Complete solutions to Exercise 14(b)

1. Very similar to **EXAMPLE 4**. We have same characteristic equation:

$$i = Ae^{-5t} + Be^{-2t}$$

By using the given initial conditions we obtain the 2 simultaneous equations

$$A + B = 0$$

$$-5A - 2B = 6$$

Solving these gives $A = -2$ and $B = 2$. Hence the solution is

$$i = 2\left(e^{-2t} - e^{-5t}\right)$$

2. Use (14.8) with the appropriate value of k

(a) $x = A\cos(3t) + B\sin(3t)$ $\quad$ [Because $k = 3$]

(b) $x = A\cos(4t) + B\sin(4t)$ $\quad$ [Because $k = 4$]

(c) $x = A\cos\left(\sqrt{2}t\right) + B\sin\left(\sqrt{2}t\right)$ $\quad$ [Because $k = \sqrt{2}$]

(d) $x = A\cos\left(\sqrt{5}t\right) + B\sin\left(\sqrt{5}t\right)$ $\quad$ [Because $k = \sqrt{5}$]

3. Characteristic equation is $r^2 + k/m = 0$, (using the characteristic equation variable r because m represents mass in this problem).

$$r^2 + \left(\sqrt{\frac{k}{m}}\right)^2 = 0$$

By (14.8), $x = A\cos(\omega t) + B\sin(\omega t)$ where $\omega = \sqrt{k/m}$

4. Characteristic equation is

$$m^2 + \left(4\times10^{-6}\right) = 0$$

$$m^2 + \left(2\times10^{-3}\right)^2 = 0$$

By (14.8)

$$v = A\cos\left[\left(2\times10^{-3}\right)t\right] + B\sin\left[\left(2\times10^{-3}\right)t\right] \qquad (\dagger)$$

Substituting $t = 0$, $v = 1$ into ($\dagger$)

$1 = A\cos(0) + B\sin(0)$ gives $A = 1$ $\quad$ [Remember $\cos(0) = 1$ and $\sin(0) = 0$]

Differentiating ($\dagger$) and substituting the second condition $t = 0$, $\frac{dv}{dt} = 2\times10^{-3}$;

$$\frac{dv}{dt} = -\left(2\times10^{-3}\right)A\sin\left[\left(2\times10^{-3}\right)t\right] + \left(2\times10^{-3}\right)B\cos\left[\left(2\times10^{-3}\right)t\right]$$

$$\left(2\times10^{-3}\right) = 0 + \left(2\times10^{-3}\right)B \text{ gives } B = 1$$

Substituting $A = 1$ and $B = 1$ into ($\dagger$) gives

$$v = \cos\left[\left(2\times10^{-3}\right)t\right] + \sin\left[\left(2\times10^{-3}\right)t\right]$$

Using (4.76) with $a = 1$ and $b = 1$ gives $v = \sqrt{2}\cos\left[\left(2\times10^{-3}\right)t - \frac{\pi}{4}\right]$

(4.76) $\quad a\cos(\theta) + b\sin(\theta) = \sqrt{a^2 + b^2}\cos(\theta - \alpha)$ where $\alpha = \tan^{-1}(b/a)$

(14.8) $\quad$ If $r^2 + k^2 = 0$ then $y = A\cos(kx) + B\sin(kx)$

5. By **EXAMPLE 6**

$$y = A\cos(kx) + B\sin(kx) \qquad (*)$$

where $k = \sqrt{P/EI}$. Substituting the boundary condition $x = L/2,\ y = e$ gives

$$e = A\cos\left(\frac{kL}{2}\right) + B\sin\left(\frac{kL}{2}\right) \qquad (\dagger)$$

Substituting $x = -L/2,\ y = e$ into (*) gives

$$e = A\cos\left(-\frac{kL}{2}\right) + B\sin\left(-\frac{kL}{2}\right)$$

$$= \underbrace{A\cos\left(\frac{kL}{2}\right)}_{\text{by (4.51)}} - \underbrace{B\sin\left(\frac{kL}{2}\right)}_{\text{by (4.50)}} \qquad (\dagger\dagger)$$

Adding $(\dagger\dagger)$ and $(\dagger)$ yields

$$2e = 2A\cos\left(\frac{kL}{2}\right)$$

$$A = \frac{e}{\cos\left(\frac{kL}{2}\right)} \underbrace{=}_{\text{by (4.11)}} e\sec\left(\frac{kL}{2}\right)$$

Substituting $A = \dfrac{e}{\cos\left(\frac{kL}{2}\right)}$ into $(\dagger)$ gives

$$e = \frac{e}{\cos\left(\frac{kL}{2}\right)}\cos\left(\frac{kL}{2}\right) + B\sin\left(\frac{kL}{2}\right)$$

$$e = e + B\sin\left(\frac{kL}{2}\right), \quad \text{hence } B = 0$$

Substituting $A = e\sec\left(\frac{kL}{2}\right)$ and $B = 0$ into (*) yields

$$y = e\sec\left(\frac{kL}{2}\right)\cos(kx)$$

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6. Characteristic equation is $r^2 + 2\omega\zeta r + \omega^2 = 0$
(using r, since m represents mass). This is a quadratic equation with variable r. Putting $a = 1,\ b = 2\omega\zeta$ and $c = \omega^2$ into (1.16) gives

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(1.16) $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(4.11) $$\frac{1}{\cos(x)} = \sec(x)$$

(4.50) $$\sin(-x) = -\sin(x)$$

(4.51) $$\cos(-x) = \cos(x)$$

$$r=\frac{-2\omega\zeta\pm\sqrt{4\omega^2\zeta^2-4\omega^2}}{2}=\frac{-2\omega\zeta\pm\sqrt{4\omega^2\left(\zeta^2-1\right)}}{2}$$

$$=\frac{-2\omega\zeta\pm2\omega\sqrt{\zeta^2-1}}{2}$$

$$=-\omega\zeta\pm\omega\sqrt{\zeta^2-1}\quad\text{[Cancelling 2's]}$$

$$r=\omega\left(-\zeta\pm\sqrt{\zeta^2-1}\right)\qquad\text{[Taking Out }\omega\text{]}$$

$$r_1=\omega\left(-\zeta+\sqrt{\zeta^2-1}\right)\text{ and }r_2=\omega\left(-\zeta-\sqrt{\zeta^2-1}\right)=-\omega\left(\zeta+\sqrt{\zeta^2-1}\right)$$

Since we have real and different roots we use (14.4)

$$x=Ae^{r_1t}+Be^{r_2t}\text{ where }r_1\text{ and }r_2\text{ are as above}.$$

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7. The characteristic equation is same as solution 6:

$$m^2+2\omega\zeta m+\omega^2=0$$

By solution to question 6

$$m=\omega\left(-\zeta\pm\sqrt{\zeta^2-1}\right)$$

For $\zeta=1,\ m=-\omega$ (equal roots) so by (14.5)

$$x=\left(A+Bt\right)e^{-\omega t}$$

Substituting the first initial condition, when $t=0,\ x=5$;

$$5=e^0\left(A+B.0\right)\text{ gives }A=5\qquad\left[\text{Remember }e^0=1\right]$$

Differentiating $x=(A+Bt)e^{-\omega t}$ by using the product rule, (6.31), yields

$$\dot{x}=-\omega e^{-\omega t}\left(A+Bt\right)+e^{-\omega t}.B$$

Substituting the other initial condition when $t=0,\ \dot{x}=0$;

$$0=-\omega e^0\left(A+B.0\right)+e^0.B$$

$$B=\omega A=5\omega\qquad\left[\text{Because }A=5\right]$$

Substituting $A=5$ and $B=5\omega$ into $x=(A+Bt)e^{-\omega t}$ gives

$$x=e^{-\omega t}\left(5+5\omega t\right)$$

$$x=5e^{-\omega t}\left(1+\omega t\right)\qquad\{\text{Taking Out 5}]$$

The following is the MAPLE output with $\theta=\omega t$

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> x:=5*exp(-theta)*(1+theta);
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$$x:=5\,e^{(-\theta)}\,(1+\theta)$$

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> plot(x,theta=0..10);
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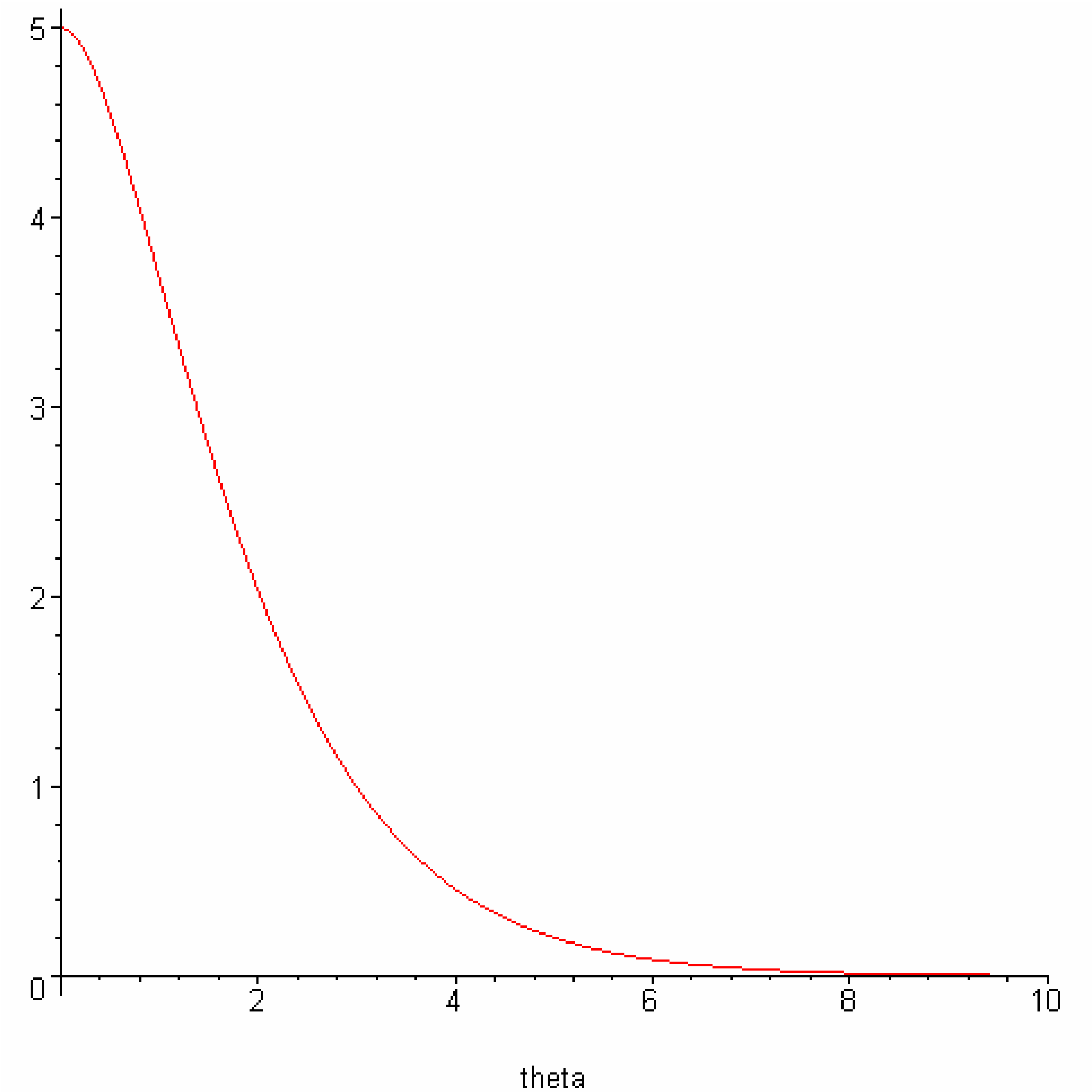

8. Since we have the same characteristic equation as solution 6 so

$$m = \omega\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

$$= \omega\left(-\zeta \pm \sqrt{-\left(1-\zeta^2\right)}\right)$$

$$= \omega\left(-\zeta \pm j\sqrt{\left(1-\zeta^2\right)}\right) \qquad \left[\text{Because } \zeta < 1 \text{ and so } \zeta^2 < 1\right]$$

$$m = -\omega\zeta \pm j\omega\beta \quad \text{where} \quad \beta = \sqrt{1-\zeta^2}$$

Since we have complex roots so by (14.6)

$$x = e^{-\zeta\omega t}\left[A\cos\left(\beta\omega t\right) + B\sin\left(\beta\omega t\right)\right] \qquad (*)$$

(14.6) If $m = \alpha \pm j\beta$ then $y = e^{\alpha x}\left[A\cos(\beta x) + B\sin(\beta x)\right]$

Substituting the initial condition, when $t = 0,\ x = 0$;

$0 = e^0\left[A\cos(0) + B\sin(0)\right]$, gives A=0 $\left(\text{because } e^0 = 1,\ \cos(0) = 1 \text{ and } \sin(0) = 0\right)$

Substituting $A = 0$ into (*) gives

$$x = Be^{-\zeta\omega t}\sin(\beta\omega t)$$

Substituting the other initial condition, when $t = 0,\ \dot{x} = \omega\beta$ means we need to differentiate $x = Be^{-\zeta\omega t}\sin(\beta\omega t)$.

$$\dot{x} = -\zeta\omega Be^{-\zeta\omega t}\sin(\beta\omega t) + B\beta\omega e^{-\zeta\omega t}\cos(\beta\omega t)$$

Substituting $t = 0$ and $\dot{x} = \omega\beta$ into this

$$\omega\beta = 0 + B\omega\beta \text{ which gives } \quad B = 1$$

Substituting $A = 0,\ B = 1$ into (*) gives

$$x = e^{-\zeta\omega t}\sin(\beta\omega t) \quad \text{where } \beta = \sqrt{1-\zeta^2}$$

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9. Dividing the characteristic equation by C gives $\mathrm{m}^2 + \frac{1}{\mathrm{RC}}\mathrm{m} + \frac{1}{\mathrm{LC}} = 0$

Substituting $a = 1,\ b = \frac{1}{RC}$ and $c = \frac{1}{LC}$ into (1.16) gives

$$m = \frac{-\frac{1}{RC} \pm \sqrt{\frac{1}{R^2C^2} - \frac{4}{LC}}}{2} = -\frac{1}{2RC} \pm \frac{1}{2}\sqrt{\frac{L-4R^2C}{R^2C^2L}}$$

$$= -\frac{1}{2RC} \pm \frac{1}{2}\sqrt{\frac{1}{R^2C^2}}\sqrt{\frac{L-4R^2C}{L}}$$

$$= -\frac{1}{2RC} \pm \frac{1}{2RC}\sqrt{\frac{L-4R^2C}{L}}$$

$$m = \frac{1}{2RC}\left[-1 \pm \sqrt{\frac{L-4R^2C}{L}}\right] \qquad (*)$$

Case (a) $\mathrm{L} = 4\mathrm{CR}^2$;

Substituting $\mathrm{L} = 4\mathrm{CR}^2$ into (*) gives $m = -\frac{1}{2RC}$ [Equal Roots]

By (14.5)

$$\mathrm{v} = (\mathrm{A} + \mathrm{Bt})\mathrm{e}^{-t/2\mathrm{RC}}$$

Case (b) $\mathrm{L} > 4\mathrm{CR}^2$;

Using (*) gives two roots m_1 and m_2 [Distinct Roots]

$$m_1 = \frac{1}{2RC}\left[-1 + \sqrt{\frac{L-4R^2C}{L}}\right],\quad m_2 = \frac{1}{2RC}\left[-1 - \sqrt{\frac{L-4R^2C}{L}}\right]$$

By (14.4) $\quad \mathrm{v} = \mathrm{Ae}^{\mathrm{m_1 t}} + \mathrm{Be}^{\mathrm{m_2 t}}$

Case (c) $L < 4CR^2$;

(1.16) $\qquad m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(14.4) $\qquad$ If m_1 and m_2 then $\mathrm{y} = \mathrm{Ae}^{\mathrm{m_1 x}} + \mathrm{Be}^{\mathrm{m_2 x}}$

(14.5) $\qquad$ Equal roots m then $\mathrm{y} = (\mathrm{A} + \mathrm{Bx})\mathrm{e}^{\mathrm{mx}}$

$$m = -\frac{1}{2RC} \pm \frac{1}{2RC}\sqrt{\frac{L-4R^2C}{L}} = -\frac{1}{2RC} \pm \frac{1}{2RC}\sqrt{-\left(\frac{4R^2C-L}{L}\right)}$$

$$m = -\frac{1}{2RC} \pm j\frac{1}{2RC}\sqrt{\left(\frac{4R^2C-L}{L}\right)} \qquad \text{[Complex Roots]}$$

Let $\alpha = -\dfrac{1}{2RC}$ and $\beta = \dfrac{1}{2RC}\sqrt{\left(\dfrac{4R^2C-L}{L}\right)}$, then substituting these into (14.6) gives

$$v = e^{\alpha t}\left[A\cos(\beta t) + B\sin(\beta t)\right]$$

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10. Same differential equation as question 9. The characteristic equation is given by

$$Cm^2 + \frac{1}{R}m + \frac{1}{L} = 0$$

The roots of this equation are given by solution 9

$$m = \frac{1}{2RC}\left[-1 \pm \sqrt{\frac{L-4R^2C}{L}}\right] \qquad (*)$$

Substituting $R = 10\times 10^3$ and $C = 1\times 10^{-9}$ gives

$$m = \frac{1}{2\times(10\times10^3)\times(1\times10^{-9})}\left[-1 \pm \sqrt{\frac{L-\left[4\times(10\times10^3)^2\times1\times10^{-9}\right]}{L}}\right]$$

$$= 50\times10^3\left[-1 \pm \sqrt{\frac{L-0.4}{L}}\right]$$

$$m = -(50\times10^3) \pm (50\times10^3)\sqrt{\frac{L-0.4}{L}}$$

Since $L < 0.4$, $m = -(50\times10^3) \pm j(50\times10^3)\sqrt{\dfrac{0.4-L}{L}}$.

Equating the imaginary part of this to the imaginary part of the roots given in the question $-(50\times10^3) \pm j(30\times10^3)$ gives

$$j(50\times10^3)\sqrt{\frac{0.4-L}{L}} = j(30\times10^3)$$

$$\sqrt{\frac{0.4-L}{L}} = 0.6 \qquad \left[\text{Dividing by } 50\times10^3\right]$$

Squaring both sides

$$\frac{0.4-L}{L} = 0.36$$

$$0.4 - L = 0.36L$$

Solving this equation gives $L = 0.294$ (correct to three d.p.) Hence $L = 0.294$ H.

11. Dividing through by C we have

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{v}{LC}=0$$

$$m^2+\frac{1}{RC}m+\frac{1}{LC}=0$$

Equating with $m^2+2\zeta\omega m+\omega^2=0$

$$\omega^2=\frac{1}{LC} \text{ gives } \omega=\frac{1}{\sqrt{LC}}$$

Equating the m terms gives $2\zeta\omega=\dfrac{1}{RC}$ and substituting $\omega=\dfrac{1}{\sqrt{LC}}$ we have

$$\frac{2\zeta}{\sqrt{LC}}=\frac{1}{RC}$$

$$\zeta=\frac{\sqrt{LC}}{2RC}=\frac{1}{2R}\sqrt{\frac{L}{C}}$$