integration is added at the end on the right hand side. We have $\int \frac{d\theta}{\theta^4 - T^4} = \int k\,dt = kt + C$, thus the required result

$$\frac{1}{4T^3}\left[\ln\left(\frac{\theta - T}{\theta + T}\right) - 2\tan^{-1}\left(\frac{\theta}{T}\right)\right] = kt + C$$

(8.26) $$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$