Complete solutions to Exercise 13(e)

1. Applying (13.12) with $h = 0.1$ gives

$$y_{n+1} = y_n + 0.1\left[f\left(x_n, y_n\right)\right]$$

$$y_{n+1} = y_n + 0.1\left(x_n + y_n^{\,2}\right) \qquad (*)$$

Putting $n = 0$ gives

$$y_1 = y_0 + 0.1\left(x_0 + y_0^{\,2}\right)$$

What is x_0 and y_0*?*

We have $y(0) = 1$ which gives $x_0 = 0$ and $y_0 = 1$. Substituting these

$$y_1 = 1 + 0.1\left(0 + 1^2\right) = 1.1$$

For $x = 0.2$. Substituting $n = 1$ into (*) gives

$$y_2 = y_1 + 0.1\left(x_1 + y_1^{\,2}\right)$$

$$= 1.1 + 0.1\left(0.1 + 1.1^2\right) = 1.231$$

For $x = 0.3$. Placing $n = 2$ into (*) gives

$$y_3 = y_2 + 0.1\left(x_2 + y_2^{\,2}\right)$$

$$= 1.231 + 0.1\left(0.2 + 1.231^2\right) = 1.4025361$$

Thus $y_1 = 1.1$, $y_2 = 1.231$ and $y_3 = 1.4025361$, the y values at $x = 0.1,\ 0.2$ and 0.3 respectively.

2. Same differential equation as **EXAMPLE 15**. We have already evaluated $y(0.3)$ in **EXAMPLE 15**. Also

$$y_{n+1} = y_n + 0.1\left(x_n^{\,2} + y_n^{\,2}\right) \qquad (*)$$

We need to find y_5. From the example $y_3 = 1.3753284$. Putting $n = 3$ into (*)

$$y_4 = y_3 + 0.1\left(x_3^{\,2} + y_3^{\,2}\right)$$

$$= 1.3753284 + 0.1\left(0.3^2 + 1.3753284^2\right) = 1.5734812$$

Putting $n = 4$ into (*) gives

$$y_5 = y_4 + 0.1\left(x_4^{\,2} + y_4^{\,2}\right)$$

$$= 1.5734812 + 0.1\left(0.4^2 + 1.5734812^2\right) = 1.8370655$$

The value of y at $x = 0.5$ is 1.8370655.

3. By applying (13.12) with $h = 1$ we have

$$y_{n+1} = y_n + 1\left[f\left(t_n, y_n\right)\right]$$

$$y_{n+1} = y_n + t_n \qquad (*)$$

For $t = 1$. Substituting $n = 0$ and $t_0 = 0$, $y_0 = 0$ (initial condition) gives

$$y_1 = y_0 + t_0 = 0 + 0 = 0$$

For $t = 2$. Substituting $n = 1$ into (*)

$$y_2 = y_1 + t_1 = 0 + 1 = 1$$

For $t = 3$. Substituting $n = 2$ into (*)

$$y_3 = y_2 + t_2 = 1 + 2 = 3$$

(13.12) $$y_{n+1} = y_n + h\left[f\left(x_n, y_n\right)\right]$$

For $t = 4$. Substituting $n = 3$ into (*)

$$y_4 = y_3 + t_3 = 3 + 3 = 6$$

For $t = 5$. Substituting $n = 4$ into (*)

$$y_5 = y_4 + t_4 = 6 + 4 = 10$$

Completed table:

t	Euler y with $h = 1$	Exact $y = 0.5t^2$
0	0	0
1	0	0.5
2	1	2
3	3	4.5
4	6	8
5	10	12.5

4. Using (13.12) with $h = 0.05$ gives

$$v_{n+1} = v_n + 0.05\left[f\left(t_n, v_n\right)\right] = v_n + 0.05\left(10 - 0.1v_n^{\ 2}\right) = v_n + 0.5 - 0.005v_n^{\ 2}$$

$$v_{n+1} = v_n - 0.005v_n^{\ 2} + 0.5 \qquad (*)$$

We need to carry out 6 iterations because $6 \times 0.05 = 0.3$.

Putting $n = 0$ into (*) and $v_0 = 0$ into the result gives

$$v_1 = v_0 - 0.005v_0^{\ 2} + 0.5 = 0 - \left(0.005 \times 0^2\right) + 0.5 = 0.5$$

Putting $n = 1$ into (*) gives

$$v_2 = v_1 - 0.005v_1^{\ 2} + 0.5 = 0.5 - \left(0.005 \times 0.5^2\right) + 0.5 = 0.99875$$

Putting $n = 2$ into (*) gives

$$v_3 = v_2 - 0.005v_2^{\ 2} + 0.5 = 0.99875 - \left(0.005 \times 0.99875^2\right) + 0.5 = 1.49376$$

Putting $n = 3$ into (*) gives

$$v_4 = v_3 - 0.005v_3^{\ 2} + 0.5 = 1.49376 - \left(0.005 \times 1.49376^2\right) + 0.5 = 1.98260$$

Putting $n = 4$ into (*) gives

$$v_5 = v_4 - 0.005v_4^{\ 2} + 0.5 = 1.98260 - \left(0.005 \times 1.98260^2\right) + 0.5 = 2.46295$$

Putting $n = 5$ into (*) gives

$$v_6 = v_5 - 0.005v_5^{\ 2} + 0.5 = 2.46295 - \left(0.005 \times 2.46295^2\right) + 0.5 = 2.93262$$

Thus $v(0.3) = 2.93262$.

For Maple solutions see next page.

(13.12) $\qquad y_{n+1} = y_n + h\left[f\left(x_n, y_n\right)\right]$

5. We have

```
> de_5:=diff(v(t),t)=10-(v(t))^2:
>
soln_1:=dsolve({de_5,v(0)=0},v(t),type=numeric,method=classical,output=array([0.5,1,1.5,2]),stepsize=0.1);
```

$$soln\_1 := \begin{bmatrix} [t, \mathrm{v}(t)] \\ \begin{bmatrix} 0.5 & 3.05662292337655916 \\ 1 & 3.16153621885291258 \\ 1.5 & 3.16227268550349816 \\ 2 & 3.16227762680160662 \end{bmatrix} \end{bmatrix}$$

```
>
soln_2:=dsolve({de_5,v(0)=0},v(t),type=numeric,method=classical,output=array([0.5,1,1.5,2]),stepsize=0.01);
```

$$soln\_2 := \begin{bmatrix} [t, \mathrm{v}(t)] \\ \begin{bmatrix} 0.5 & 2.91998426966538282 \\ 1 & 3.15266079012819888 \\ 1.5 & 3.16191037745882974 \\ 2 & 3.16226365415457700 \end{bmatrix} \end{bmatrix}$$

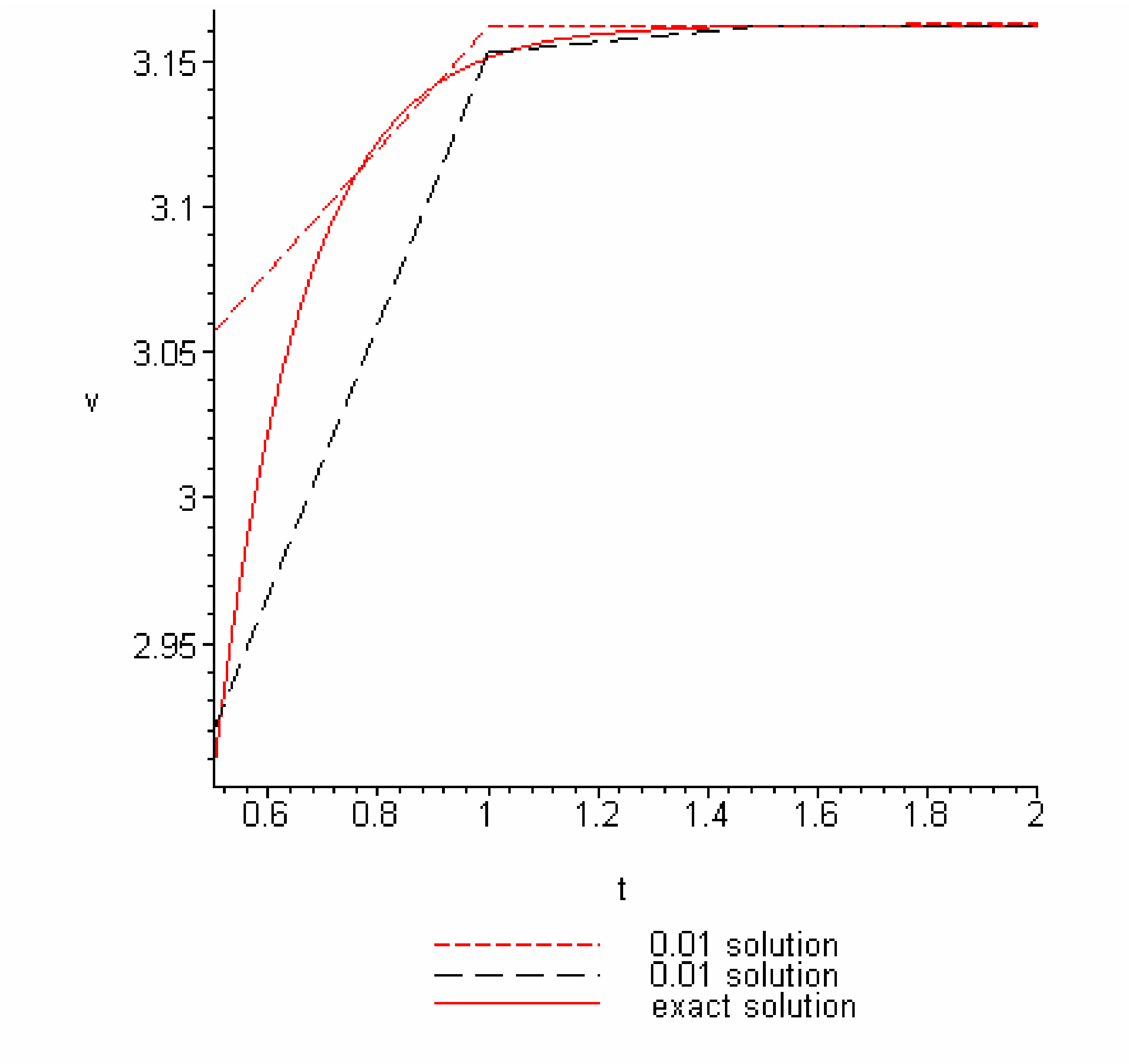
```
> soln:=dsolve({de_5,v(0)=0},v(t));
```

$$soln := \mathrm{v}(t) = \sqrt{10}\,\tanh(\sqrt{10}\,t)$$

```
> p_1:=plot(rhs(soln),t=0.5..2):
> with(plots):
```

Warning, the name changecoords has been redefined

```
> p_2:=plots[odeplot](soln_1,[t,v(t)],linestyle=3):
>
p_3:=plots[odeplot](soln_2,[t,v(t)],linestyle=6,color=black):
> display({p_1,p_2,p_3});
```




```
> evalf(subs(t=2,rhs(soln)));
```

$$3.162257356$$

The pecentage error at t=2 for h=0.1 and h=0.01 is 0.0006and 0.0002 respectively.

---

6. We have (remember at x=0, y=0).

```
> de_1:=diff(y(x),x)=sqrt(x+y(x));
```

$$de\_1 := \frac{d}{dx}\,y(x) = \sqrt{x + y(x)}$$

```
>
soln_01:=dsolve({de_1,y(0)=0},y(x),type=numeric,method=classical,output=array([0.1,0.2,0.3,0.4,0.5]),stepsize=0.1);
```

$$soln\_01 := \begin{bmatrix} [x, y(x)] \\ \begin{bmatrix} 0.1 & 0. \\ 0.2 & 0.03162277660168379 84 \\ 0.3 & 0.0797499806137270950 \\ 0.4 & 0.141373838056581741 \\ 0.5 & 0.214951948799966652 \end{bmatrix} \end{bmatrix}$$

```
> soln_001:=dsolve({de_1,y(0)=0},y(x),type=numeric,method=classical,output=array([0.1,0.2,0.3,0.4,0.5]),stepsize=0.01);
```

soln_001 :=

[x, y(x)]	
0.1	0.0205462269583625326
0.2	0.0630905726038878401
0.3	0.120651906071559650
0.4	0.191016516061380537
0.5	0.272928123095270370

```
> soln_005:=dsolve({de_1,y(0)=0},y(x),type=numeric,method=classical,output=array([0.1,0.2,0.3,0.4,0.5]),stepsize=0.005);
```

soln_005 :=

[x, y(x)]	
0.1	0.0216877676232933130
0.2	0.0648149958060607878
0.3	0.122886561855593754
0.4	0.193726383457397122
0.5	0.276092300626568810

---

7. We have

```
> de_1:=diff(x(t),t)=t^2+x(t)*exp(-t);
```

$$de\_1 := \frac{d}{dt}\,x(t) = t^2 + x(t)\,e^{(-t)}$$

```
> sol:=dsolve({de_1,x(0.1)=-1},x(t)):
> sol_0||25:=evalf(dsolve({de_1,x(0.1)=-1},x(t),type=numeric,method=classical,output=array([0.5,1,1.5,2,2.5]),stepsize=0.25));
```

sol_025 :=

[t, x(t)]	
0.5	-1.334684366
1.	-1.508075041
1.5	-1.106192996
2.	0.1339250949
2.5	2.434079395

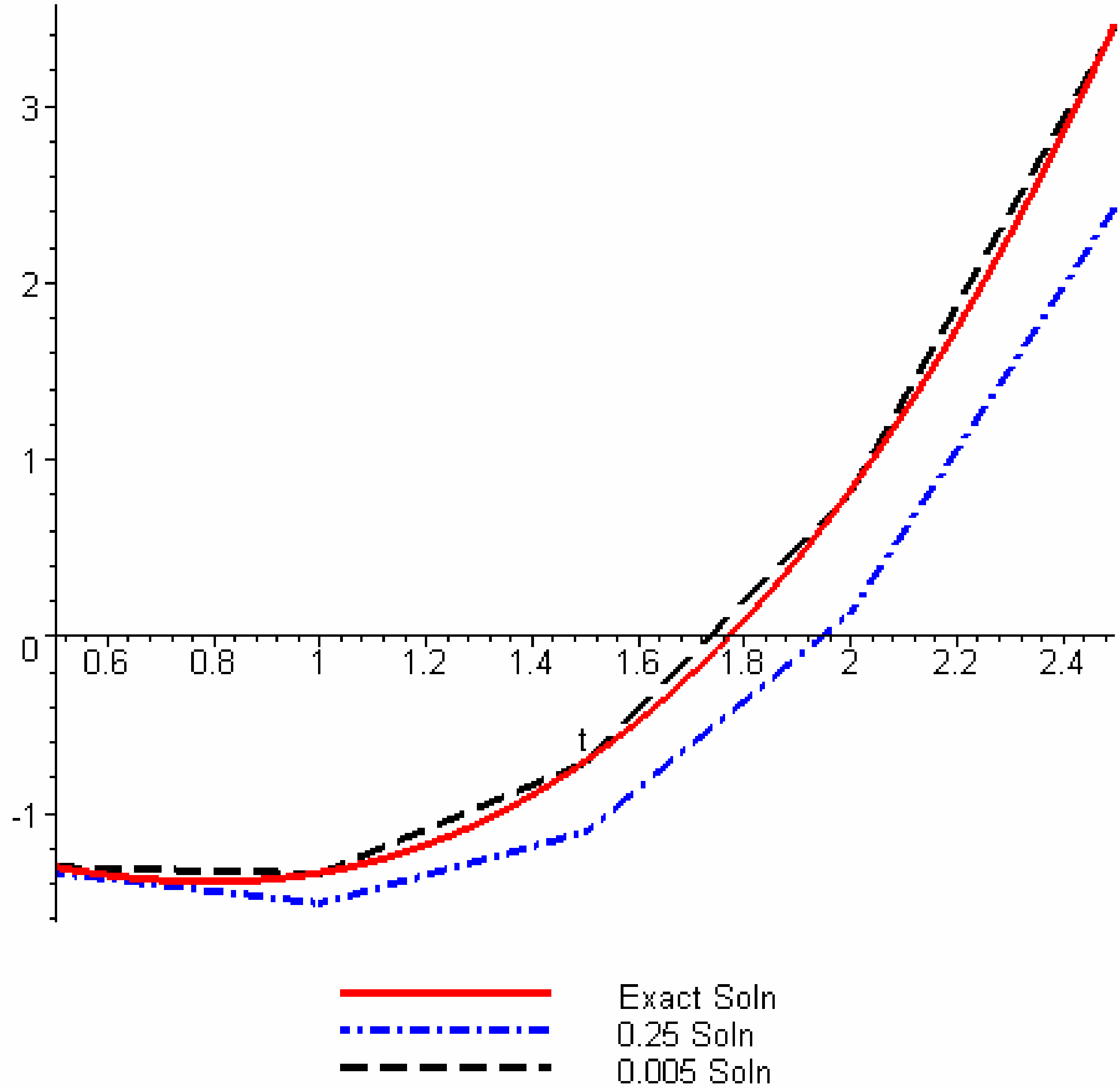
```
> sol_0||005:=evalf(dsolve({de_1,x(0.1)=-1},x(t),type=numeric,method=classical,output=array([0.5,1,1.5,2,2.5]),stepsize=0.005));
```

sol_05 :=

[t, x(t)]	
0.5	-1.303585334
1.	-1.339936957
1.5	-0.7141322648
2.	0.8139640537
2.5	3.451985054

```
> with(plots):p_1:=plot(rhs(sol),t=0.5..2.5):
```

```
> p_2:=plots[odeplot](sol_0||25,[t,x(t)],linestyle=4,color=blue):
> p_3:=plots[odeplot](sol_0||005,[t,x(t)],linestyle=6,color=black):
> display({p_1,p_2,p_3});
```




```
> evalf(subs(t=2,rhs(soln)));
```

0.829043979

Percentage error is 1.82% for h=0.005 and 83.85% for h=0.25.