Complete solutions to Exercise 13(f)

1. Substituting $h = 0.1$ into (13.14) gives

$$y_{n+1} = y_n + \frac{0.1}{2}\left[y_n' + f\left(x_n + 0.1,\ y_n + 0.1y_n'\right)\right] \qquad (*)$$

Putting $n = 0$ into (*) gives the formula for the y value at $x = 0.1$, y_1:

$$y_1 = y_0 + \frac{0.1}{2}\left[y_0' + f\left(x_0 + 0.1,\ y_0 + 0.1y_0'\right)\right]$$

What is x_0 and y_0 equal to?

Given by the initial condition, $x_0 = 0$ and $y_0 = 2$. Substituting these

$$y_1 = 2 + 0.05\left[y_0' + f\left(0.1,\ 2 + 0.1y_0'\right)\right] \qquad (\dagger)$$

What is the value of y_0'?

$$y' = x + y$$

$$y_0' = x_0 + y_0 = 0 + 2 = 2$$

Substituting $y_0' = 2$ into $(\dagger)$ and using $f(x, y) = x + y$ gives

$$y_1 = 2 + 0.05\left[2 + f\left(0.1,\ 2 + (0.1 \times 2)\right)\right]$$

$$= 2 + 0.05\left[2 + f\left(0.1,\ 2.2\right)\right] = 2 + 0.05\left[2 + (0.1 + 2.2)\right] = 2.215$$

At $x = 0.2$ we approximate y with y_2. Replacing n with 1 in (*) gives

$$y_2 = y_1 + 0.05\left[y_1' + f\left(x_1 + 0.1,\ y_1 + 0.1y_1'\right)\right]$$

We have $x_1 = 0.1$, $y_1 = 2.215$. Substituting these

$$y_2 = 2.215 + 0.05\left[y_1' + f\left(0.1 + 0.1,\ 2.215 + 0.1y_1'\right)\right]$$

What is y_1' equal to?

$$y_1' = x_1 + y_1 = 0.1 + 2.215 = 2.315$$

Substituting $y_1' = 2.315$ into y_2

$$y_2 = 2.215 + 0.05\left[2.315 + f\left(0.2,\ 2.215 + (0.1 \times 2.315)\right)\right]$$

$$= 2.215 + 0.05\left[2.315 + f\left(0.2,\ 2.4465\right)\right]$$

$$y_2 = 2.215 + 0.05\left[2.315 + (0.2 + 2.4465)\right] = 2.463075$$

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2. Same differential equation as **EXAMPLE 17**. In **EXAMPLE 17** we have already established

$$y_{n+1} = y_n + 0.05\left[y_n' + f\left(x_n + 0.1,\ y_n + 0.1y_n'\right)\right] \qquad (*)$$

Moreover we have $x_2 = 0.2$, $y_2 = 1.251531$. We only need to find y_3, the y value at $x = 0.3$. Substituting $n = 2$ into (*) gives

$$y_3 = y_2 + 0.05\left[y_2' + f\left(x_2 + 0.1,\ y_2 + 0.1y_2'\right)\right]$$

$$= 1.251531 + 0.05\left[y_2' + f\left(0.2 + 0.1,\ 1.251531 + 0.1y_2'\right)\right] = y_3$$

What is y_2' equal to?

$$y_2' = x_2{}^2 + y_2{}^2$$

$$= 0.2^2 + 1.251531^2 = 1.606330$$

(13.14) $$y_{n+1} = y_n + \frac{h}{2}\left[y_n' + f\left(x_n + h,\ y_n + hy_n'\right)\right]$$

Substituting $y_2' = 1.606330$ into y_3 establishes:

$$y_3 = 1.251531 + 0.05\left[1.60633 + f\left(0.3,\ 1.251531 + \left(0.1\times 1.60633\right)\right)\right]$$

$$= 1.251531 + 0.05\left[1.60633 + f\left(0.3,\ 1.412164\right)\right]$$

$$= 1.251531 + 0.05\left[1.60633 + \left(0.3^2 + 1.412164^2\right)\right] = 1.436058$$

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3. We need to determine y_2. We cannot obtain y_2 without first finding y_1. Putting $h = 0.2$ and $n = 0$ into (13.14) gives

$$y_1 = y_0 + \frac{0.2}{2}\left[y_0' + f\left(x_0 + 0.2,\ y_0 + 0.2y_0'\right)\right]$$

From the initial condition we have $x_0 = 1.2,\ y_0 = 1$. Substituting these

$$y_1 = 1 + 0.1\left[y_0' + f\left(1.4,\ 1 + 0.2y_0'\right)\right] \qquad (*)$$

What is the value of y_0'?

$$y' = \ln\left|x + y\right|$$

$$y_0' = \ln\left|x_0 + y_0\right| = \ln\left|1.2 + 1\right| = 0.7885$$

Note that since $1.2 + 1$ is positive we have

$$\ln\left|1.2 + 1\right| = \ln\left(1.2 + 1\right) = \ln\left(2.2\right) = 0.7885$$

Substituting $y_0' = 0.7885$ into (*) gives

$$y_1 = 1 + 0.1\left[0.7885 + f\left(1.4,\ 1 + \left(0.2\times 0.7885\right)\right)\right]$$

$$= 1 + 0.1\left[0.7885 + f\left(1.4,\ 1.1577\right)\right] = 1 + 0.1\left[0.7885 + \ln\left|1.4 + 1.1577\right|\right] = 1.1728 = y_1$$

To find y_2 we need to place $n = 1$ and $h = 0.2$ into (13.14):

$$y_2 = y_1 + 0.1\left[y_1' + f\left(x_1 + 0.2,\ y_1 + 0.2y_1'\right)\right]$$

Substituting $x_1 = 1.4$ and $y_1 = 1.1728$ we have

$$y_2 = 1.1728 + 0.1\left[y_1' + f\left(1.6,\ 1.1728 + 0.2y_1'\right)\right] \qquad (**)$$

The value of y_1' is

$$y_1' = \ln\left|x_1 + y_1\right| = \ln\left|1.4 + 1.1728\right| = 0.945$$

Placing $y_1' = 0.945$ into (**) we have

$$y_2 = 1.1728 + 0.1\left[0.945 + f\left(1.6,\ 1.1728 + \left(0.2\times 0.945\right)\right)\right]$$

$$= 1.1728 + 0.1\left[0.945 + f\left(1.6,\ 1.3618\right)\right]$$

$$y_2 = 1.1728 + 0.1\left[0.945 + \ln\left|1.6 + 1.3618\right|\right] = 1.3759$$

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4. For Maple solutions see the next page

(13.14) $$y_{n+1} = y_n + \frac{h}{2}\left[y_n' + f\left(x_n + h,\ y_n + hy_n'\right)\right]$$

4. We have

```
> de_4:=diff(x(t),t)=(t^2)+x(t)*exp(-t);
```

$$de\_4 := \frac{d}{dt}\,\mathrm{x}(t) = t^2 + \mathrm{x}(t)\,\mathbf{e}^{(-t)}$$

```
> soln:=dsolve({de_4,x(0.1)=-1},x(t)):
> soln_Euler:=evalf(dsolve({de_4,x(0.1)=-
1},x(t),type=numeric,method=classical,output=array([.5,1,
1.5,2,2.5]),stepsize=0.25));
```

soln_Euler :=

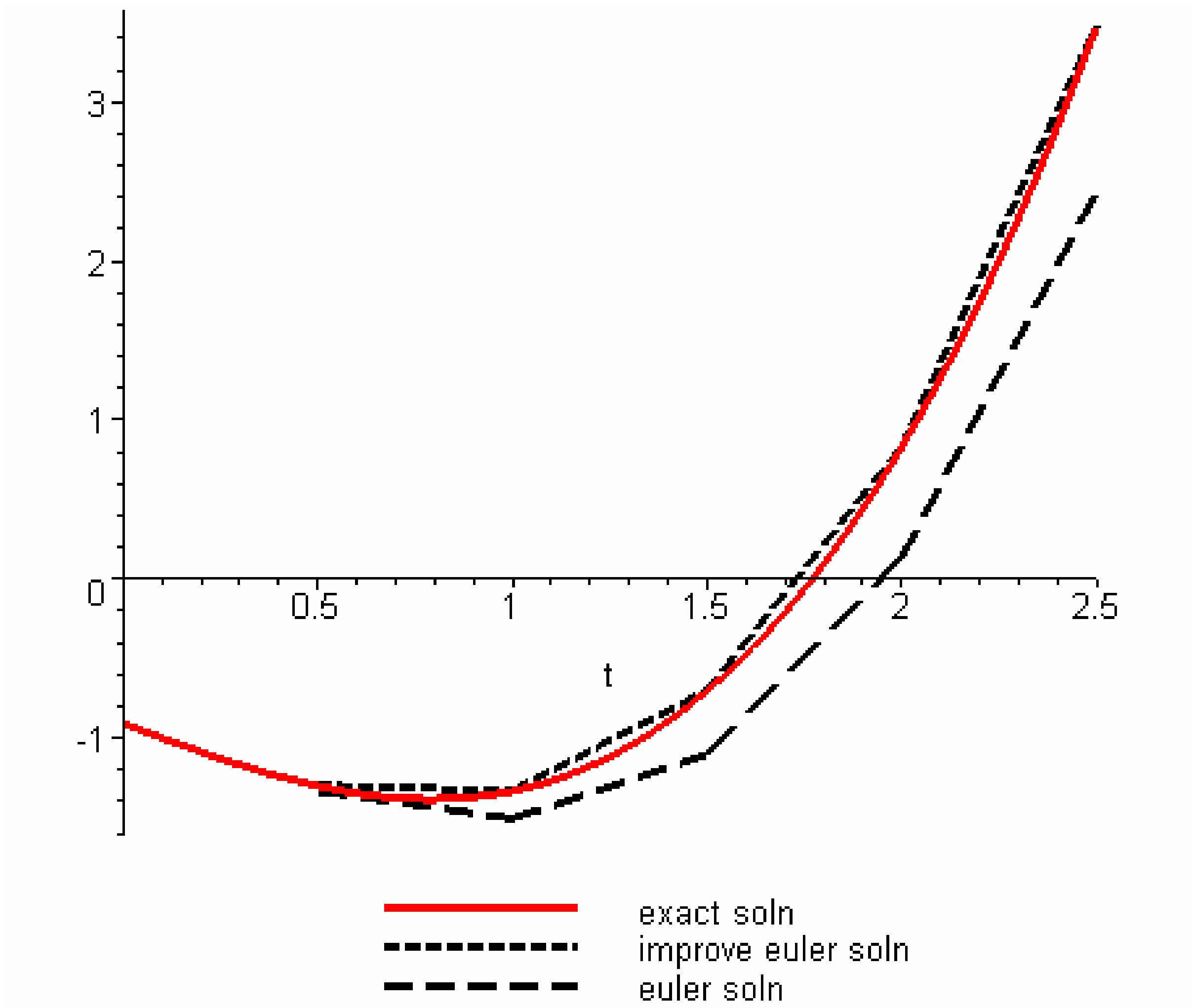
t	x(t)
0.5	-1.334684366
1.	-1.508075041
1.5	-1.106192996
2.	0.1339250949
2.5	2.434079395

```
> soln_IMPROVEul:=evalf(dsolve({de_4,x(0.1)=-
1},x(t),type=numeric,method=classical[heunform],output=ar
ray([.5,1,1.5,2,2.5]),stepsize=0.25));
```

soln_IMPROVEul :=

t	x(t)
0.5	-1.299787303
1.	-1.332282914
1.5	-0.7022746568
2.	0.8316491078
2.5	3.477565346

```
> with(plots):
>
p_1:=plots[odeplot](soln_Euler,[t,x(t)],linestyle=6,color
=black):
>
p_2:=plots[odeplot](soln_IMPROVEul,[t,x(t)],linestyle=3,c
olor=black):
> p_3:=plot(rhs(soln),t=0..2.5):
> display({p_1,p_2,p_3});
```



5. We have

```
> de_5:=diff(y(x),x)=y(x)^2+x^3;
```

$$de\_5 := \frac{d}{dx}\,y(x) = y(x)^2 + x^3$$

```
> soln:=dsolve({de_5,y(0)=-1},y(x)):
> soln_Euler:=dsolve({de_5,y(0)=-
1},y(x),type=numeric,method=classical,output=array([0,0.5
,1,1.5,2]),stepsize=0.01);
```

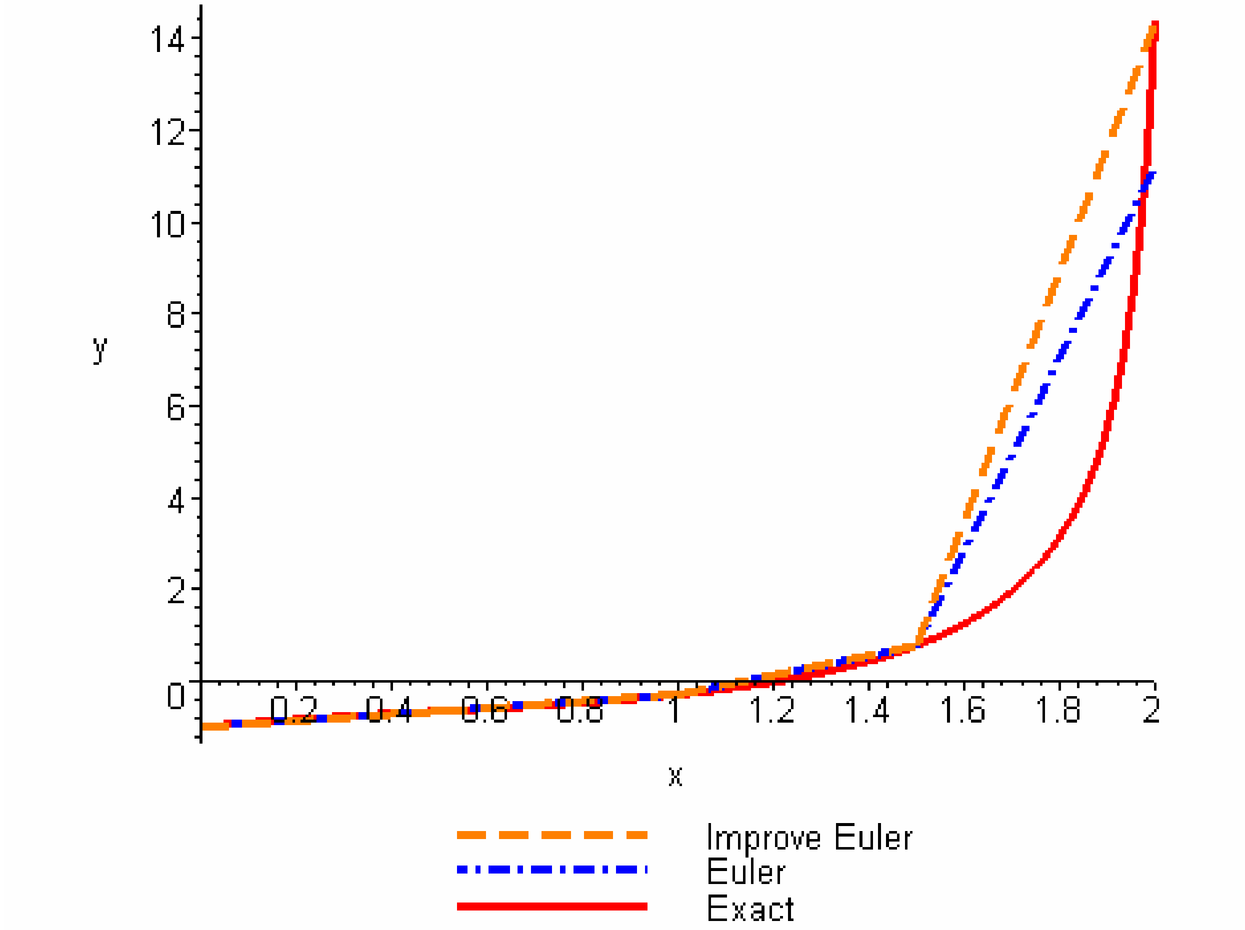
$soln\_Euler :=$ $[x, y(x)]$

0	-1.
0.5	-0.651627554586721835
1	-0.292947356106644852
1.5	0.760308174940523718
2	11.2213047551889993

```
> soln_IMPROVEul:=dsolve({de_5,y(0)=-
1},y(x),type=numeric,method=classical[heunform],output=ar
ray([0,0.5,1,1.5,2]),stepsize=0.01);
```

$$soln\_IMPROVEul := \begin{bmatrix} [x, y(x)] \\ \begin{bmatrix} 0 & -1. \\ 0.5 & -0.652999462841611256 \\ 1 & -0.291407012838878932 \\ 1.5 & 0.778361927362244854 \\ 2 & 14.3308353540789018 \end{bmatrix} \end{bmatrix}$$

```
> with(plots):
>
p_1:=plots[odeplot](soln_Euler,[x,y(x)],linestyle=4,color
=blue):
>
p_2:=plots[odeplot](soln_IMPROVEul,[x,y(x)],linestyle=6,c
olor=coral):
> p_3:=plot(rhs(soln),x=0..2):
> display({p_1,p_2,p_3});
```



6. We have

```
> de_6:=diff(x(t),t)=10-(x(t))^(2);
```

$$de\_6 := \frac{d}{dt}\,x(t) = 10 - x(t)^2$$

```
> soln:=dsolve({de_6,x(0)=0},x(t));
```

$$soln := x(t) = \sqrt{10}\,\tanh(\sqrt{10}\,t)$$

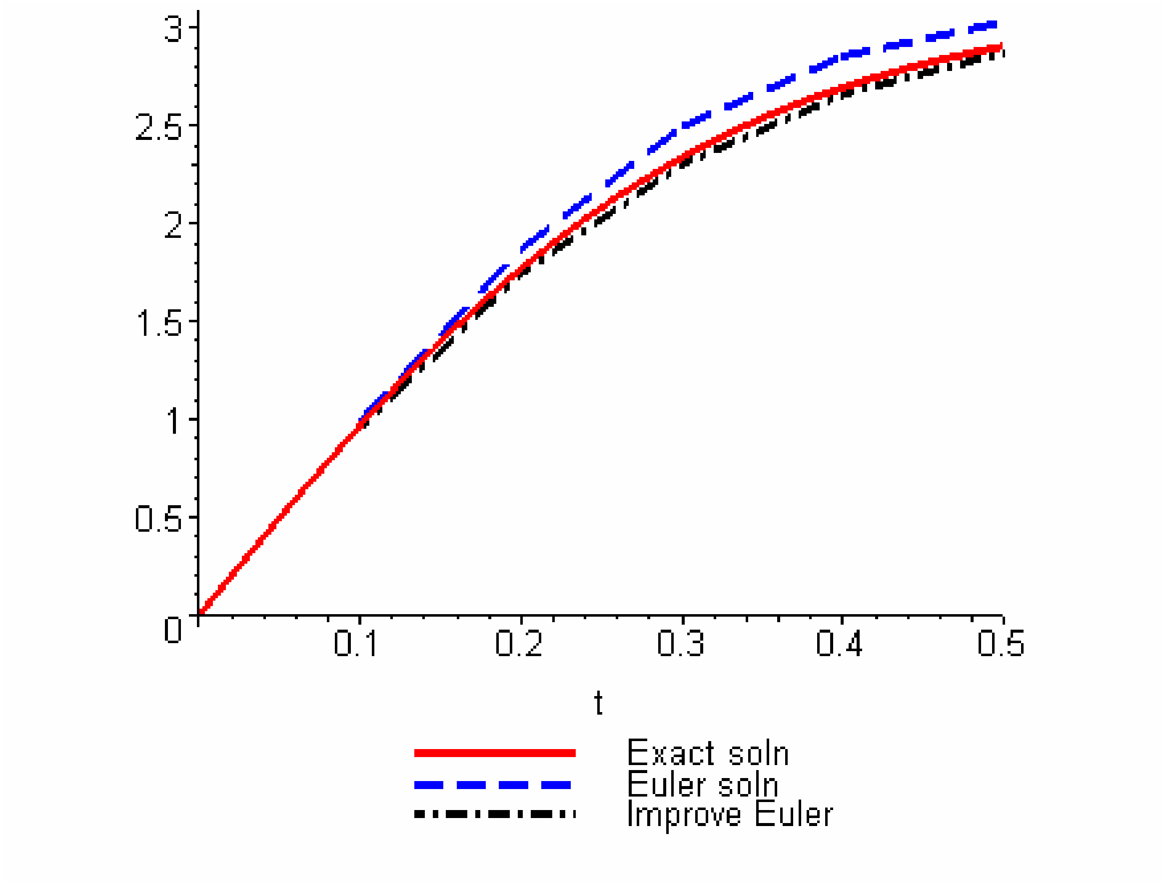
```
> soln_Eul:=evalf(dsolve({de_6,x(0)=0},x(t),type=numeric,method=classical,output=array([0.1,0.2,0.3,0.4,0.5]),stepsize=0.09));
```

$$soln\_Eul := \left[\begin{array}{c} [t, x(t)] \\ \left[\begin{array}{cc} 0.1 & 0.9919000000 \\ 0.2 & 1.870831307 \\ 0.3 & 2.495519396 \\ 0.4 & 2.854659691 \\ 0.5 & 3.029963264 \end{array}\right] \end{array}\right]$$

```
> soln_IMPROVEul:=evalf(dsolve({de_6,x(0)=0},x(t),type=numeric,method=classical[heunform],output=array([0.1,0.2,0.3,0.4,0.5]),stepsize=0.09));
```

$$soln\_IMPROVEul := \left[\begin{array}{c} [t, x(t)] \\ \left[\begin{array}{cc} 0.1 & 0.9552508396 \\ 0.2 & 1.743499134 \\ 0.3 & 2.302041007 \\ 0.4 & 2.659438465 \\ 0.5 & 2.874583966 \end{array}\right] \end{array}\right]$$

```
> with(plots):
> p_1:=plots[odeplot](soln_Eul,[t,x(t)],linestyle=6,color=blue):
> p_2:=plots[odeplot](soln_IMPROVEul,[t,x(t)],linestyle=4,color=black):
> p_3:=plot(rhs(soln),t=0..0.5):
> display({p_1,p_2,p_3});
```




```
> evalf(subs(t=0.5,rhs(soln)));
```

2.905436073

The percentage error with Euler is 4.286% and with improved Euler it is 1.062%.