Complete solutions to Exercise 13(g)

1. Putting $n=0$ and $h=0.1$ into (13.15) gives

$$y_1 = y_0 + \frac{0.1}{6}\left[K_1 + 2\left(K_2 + K_3\right) + K_4\right] \qquad (*)$$

We need to find K_1, K_2, K_3 and K_4. With $n=0$, $x_0=0$ and $y_0=2$ gives

$$K_1 = f\left(x_0, y_0\right) = f\left(0,2\right) = 0+2 = 2$$

What is K_2 equal to?

$$K_2 = f\left(x_0 + \frac{1}{2}(0.1),\ y_0 + \frac{1}{2}(0.1\times 2)\right) = f\left(0 + \frac{1}{2}(0.1),\ 2+0.1\right)$$

$$= f\left(0.05,\ 2.1\right) = 0.05+2.1 = 2.15 = K_2$$

Need to determine K_3.

$$K_3 = f\left(0 + \frac{1}{2}(0.1),\ 2 + \frac{1}{2}(0.1\times 2.15)\right)$$

$$= f\left(0.05,\ 2.1075\right) = 0.05+2.1075 = 2.1575 = K_3$$

Lastly we find K_4.

$$K_4 = f\left(0+0.1,\ 2+\left(0.1\times 2.1575\right)\right)$$

$$= f\left(0.1,\ 2.21575\right) = 0.1+2.21575 = 2.31575 = K_4$$

Substituting $y_0=2$, $K_1=2$, $K_2=2.15$, $K_3=2.1575$ and $K_4=2.31575$ into (*)

$$y_1 = 2 + \frac{0.1}{6}\left[2+2\left(2.15+2.1575\right)+2.31575\right] = 2.215513 = y_1$$

We evaluate y_2 in a similar fashion. Putting $n=1$, $h=0.1$ into (13.15) gives

$$y_2 = y_1 + \frac{0.1}{6}\left[K_1 + 2\left(K_2 + K_3\right) + K_4\right] \qquad (**)$$

Need to find K_1, K_2, K_3 and K_4. Substituting $x_1=0.1$ and $y_1=2.215513$

$$K_1 = f\left(x_1, y_1\right) = f\left(0.1, 2.215513\right) = 0.1+2.215513 = 2.315513$$

Similarly with $x_1=0.1$ and $K_1=2.315513$ we have

$$K_2 = f\left(x_1 + \frac{1}{2}(0.1),\ y_1 + \frac{1}{2}(0.1\times 2.315513)\right) = f\left(0.1+0.05,\ 2.215513+0.115776\right)$$

$$= f\left(0.15,\ 2.331289\right) = 2.481289 = K_2$$

Need to determine K_3.

$$K_3 = f\left(0.15,\ 2.215513 + \frac{1}{2}(0.1\times 2.481289)\right)$$

$$= f\left(0.15,\ 2.339577\right) = 0.15+2.339577 = 2.489577 = K_3$$

Lastly,

$$K_4 = f\left(0.2,\ 2.215513+\left(0.1\times 2.489577\right)\right)$$

$$= f\left(0.2,\ 2.464471\right) = 0.2+2.464471 = 2.664471 = K_4$$

Substituting $y_1=2.215513$, $K_1=2.315513$, $K_2=2.481289$, $K_3=2.489577$ and $K_4=2.664471$ into (**) gives

---

(13.15) $$y_{n+1} = y_n + \frac{h}{6}\left[K_1 + 2\left(K_2 + K_3\right) + K_4\right]$$

$$y_2 = 2.215513 + \frac{0.1}{6}\left[2.315513 + 2\left(2.481289 + 2.489577\right) + 2.664471\right] = 2.464208$$

2. Similar solution to question 1 with $f(x, y) = \sqrt{x+y}$. Placing $n = 0,\ h = 0.2$ into (13.15):

$$y_1 = y_0 + \frac{0.2}{6}\left[K_1 + 2\left(K_2 + K_3\right) + K_4\right]$$

We find $K_1 = 0,\ K_2 = 0.3162,\ K_3 = 0.3628$ and $K_4 = 0.5221$. Thus with $y_0 = 0$

$$y_1 = 0 + \frac{0.2}{6}\left[0 + 2\left(0.3162 + 0.3628\right) + 0.5221\right] = 0.0627$$

Similarly for y_2 with $y_1 = 0.0627$ we have

$$y_2 = 0.0627 + \frac{0.2}{6}\left[K_1 + 2\left(K_2 + K_3\right) + K_4\right]$$

We find $K_1 = 0.5125,\ K_2 = 0.6434,\ K_3 = 0.6535$ and $K_4 = 0.7703$. Hence

$$y_2 = 0.0627 + \frac{0.2}{6}\left[0.5125 + 2\left(0.6434 + 0.6535\right) + 0.7703\right] = 0.1919$$

3. Substituting $h = 0.2,\ n = 0$ into (13.15) we have

$$y_1 = y_0 + \frac{0.2}{6}\left[K_1 + 2\left(K_2 + K_3\right) + K_4\right] \qquad (*)$$

Using the initial condition we have $x_0 = 1.2,\ y_0 = 1$. Substituting these with $f(x, y) = \ln(x + y)$ gives $K_1 = f(1.2, 1) = \ln(1.2 + 1) = 0.7885$

What is K_2 equal to?

$$K_2 = f\left(x_0 + \frac{1}{2}(0.2),\ y_0 + \frac{1}{2}\left[0.2 \times 0.7885\right]\right) = f\left(1.2 + 0.1,\ 1 + 0.0789\right)$$

$$K_2 = f\left(1.3,\ 1.0789\right) = \ln\left(1.3 + 1.0789\right) = 0.8666$$

Also

$$K_3 = f\left(1.3,\ 1 + \frac{1}{2}\left(0.2 \times 0.8666\right)\right)$$

$$= f\left(1.3,\ 1.0867\right) = \ln\left(1.3 + 1.0867\right) = 0.8699$$

Lastly

$$K_4 = f\left(1.4,\ 1 + \left(0.2 \times 0.8699\right)\right)$$

$$= f\left(1.4,\ 1.1740\right) = \ln\left(1.4 + 1.174\right) = 0.9455$$

Substituting $y_0 = 1,\ K_1 = 0.7885,\ K_2 = 0.8666,\ K_3 = 0.8699$ and $K_4 = 0.9455$ into (*) gives

$$y_1 = 1 + \frac{0.2}{6}\left[0.7885 + 2\left(0.8666 + 0.8699\right) + 0.9455\right] = 1.1736$$

Similarly we determine y_2 by using $x_1 = 1.4$ and $y_1 = 1.1736$. Hence

$$K_1 = 0.9453,\ K_2 = 1.0182,\ K_3 = 1.0208 \text{ and } K_4 = 1.0912$$

Substituting these into (13.15) with $n = 1$ yields

$$y_2 = 1.1736 + \frac{0.2}{6}\left[0.9453 + 2\left(1.0182 + 1.0208\right) + 1.0912\right] = 1.3774$$

(13.15) $$y_{n+1} = y_n + \frac{h}{6}\left[K_1 + 2\left(K_2 + K_3\right) + K_4\right]$$

4. Using (13.15) with $h=1$ and $n=0$ gives

$$v_1 = v_0 + \frac{1}{6}\left[K_1 + 2\left(K_2 + K_3\right) + K_4\right] \qquad (*)$$

We know $t_0 = 0,\ v_0 = 0$ and $f(t,v) = 9.8 - \frac{v^2}{10}$. Thus

$$K_1 = f(0,0) = 9.8 - \frac{0^2}{10} = 9.8$$

For K_2.

$$K_2 = f\left(0 + \frac{1}{2}(1),\ 0 + \frac{1}{2}(1\times 9.8)\right)$$

$$= f(0.5,\ 4.9) = 9.8 - \frac{4.9^2}{10} = 7.399 = K_2$$

For K_3.

$$K_3 = f\left(0.5,\ 0 + \frac{1}{2}(1\times 7.399)\right)$$

$$= f(0.5,\ 3.6995) = 9.8 - \frac{3.6995^2}{10} = 8.4314 = K_3$$

For K_4.

$$K_4 = f\left(1,\ 0 + (1\times 8.4314)\right)$$

$$= f(1,\ 8.4314) = 9.8 - \frac{8.4314^2}{10} = 2.6911 = K_4$$

Putting $v_0 = 0,\ K_1 = 9.8,\ K_2 = 7.399,\ K_3 = 8.4314$ and $K_4 = 2.6911$ into (*)

$$v_1 = 0 + \frac{1}{6}\left[9.8 + 2\left(7.399 + 8.4314\right) + 2.6911\right] = 7.3587$$

Similarly we obtain v_2:

$$v_2 = v_1 + \frac{1}{6}\left[K_1 + 2\left(K_2 + K_3\right) + K_4\right]$$

We substitute $v_1 = 7.3587$ and evaluate the K's as above:

$$K_1 = 4.3850,\ K_2 = 0.6775,\ K_3 = 3.8748 \text{ and } K_4 = -2.8192$$

Substituting these into v_2 gives

$$v_2 = 7.3587 + \frac{1}{6}\left[4.3850 + 2\left(0.6775 + 3.8748\right) + \left(-2.8192\right)\right] = 9.1371$$

Thus the velocity after 2 seconds is approximately $9.1371\ \mathrm{m/s}$.

---

5. We have

```
> de_5:=diff(y(x),x)=(x^4)+(y(x)^2);
```

$$de\_5 := \frac{d}{dx}\mathrm{y}(x) = x^4 + \mathrm{y}(x)^2$$

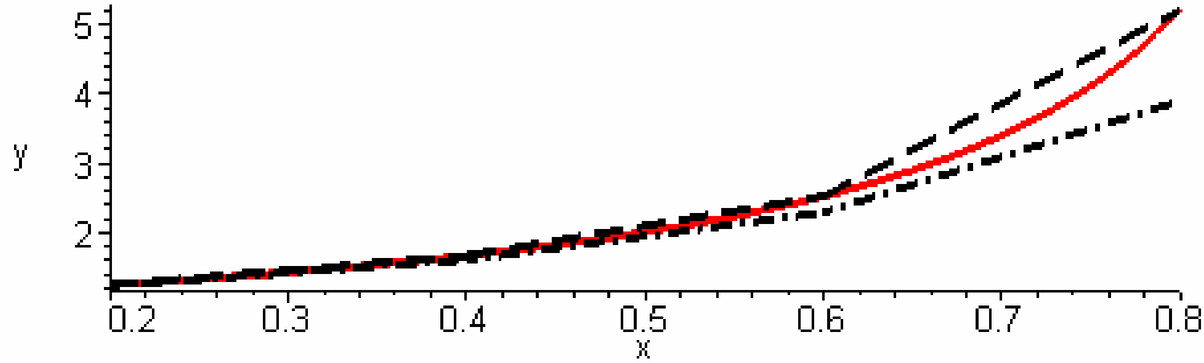
```
> soln:=dsolve({de_5,y(0)=1},y(x)):
>
e_soln:=dsolve({de_5,y(0)=1},y(x),type=numeric,method=cla
ssical,output=array([0.2,0.4,0.6,0.8]),stepsize=0.05);
```

$$e\_soln := \begin{bmatrix} [x, y(x)] \\ \begin{bmatrix} 0.2 & 1.23422130956856435 \\ 0.4 & 1.60686630371344208 \\ 0.6 & 2.29119418751065762 \\ 0.8 & 3.89809775099833766 \end{bmatrix} \end{bmatrix}$$

```
> rk_soln:=dsolve({de_5,y(0)=1},y(x),type=numeric,method=classical[rk4],output=array([0.2,0.4,0.6,0.8]),stepsize=0.05);
```

$$rk\_soln := \begin{bmatrix} [x, y(x)] \\ \begin{bmatrix} 0.2 & 1.25006939886003066 \\ 0.4 & 1.66921249191167220 \\ 0.6 & 2.52502300797200441 \\ 0.8 & 5.20555806331606341 \end{bmatrix} \end{bmatrix}$$

```
> with(plots):
> p_1:=plots[odeplot](e_soln,[x,y(x)],linestyle=4,color=black):
> p_2:=plots[odeplot](rk_soln,[x,y(x)],linestyle=6,color=black):
> p_3:=plot(rhs(soln),x=0.2..0.8):
> plots[display]({p_1,p_2,p_3});
```



6. We have

```
> de_6:=diff(y(x),x)=sin(x)*exp(y(x));
```

$$de\_6 := \frac{d}{dx}\, y(x) = \sin(x)\, e^{y(x)}$$

```
> soln:=dsolve({de_6,y(0)=-1},y(x)):
> e_soln:=evalf(dsolve({de_6,y(0)=-1},y(x),type=numeric,method=classical[heunform],output=array([1,2,3,4]),stepsize=0.2));
```

$$e\_soln := \begin{bmatrix} [x, y(x)] \\ \begin{bmatrix} 1. & -0.8155938368 \\ 2. & -0.2687066008 \\ 3. & 0.3044288387 \\ 4. & -0.07072140415 \end{bmatrix} \end{bmatrix}$$

```
> rk_soln:=evalf(dsolve({de_6,y(0)=-
1},y(x),type=numeric,method=classical[rk4],output=array([
1,2,3,4]),stepsize=0.2));
```

$$rk\_soln := \begin{bmatrix} [x, y(x)] \\ \begin{bmatrix} 1. & -0.8147376210 \\ 2. & -0.2640035919 \\ 3. & 0.3170630309 \\ 4. & -0.06263495332 \end{bmatrix} \end{bmatrix}$$

```
> with(plots):
>
p_1:=plots[odeplot](e_soln,[x,y(x)],linestyle=4,color=bla
ck):
>
p_2:=plots[odeplot](rk_soln,[x,y(x)],linestyle=6,color=co
ral):
> p_3:=plot(rhs(soln),x=0..4):
> plots[display]({p_1,p_2,p_3});
```