Complete solutions to Exercise 14(d)

1. We have

$$\frac{d^2i}{dt^2}+8\frac{di}{dt}+15i=150$$

Characteristic equation is

$$m^2+8m+15=0$$
$$(m+3)(m+5)=0$$
$$m_1=-3,\ m_2=-5 \quad \text{(distinct roots)}$$

Thus complementary function is given by $i_c=Ae^{-3t}+Be^{-5t}$

Particular integral I; since the right hand side is 150, a constant, so

$$I=C \quad \text{where } C \text{ is a constant}$$

Hence

$$15C=150 \text{ gives } C=10$$

The general solution is

$$i=Ae^{-3t}+Be^{-5t}+10 \qquad (*)$$

Substituting $t=0,\ i=0$ into $i=Ae^{-3t}+Be^{-5t}+10$ gives

$$0=A+B+10$$
$$A+B=-10$$

Differentiating (*) gives $\dfrac{di}{dt}=-3Ae^{-3t}-5Be^{-5t}$

Substituting $t=0,\ \dfrac{di}{dt}=0$ yields

$$0=-3A-5B \text{ gives } B=-\frac{3}{5}A$$

Substituting $B=-\frac{3}{5}A$ into $A+B=-10$

$$A-\frac{3}{5}A=-10$$
$$\frac{2}{5}A=-10 \text{ gives } A=-25$$

Hence $B=15$. Substituting $A=-25$ and $B=15$ into $i=Ae^{-3t}+Be^{-5t}+10$ gives

$$i=-25e^{-3t}+15e^{-5t}+10=5\left(2+3e^{-5t}-5e^{-3t}\right)$$

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2. For complementary function we consider the homogeneous equation

$$\frac{d^2i}{dt^2}+3\frac{di}{dt}+2i=0$$

The characteristic equation is

$$m^2+3m+2=0$$
$$(m+2)(m+1)=0$$
$$m_1=-2,\ m_2=-1$$

By (14.4) the complementary function, i_c, is

$$i_c=Ae^{-2t}+Be^{-t}$$

(14.4) m_1 and m_2 gives $i=Ae^{m_1t}+Be^{m_2t}$

For particular integral; since $f(t)=5e^{-3t}$ so our trial function is

$$I=Ce^{-3t}$$

Differentiating $\frac{dI}{dt}=-3Ce^{-3t}$, $\frac{d^2I}{dt^2}=9Ce^{-3t}$

Substituting into $\frac{d^2I}{dt^2}+3\frac{dI}{dt}+2I=5e^{-3t}$ gives

$$9Ce^{-3t}+3\left(-3Ce^{-3t}\right)+2Ce^{-3t}=5e^{-3t}$$

$$(9C-9C+2C)e^{-3t}=5e^{-3t}$$

$2C=5$ gives $C=2.5$ and so $I=2.5e^{-3t}$

The general solution, $i=i_c+I$, is

$$i=Ae^{-2t}+Be^{-t}+2.5e^{-3t} \qquad (\dagger)$$

To find A and B we use the given initial conditions;
Substituting $t=0$, $i=0$ into $(\dagger)$

$$0=A+B+2.5$$

$$A+B=-2.5$$

Differentiating $(\dagger)$ yields $\frac{di}{dt}=-2Ae^{-2t}-Be^{-t}-7.5e^{-3t}$

Putting $t=0$ and $\frac{di}{dt}=5$ gives

$$-2A-B-7.5=5$$

$$-2A-B=12.5$$

Solving the simultaneous equations

$$A+B=-2.5$$

$$-2A-B=12.5$$

gives $A=-10$ and $B=7.5$

Substituting $A=-10$ and $B=7.5$ into $i=Ae^{-2t}+Be^{-t}+2.5e^{-3t}$ yields

$$i=7.5e^{-t}+2.5e^{-3t}-10e^{-2t}$$

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3. Very similar to **EXAMPLE 13**, $x=2\cos(3t)+\sin(3t)-2$

4. We have

$$m\ddot{x}+kx=F\sin(\alpha t) \qquad (*)$$

For complementary function

$$m\ddot{x}+kx=0$$

$$mr^2+k=0$$

$$r^2+\left(\sqrt{\frac{k}{m}}\right)^2=0$$

By (14.8) $x_c=A\cos(\omega t)+B\sin(\omega t)$ where $\omega=\sqrt{k/m}$

We need to find the value of the constant C in the given trial function

$$X=C\sin(\alpha t)$$

$$\dot{X}=\alpha C\cos(\alpha t)$$

$$\ddot{X}=-\alpha^2 C\sin(\alpha t)$$

(14.8) $r^2+\omega^2=0$ gives $x=A\cos(\omega t)+B\sin(\omega t)$

Substituting these into $m\ddot{X}+kX=F\sin(\alpha t)$ gives

$$m\left[-\alpha^2 C\sin(\alpha t)\right]+kC\sin(\alpha t)=F\sin(\alpha t)$$

$$\left(-m\alpha^2+k\right)C\sin(\alpha t)=F\sin(\alpha t)$$

$$C=\frac{F}{k-m\alpha^2} \text{ and so } X=\frac{F}{k-m\alpha^2}\sin(\alpha t)$$

Remember $x=x_c+X$, we have

$$x=A\cos(\omega t)+B\sin(\omega t)+\frac{F}{k-m\alpha^2}\sin(\alpha t) \qquad (**)$$

Substituting the initial condition $t=0,\ x=0$ gives $0=A$
Differentiating (**)

$$\dot{x}=-\omega A\sin(\omega t)+\omega B\cos(\omega t)+\frac{\alpha F}{k-m\alpha^2}\cos(\alpha t)$$

Putting in $t=0$ and $\dot{x}=0$ gives

$$0=0+\omega B+\frac{\alpha F}{k-m\alpha^2}$$

$$\omega B=-\frac{\alpha F}{k-m\alpha^2} \quad \text{hence} \quad B=-\frac{\alpha F}{\omega\left(k-m\alpha^2\right)}$$

Substituting $A=0$ and $B=-\dfrac{\alpha F}{\omega\left(k-m\alpha^2\right)}$ into (**) gives

$$x=-\frac{\alpha F}{\omega\left(k-m\alpha^2\right)}\sin(\omega t)+\frac{F}{k-m\alpha^2}\sin(\alpha t)$$

$$=\frac{F}{\omega\left(k-m\alpha^2\right)}\left[\omega\sin(\alpha t)-\alpha\sin(\omega t)\right]$$

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5. Dividing the given differential equation by EI yields

$$\frac{d^2y}{dx^2}+\frac{P}{EI}y=\frac{F}{EI}x \qquad (*)$$

Complementary function y_c;

$$\frac{d^2y_c}{dx^2}+\frac{P}{EI}y_c=0$$

We have

$$y_c=A\cos(kx)+B\sin(kx) \qquad \text{where } k=\sqrt{P/EI}$$

Particular integral Y; Using the trial function

$$Y=Cx+D$$

$$\frac{dY}{dx}=C, \quad \frac{d^2Y}{dx^2}=0$$

Substituting these into $\dfrac{d^2Y}{dx^2}+\dfrac{P}{EI}Y=\dfrac{F}{EI}x$ gives

$$\frac{P}{EI}(Cx+D)=\frac{F}{EI}x$$

Equating coefficients of x: $\dfrac{PC}{EI}=\dfrac{F}{EI}$ gives $C=\dfrac{F}{P}$

Equating constants gives $D=0$

Putting $C=\dfrac{F}{P}$ and $D=0$ into $Y=Cx+D$ gives $Y=\dfrac{Fx}{P}$

Since $y = y_c + Y$ so we have

$$y = A\cos(kx) + B\sin(kx) + \frac{Fx}{P} \qquad (**)$$

Substituting the boundary condition $x = 0,\ y = 0$ into (**) yields

$$0 = A + 0 + 0, \text{ hence } A = 0$$

$$\frac{dy}{dx} = -kA\sin(kx) + kB\cos(kx) + \frac{F}{P}$$

Substituting the other boundary condition $x = L,\ \frac{dy}{dx} = 0$ gives

$$0 = 0 + kB\cos(kL) + \frac{F}{P}$$

$$Bk\cos(kL) = -\frac{F}{P}$$

$$B = -\frac{F}{Pk\cos(kL)} = -\frac{F}{kP}\underbrace{\sec(kL)}_{\text{by (4.11)}}$$

Substituting $A = 0$ and $B = -\frac{F}{kP}\sec(kL)$ into (**) gives

$$y = \frac{Fx}{P} - \frac{F}{kP}\sec(kL)\sin(kx)$$

$$= \frac{F}{kP}\left[kx - \sec(kL)\sin(kx)\right]$$

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6. Rearranging the differential equation

$$EI\frac{d^2y}{dx^2} + Py = P(e+d)$$

Dividing by EI

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{P}{EI}(e+d) \qquad (*)$$

Complementary function, y_c, as above is

$$y_c = A\cos(kx) + B\sin(kx) \text{ where } k = \sqrt{\frac{P}{EI}}$$

Particular integral Y; Since there is no x term on the right hand side so it is a constant. Hence our trial function is

$$Y = C \quad \text{(constant)}$$

Substituting into $\frac{d^2Y}{dx^2} + \frac{P}{EI}Y = \frac{P}{EI}(e+d)$ gives

$$0 + \frac{P}{EI}C = \frac{P}{EI}(e+d)$$

Hence $C = e + d$

$$Y = e + d$$

Since $y = y_c + Y$, we have

$$y = A\cos(kx) + B\sin(kx) + (e+d) \qquad (**)$$

Substituting the initial condition $x = 0,\ y = 0$ into (**) gives

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(4.11) $$\frac{1}{\cos(\theta)} = \sec(\theta)$$

$$0 = A\cos(0) + B\sin(0) + (e+d)$$
$$= A + (e+d)$$
$$-(e+d) = A$$

Differentiating (**)

$$\frac{dy}{dx} = -kA\sin(kx) + kB\cos(kx)$$

Substituting the other initial condition $x = 0,\ \frac{dy}{dx} = 0$ yields

$$0 = 0 + kB \text{ gives } B = 0$$

Substituting $A = -(e+d)$ and $B = 0$ into (**)

$$y = (e+d) - (e+d)\cos(kx)$$
$$y = (e+d)[1 - \cos(kx)]$$

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7. The differential equation can be rearranged to

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{M}{EI} \qquad (*)$$

Complementary function y_c;

$$y_c = A\cos(kx) + B\sin(kx) \qquad \text{where } k = \sqrt{\frac{P}{EI}}$$

Particular integral Y;

$$Y = C \text{ (a constant)}$$

Substituting into $\frac{d^2Y}{dx^2} + \frac{P}{EI}Y = \frac{M}{EI}$

$$0 + \frac{PC}{EI} = \frac{M}{EI} \text{ gives } C = \frac{M}{P} \text{ and so } Y = \frac{M}{P}$$

We have $y = y_c + Y$, hence

$$y = A\cos(kx) + B\sin(kx) + \frac{M}{P} \qquad (**)$$

Substituting the initial condition $x = 0,\ y = 0$ into (**)

$$0 = A + 0 + \frac{M}{P}, \text{ hence } A = -\frac{M}{P}$$

Differentiating (**) gives

$$\frac{dy}{dx} = -kA\sin(kx) + kB\cos(kx)$$

Substituting $x = 0,\ \frac{dy}{dx} = 0$ gives

$$0 = 0 + kB \text{ gives } B = 0$$

Putting $A = -\frac{M}{P}$ and $B = 0$ into $y = A\cos(kx) + B\sin(kx) + \frac{M}{P}$ yields

$$y = -\frac{M}{P}\cos(kx) + \frac{M}{P} = \frac{M}{P}[1 - \cos(kx)]$$