Complete solutions to Exercise 14(c)

1. (a) Since $f(x)=18$ is a constant so $Y=C$, differentiating this gives zero. Substituting for Y into

$$\frac{d^2Y}{dx^2}-\frac{dY}{dx}-2Y=18$$

gives

$$0-0-2C=18, \text{ hence } C=-9$$

The particular integral $Y=-9$. (This particular integral can be seen clearly without going through all the mechanism above. Only the last term on the left hand side $-2y$ plays a part. Since you need 18, $Y=-9$).

(b) Since $f(x)=2x+3$ is a linear function so

$$Y=ax+b$$

$$\frac{dY}{dx}=a, \quad \frac{d^2Y}{dx^2}=0$$

Substituting these into the given differential equation

$$\frac{d^2Y}{dx^2}-\frac{dY}{dx}-2Y=2x+3$$

gives

$$0-a-2(ax+b)=2x+3$$

$$-2ax-(a+2b)=2x+3$$

Equating coefficients of

x: $\quad -2a=2$ hence $a=-1$

constants: $\quad -(a+2b)=3$

$$-(-1+2b)=3$$

$$-2b=2 \text{ hence } b=-1$$

Substituting $a=-1$ and $b=-1$ into $Y=ax+b$ gives

$$Y=-x-1$$

(c) Since $f(x)=130\sin(3x)$ our trial function is (see (14.16))

$$Y=a\cos(3x)+b\sin(3x)$$

$$\frac{dY}{dx}=\underbrace{-3a\sin(3x)}_{\text{by (6.13)}}+\underbrace{3b\cos(3x)}_{\text{by (6.12)}}$$

$$\frac{d^2Y}{dx^2}=\underbrace{-9a\cos(3x)}_{\text{by (6.12)}}-\underbrace{9b\sin(3x)}_{\text{by (6.13)}}$$

Substituting into

$$\frac{d^2Y}{dx^2}-\frac{dY}{dx}-2Y=130\sin(3x)$$

gives

$$-9a\cos(3x)-9b\sin(3x)-[-3a\sin(3x)+3b\cos(3x)]-2[a\cos(3x)+b\sin(3x)]=130\sin(3x)$$

(6.12) $\quad [\sin(kx)]'=k\cos(kx)$

(6.13) $\quad [\cos(kx)]'=-k\sin(kx)$

$$(-9a-3b-2a)\cos(3x)+(3a-9b-2b)\sin(3x)=130\sin(3x)$$
$$(-11a-3b)\cos(3x)+(3a-11b)\sin(3x)=130\sin(3x)$$

Equating coefficients of
$\cos(3x)$: $\quad -11a-3b=0$
$\sin(3x)$: $\quad 3a-11b=130$
Solving these simultaneous equations gives $a=3$ and $b=-11$. Putting $a=3$ and $b=-11$ into our trial function $Y=a\cos(3x)+b\sin(3x)$ gives the particular integral $Y=3\cos(3x)-11\sin(3x)$
(d) Since $f(x)=\cos(x)$ our trial function is given by

$$Y=a\cos(x)+b\sin(x) \qquad (\dagger)$$
$$\frac{dY}{dx}=-a\sin(x)+b\cos(x)$$
$$\frac{d^2Y}{dx^2}=-a\cos(x)-b\sin(x)$$

Substituting into

$$\frac{d^2Y}{dx^2}-\frac{dY}{dx}-2Y=\cos(x)$$

gives

$$-a\cos(x)-b\sin(x)-[-a\sin(x)+b\cos(x)]-2[a\cos(x)+b\sin(x)]=\cos(x)$$
$$(-a-b-2a)\cos(x)+(-b+a-2b)\sin(x)=\cos(x)$$
$$(-3a-b)\cos(x)+(a-3b)\sin(x)=\cos(x)$$

Equating coefficients of
$\cos(x)$: $\quad -3a-b=1$
$\sin(x)$: $\quad a-3b=0,\ \ a=3b$
Solving these simultaneous equations gives $a=-3/10,\ b=-1/10$.
Substituting these into (†)

$$Y=-\frac{3}{10}\cos(x)-\frac{1}{10}\sin(x)=-\frac{1}{10}[3\cos(x)+\sin(x)]$$

(e) Straightforward, $y=e^{3x}/4$

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2. The first two, (a) and (b), are uncomplicated.
(a) $y=Ae^{-2x}+Be^{-x}+3$ (b) $y=Ae^{-x/3}+Be^{x}-2x+7$
(c) First we find the complementary function y_c, (right hand side is equal to zero). The characteristic equation is

$$m^2+2m+1=0$$
$$(m+1)^2=0,\ m=-1 \ \ \text{(equal roots)}$$

By (14.5) $y_c=(A+Bx)e^{-x}$. Since $f(x)=36e^{5x}$ so by (14.17), $Y=Ce^{5x}$
Differentiating gives

(14.5) Equal roots, m, gives $y=(A+Bx)e^{mx}$
(14.7) If $f(x)=Ae^{kx}$ then $y=Ce^{kx}$

$$\frac{dY}{dx} = 5Ce^{5x}$$
$$\frac{d^2Y}{dx^2} = 25Ce^{5x}$$

Substituting into $\frac{d^2Y}{dx^2} + 2\frac{dY}{dx} + Y = 36e^{5x}$ gives

$$25Ce^{5x} + 2\left(5Ce^{5x}\right) + Ce^{5x} = 36e^{5x}$$
$$36Ce^{5x} = 36e^{5x} \text{ which gives } \quad C = 1$$

Substituting $C = 1$ into $Y = Ce^{5x}$ gives the particular integral $Y = e^{5x}$
Adding the complementary function, y_c, and the particular integral gives the general solution

$$y = (A + Bx)e^{-x} + e^{5x}$$

(d) Same procedure as part (a). The characteristic equation is

$$m^2 + 3m - 4 = 0$$
$$(m+4)(m-1) = 0$$
$$m_1 = -4, \quad m_2 = 1$$

Since we have 2 distinct roots so

$$y_c = Ae^{-4x} + Be^{x}$$

Particular integral Y:
Since $f(x) = -34\sin(x)$ we try the function

$$Y = a\cos(x) + b\sin(x)$$
$$\frac{dY}{dx} = -a\sin(x) + b\cos(x)$$
$$\frac{d^2Y}{dx^2} = -a\cos(x) - b\sin(x)$$

Substituting into $\frac{d^2Y}{dx^2} + 3\frac{dY}{dx} - 4Y = -34\sin(x)$ gives

$$\left[-a\cos(x) - b\sin(x)\right] + 3\left[-a\sin(x) + b\cos(x)\right] - 4\left[a\cos(x) + b\sin(x)\right] = -34\sin(x)$$
$$(-a + 3b - 4a)\cos(x) + (-b - 3a - 4b)\sin(x) = -34\sin(x)$$
$$(3b - 5a)\cos(x) + (-5b - 3a)\sin(x) = -34\sin(x)$$

Equating coefficients of

$\cos(x)$: $\quad 3b - 5a = 0$

$\sin(x)$: $\quad -5b - 3a = -34$

Solving these simultaneous equations gives $a = 3,\ b = 5$. Substituting for a and b into our trial function, $Y = a\cos(x) + b\sin(x)$, gives

$$Y = 3\cos(x) + 5\sin(x)$$

and the general solution, $\mathrm{y} = \mathrm{y_c} + \mathrm{Y}$, is

$$y = Ae^{-4x} + Be^{x} + 3\cos(x) + 5\sin(x)$$

3. By adding kx to both sides, the given differential equation becomes

$$m\ddot{x}+kx=mg$$

Dividing through by m gives

$$\ddot{x}+\frac{k}{m}x=g \qquad (*)$$

Complementary function x_c:

$$\ddot{x}_c+\frac{k}{m}x_c=0$$

The characteristic equation

$$r^2+\frac{k}{m}=0$$

$$r^2+\left(\sqrt{\frac{k}{m}}\right)^2=0$$

By (14.8) $x_c=A\cos(\omega t)+B\sin(\omega t)$ where $\omega=\sqrt{\frac{k}{m}}$

Particular integral X: Since

$$\ddot{x}+\frac{k}{m}x=g$$

and g is a constant so X=C, $\dot{X}=\ddot{X}=0$. Substituting these into (*)

$$\frac{k}{m}C=g \quad \text{which gives} \quad C=\frac{mg}{k}$$

Hence $X=\frac{mg}{k}$. The general solution is $x=x_c+X$ so we have

$$x=A\cos(\omega t)+B\sin(\omega t)+\frac{mg}{k} \quad \text{where} \quad \omega=\sqrt{\frac{k}{m}}$$

4. We have

$$\left(1\times10^3\right)\frac{d^2x}{dt^2}=(0.5+4x)\times10^3$$

$$\frac{d^2x}{dt^2}=(0.5+4x) \qquad \left[\text{dividing by } 10^3\right]$$

$$\frac{d^2x}{dt^2}-4x=0.5 \qquad (*)$$

Characteristic equation is $r^2-4=0$, solving gives $r_1=2$ and $r_2=-2$.

By (14.4) $x_c=Ae^{2t}+Be^{-2t}$

Since $f(x)=0.5$ our trial function is a constant, $X=C$. Substituting into (*) gives

$$-4C=0.5 \text{ gives } C=-0.125$$

The general solution is found by adding together the complementary function, $x_c=Ae^{2t}+Be^{-2t}$, and the particular integral, $X=-0.125$:

$$x=Ae^{2t}+Be^{-2t}-0.125$$

(14.4) If r_1 and r_2 then $y=Ae^{r_1x}+Be^{r_2x}$

(14.8) $r^2+k^2=0$ gives $y=A\cos(kx)+B\sin(kx)$

5. By adding kx to both sides, the differential equation becomes

$$m\ddot{x}+kx=F\sin\left(\alpha t\right)$$

Complementary function x_c:

$$m\ddot{x}_c+kx_c=0,\quad \ddot{x}_c+\frac{k}{m}x_c=0$$

Thus the characteristic equation is

$$r^2+\left(\sqrt{\frac{k}{m}}\right)^2=0$$

By (14.8) $\quad x_c=A\cos\left(\omega t\right)+B\sin\left(\omega t\right)$ where $\omega=\sqrt{\frac{k}{m}}$

Particular integral X:
By (14.16)

$$X=a\cos\left(\alpha t\right)+b\sin\left(\alpha t\right)$$

Differentiating gives

$$\dot{X}=\alpha\left[-a\sin\left(\alpha t\right)+b\cos\left(\alpha t\right)\right]$$

$$\ddot{X}=\alpha^2\left[-a\cos\left(\alpha t\right)-b\sin\left(\alpha t\right)\right]=-\alpha^2\left[a\cos\left(\alpha t\right)+b\sin\left(\alpha t\right)\right]$$

Substituting into

$$\ddot{X}+\frac{k}{m}X=\frac{F}{m}\sin\left(\alpha t\right)$$

gives

$$-\alpha^2\left[a\cos\left(\alpha t\right)+b\sin\left(\alpha t\right)\right]+\frac{k}{m}\left[a\cos\left(\alpha t\right)+b\sin\left(\alpha t\right)\right]=\frac{F}{m}\sin\left(\alpha t\right)$$

$$\left(\frac{k}{m}-\alpha^2\right)a\cos\left(\alpha t\right)+\left(\frac{k}{m}-\alpha^2\right)b\sin\left(\alpha t\right)=\frac{F}{m}\sin\left(\alpha t\right)$$

Equating coefficients of

$\cos(\alpha t)$: $\quad \left(\frac{k}{m}-\alpha^2\right)a=0$, gives $a=0$ because $\alpha^2\neq\frac{k}{m}$

$\sin(\alpha t)$: $\quad \left(\frac{k}{m}-\alpha^2\right)b=\frac{F}{m}$

$$b=\frac{F}{\cancel{m}\left(\frac{k-m\alpha^2}{\cancel{m}}\right)}=\frac{F}{k-m\alpha^2}$$

Substituting $a=0$ and $b=\frac{F}{k-m\alpha^2}$ into $X=a\cos(\alpha t)+b\sin(\alpha t)$ gives

$$X=\frac{F}{k-m\alpha^2}\sin\left(\alpha t\right)$$

The general solution, $x=x_c+X$, is

$$x=A\cos\left(\omega t\right)+B\sin\left(\omega t\right)+\frac{F}{k-m\alpha^2}\sin\left(\alpha t\right)\qquad \text{where } \omega=\sqrt{\frac{k}{m}}$$

(14.16) $\quad$ If $f(x)=A\sin\left(kx\right)$ then $Y=a\cos\left(kx\right)+b\sin\left(kx\right)$

6. Very similar to solution 5.

7. Rearranging the equation to

$$m\ddot{x}+c\dot{x}+kx=mg+kL \qquad (*)$$

Two parts, first find the complementary function;

$$m\ddot{x}_c+c\dot{x}_c+kx_c=0$$

Dividing by m

$$\ddot{x}_c+\frac{c}{m}\dot{x}_c+\frac{k}{m}x_c=0$$

The characteristic equation is

$$r^2+\frac{c}{m}r+\frac{k}{m}=0$$

Using the quadratic equation formula, (1.16), gives

$$r=\frac{-c/m\pm\sqrt{c^2/m^2-4k/m}}{2}$$

$$=-\frac{c}{2m}\pm\frac{1}{2}\sqrt{\frac{c^2-4km}{m^2}}$$

$$=-\frac{c}{2m}\pm\frac{1}{2m}\sqrt{c^2-4km}$$

$$=\frac{1}{2m}\left[-c\pm\sqrt{c^2-4km}\right]$$

$$=\frac{1}{2m}\left[-c\pm\sqrt{-\left(4km-c^2\right)}\right]$$

$$r=\frac{1}{2m}\left[-c\pm\underbrace{j}_{\substack{\text{because}\\ c^2<4km}}\sqrt{4km-c^2}\right]=-\frac{c}{2m}\pm j\frac{1}{2m}\sqrt{4km-c^2}$$

Using (14.6) with $\alpha=-\frac{c}{2m}$ and $\beta=\frac{1}{2m}\sqrt{4km-c^2}$ gives

$$x_c=e^{-\frac{c}{2m}t}\left[A\cos(\beta t)+B\sin(\beta t)\right]$$

where x_c represents the complementary function. We need to find the particular integral; Since the right hand side of (*) is a constant, $mg+kL$, so by (14. 11)

$$X=C,\ \dot{X}=0 \text{ and } \ddot{X}=0$$

Substituting into $m\ddot{X}+c\dot{X}+kX=mg+kL$ gives

$$kC=mg+kL$$

$$C=\frac{mg}{k}+L$$

(1.16) $\qquad r=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

(14.6) $\qquad$ If $m=\alpha\pm j\beta$ then $x=e^{\alpha t}\left[A\cos(\beta t)+B\sin(\beta t)\right]$

(14.11) $\qquad$ If $f(x)=\text{constant then } Y=C$

Hence the particular integral is $X = \frac{mg}{k} + L$.

The general solution, $x = x_c + X$, is

$$x = e^{-\frac{c}{2m}t}\left[A\cos(\beta t) + B\sin(\beta t)\right] + \frac{mg}{k} + L$$

where $\beta = \frac{1}{2m}\sqrt{4km - c^2}$.

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8. Dividing the given differential equation by CL we have

$$\frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I}{LC} \qquad (*)$$

Putting $R = 1\times 10^3$, $C = 10\times 10^{-9}$ and $L = 50\times 10^{-3}$

$$\frac{1}{RC} = \frac{1}{1\times 10^3 \times 10\times 10^{-9}} = 1\times 10^5$$

$$\frac{1}{LC} = \frac{1}{50\times 10^{-3}\times 10\times 10^{-9}} = 2\times 10^9$$

Substituting these and $I = 50\times 10^{-3}$ into (*) gives

$$\frac{d^2 i}{dt^2} + \left(1\times 10^5\right)\frac{di}{dt} + \left(2\times 10^9\right)i = \left(2\times 10^9\right)\left(50\times 10^{-3}\right) \qquad (**)$$

Complementary function i_c:

The characteristic equation is

$$m^2 + \left(1\times 10^5\right)m + \left(2\times 10^9\right) = 0$$

Putting $a = 1$, $b = 1\times 10^5$ and $c = 2\times 10^9$ into (1.16) gives

$$m = \frac{-\left(1\times 10^5\right) \pm \sqrt{\left(1\times 10^5\right)^2 - \left(4\times 2\times 10^9\right)}}{2}$$

$$= -\left(5\times 10^4\right) \pm \left(2.236\times 10^4\right)$$

$$m_1 = -2.764\times 10^4 \text{ and } m_2 = -7.236\times 10^4$$

By (14.4)

$$i_c = Ae^{-\left(2.764\times 10^4\right)t} + Be^{-\left(7.236\times 10^4\right)t}$$

Particular integral I:

Since the right hand side of (**) is $\left(2\times 10^9\right)\left(50\times 10^{-3}\right)$, a constant, so our trial function.

$$I = K \qquad (K \text{ is constant})$$

Substituting into $\frac{d^2 I}{dt^2} + \left(1\times 10^5\right)\frac{dI}{dt} + \left(2\times 10^9\right)I = \left(2\times 10^9\right)\left(50\times 10^{-3}\right)$ gives

$$0 + 0 + \left(2\times 10^9\right)K = \left(2\times 10^9\right)\left(50\times 10^{-3}\right)$$

$$K = 50\times 10^{-3}$$

$$I = 50\times 10^{-3}$$

(1.16) $$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(14.4) If m_1 and m_2 then $y = Ae^{m_1 t} + Be^{m_2 t}$

We know $i = i_c + I$

$$i = Ae^{-(2.764\times10^4)t} + Be^{-(7.236\times10^4)t} + (50\times10^{-3})$$

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9. By looking at **EXAMPLE 7** of this chapter, we know the complementary function $y_c = Ae^{2x} + Be^{-x}$. (Same characteristic equation). The trial function is $Y = Cxe^{-x}$ because Ce^{-x} is already part of the complementary function.

$$Y = Cxe^{-x} \qquad (\dagger)$$

Differentiating by using the product rule (6.31) gives

$$\frac{dY}{dx} = Ce^{-x} - Cxe^{-x}$$

$$\frac{d^2Y}{dx^2} = -Ce^{-x} - \left(Ce^{-x} - Cxe^{-x}\right) = -2Ce^{-x} + Cxe^{-x}$$

Substituting into

$$\frac{d^2Y}{dx^2} - \frac{dY}{dx} - 2Y = 5e^{-x}$$

gives

$$-2Ce^{-x} + Cxe^{-x} - Ce^{-x} + Cxe^{-x} - 2Cxe^{-x} = 5e^{-x}$$

$$\left(-2C + Cx - C + Cx - 2Cx\right)e^{-x} = 5e^{-x}$$

$$-3C = 5 \text{ gives } C = -5/3$$

Hence $Y = -\frac{5}{3}xe^{-x}$ $\left(\text{putting } C = -\frac{5}{3} \text{ into } (\dagger)\right)$

The general solution is given by $y = y_c + Y$, so we have

$$y = Ae^{2x} + Be^{-x} - \frac{5}{3}xe^{-x}$$

10.(a) Characteristic equation is

$$m^2 + 4m + 3 = 0$$

$$(m+3)(m+1) = 0$$

$$m_1 = -3, \quad m_2 = -1$$

Since we have two distinct roots so $y_c = Ae^{-3x} + Be^{-x}$

Hence the trial function is $Y = Cxe^{-3x}$

(b) Characteristic equation is

$$m^2 + 9 = 0$$

$$m^2 + 3^2 = 0$$

By (14.8), $y_c = A\cos(3x) + B\sin(3x)$. The trial function could be

$$a\cos(3x) + b\sin(3x)$$

Since $a\cos(3x)$ is already part of the complementary function, so we try

$$Y = x\left[a\cos(3x) + b\sin(3x)\right]$$

(6.31) $(uv)' = u'v + v'u$

(14.8) $m^2 + k^2 = 0$ then $y = A\cos(kx) + B\sin(kx)$

11. The differential equation can be rearranged to

$$\frac{d^2y}{dx^2}+\frac{P}{EI}y=-\frac{Pm}{EI}\sin\left(\frac{\pi x}{L}\right) \qquad (*)$$

Complementary function y_c; As before

$$y_c = A\cos(kx)+B\sin(kx) \text{ where } k=\sqrt{\frac{P}{EI}}$$

Particular integral Y; Since the right hand side of (*) is $-\frac{Pm}{EI}\sin\left(\frac{\pi x}{L}\right)$ so by (14.16) our trial function is

$$Y = C\cos\left(\frac{\pi x}{L}\right)+D\sin\left(\frac{\pi x}{L}\right) \qquad (**)$$

$$\frac{dY}{dx}=\frac{\pi}{L}\left[-C\sin\left(\frac{\pi x}{L}\right)+D\cos\left(\frac{\pi x}{L}\right)\right]$$

$$\frac{d^2Y}{dx^2}=-\frac{\pi^2}{L^2}\left[C\cos\left(\frac{\pi x}{L}\right)+D\sin\left(\frac{\pi x}{L}\right)\right]$$

Substituting into $\frac{d^2Y}{dx^2}+\frac{P}{EI}Y=-\frac{Pm}{EI}\sin\left(\frac{\pi x}{L}\right)$ gives

$$-\frac{\pi^2}{L^2}\left[C\cos\left(\frac{\pi x}{L}\right)+D\sin\left(\frac{\pi x}{L}\right)\right]+\frac{P}{EI}\left[C\cos\left(\frac{\pi x}{L}\right)+D\sin\left(\frac{\pi x}{L}\right)\right]$$

$$=-\frac{Pm}{EI}\sin\left(\frac{\pi x}{L}\right)$$

Equating coefficients of $\cos\left(\frac{\pi x}{L}\right)$;

$$C\left[\frac{P}{EI}-\frac{\pi^2}{L^2}\right]=0 \text{ gives } C=0$$

Equating coefficients of $\sin\left(\frac{\pi x}{L}\right)$;

$$D\left(\frac{P}{EI}-\frac{\pi^2}{L^2}\right)=-\frac{Pm}{EI}$$

$$D\left(\frac{PL^2-\pi^2EI}{(EI)L^2}\right)=-\frac{Pm}{EI}$$

Multiply through by EI gives

$$\frac{D\left(PL^2-\pi^2EI\right)}{L^2}=-Pm$$

$$D=\frac{PL^2m}{EI\pi^2-PL^2}$$

Substituting $C=0$ and $D=\frac{PL^2m}{EI\pi^2-PL^2}$ into (**) gives

(14.16) If $f(x)=A\sin(kx)$ then $Y=a\cos(kx)+b\sin(kx)$

$$Y = \frac{PL^2 m}{EI\pi^2 - PL^2}\sin\left(\frac{\pi x}{L}\right)$$

The general solution, y, is given by

$$y = A\cos(kx) + B\sin(kx) + \frac{PL^2 m}{EI\pi^2 - PL^2}\sin\left(\frac{\pi x}{L}\right)$$