Complete solutions to Exercise 15(a)

1. Very similar to **EXAMPLE 1**.

(a) $\frac{\partial f}{\partial x}=2x,\quad \frac{\partial f}{\partial y}=2y$ (b) $\frac{\partial f}{\partial x}=\cos(x),\quad \frac{\partial f}{\partial y}=-\sin(y)$

(c) $\frac{\partial f}{\partial x}=3x^2+3y,\quad \frac{\partial f}{\partial y}=3y^2+3x$ (d) $\frac{\partial f}{\partial x}=\frac{1000}{x^2},\quad \frac{\partial f}{\partial y}=2$

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2. We are given $M=Px+\frac{wx^3}{3}$. Thus

$$\frac{\partial M}{\partial P}=x \quad\text{and}\quad \frac{\partial M}{\partial x}=P+\frac{3wx^2}{3}=P+wx^2$$

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3. We have

(a) $$\Delta=\frac{\partial U}{\partial P}=\frac{\partial}{\partial P}\left(\frac{P^2L}{2AE}\right)=\frac{2PL}{2AE}=\frac{PL}{AE}$$

(b) $$\Delta=\frac{\partial U}{\partial P}=\frac{\partial}{\partial P}\left(\frac{9P^2L^3}{96EI}\right)=\frac{18PL^3}{96EI}=\frac{3PL^3}{16EI}$$

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4. Given $\Omega=Ax^2y^4+Bx^4y^2$ we have

$$\frac{\partial\Omega}{\partial y}=4Ax^2y^3+2Bx^4y$$

$$\sigma_x=\frac{\partial^2\Omega}{\partial y^2}=12Ax^2y^2+2Bx^4$$

$$\frac{\partial\Omega}{\partial x}=2Axy^4+4Bx^3y^2$$

$$\sigma_y=\frac{\partial^2\Omega}{\partial x^2}=2Ay^4+12Bx^2y^2$$

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5. (a) Given $u=y^2-x^2,\quad v=2xy$ we have

$$\frac{\partial u}{\partial x}=-2x \qquad \frac{\partial v}{\partial y}=2x$$

Hence $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=-2x+2x=0$

(b) We are given $u=\tan^{-1}\left(\frac{y}{x}\right)$. We find $\frac{\partial u}{\partial x}$ by using $\frac{\partial}{\partial x}\left[\tan^{-1}(P)\right]=\frac{1}{1+P^2}\frac{\partial P}{\partial x}$

$$\frac{\partial u}{\partial x}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)=\frac{-y}{\left(\frac{x^2+y^2}{x^2}\right)x^2}=-\frac{y}{x^2+y^2}$$

Consider $v=\frac{1}{2}\ln\left(x^2+y^2\right)$ then using $\frac{\partial}{\partial y}\left[\ln(P)\right]=\frac{1}{P}\frac{\partial P}{\partial y}$ gives

$$\frac{\partial v}{\partial y}=\frac{1}{2\left(x^2+y^2\right)}.2y=\frac{y}{x^2+y^2}$$

Hence $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=-\frac{y}{x^2+y^2}+\frac{y}{x^2+y^2}=0$

(c) We have $u = -2y\left[(1+x)^2 + y^2\right]^{-1}$

$$\frac{\partial u}{\partial x} = 2y\left[(1+x)^2 + y^2\right]^{-2} 2(1+x) = \frac{4y(1+x)}{\left[(1+x)^2 + y^2\right]^2}$$

We have $v = \frac{1 - x^2 - y^2}{(1+x)^2 + y^2}$. Using the quotient rule, (15.7), with

$$p = 1 - x^2 - y^2 \qquad q = (1+x)^2 + y^2$$

$$\frac{\partial p}{\partial y} = -2y \qquad \frac{\partial q}{\partial y} = 2y$$

gives

$$\frac{\partial v}{\partial y} = \frac{-2y\left[(1+x)^2 + y^2\right] - (1 - x^2 - y^2)2y}{\left((1+x)^2 + y^2\right)^2}$$

$$= \frac{-2y\left[1 + 2x + x^2 + y^2\right] - 2y + 2yx^2 + 2y^3}{\left((1+x)^2 + y^2\right)^2}$$

$$= \frac{-2y - 4yx - 2yx^2 - 2y^3 - 2y + 2yx^2 + 2y^3}{\left((1+x)^2 + y^2\right)^2}$$

$$= \frac{-4y - 4yx}{\left((1+x)^2 + y^2\right)^2} = \frac{-4y(1+x)}{\left((1+x)^2 + y^2\right)^2}$$

Hence $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

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6. Very similar to **EXAMPLE 8**. We have $u = \frac{y}{x^2 + y^2}$ and $v = -\frac{x}{x^2 + y^2}$

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7. We need to find κ where $\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)$. We are given the ideal gas equation.

$$V = \frac{RT}{P} = RTP^{-1} \qquad (*)$$

$$\frac{\partial V}{\partial P} = -\frac{RT}{P^2}$$

$$\kappa = -\frac{1}{V}\left(-\frac{RT}{P^2}\right)$$

$$= \frac{RT}{P^2 V} = \frac{RT}{P(PV)} \underbrace{=}_{\text{by } (*)} \frac{V}{PV} = \frac{1}{P}$$

(15.7) $$\frac{\partial}{\partial y}\left(\frac{p}{q}\right) = \frac{\frac{\partial p}{\partial y} q - p\frac{\partial q}{\partial y}}{q^2}$$

We need to find β where $\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)$. For $V = \frac{RT}{P}$

$$\frac{\partial V}{\partial T} = \frac{R}{P}$$

$$\beta = \frac{1}{V}\frac{R}{P} = \frac{R}{PV} \underbrace{=}_{\text{by transposing (*)}} \frac{1}{T}$$

8. Rearranging the given equation we have

$$V = \frac{RT}{P} - \frac{K}{RT} = \frac{RT}{P} - \frac{KT^{-1}}{R}$$

$$\frac{\partial V}{\partial T} = \frac{R}{P} + \frac{K}{RT^2}$$

Rearranging $\frac{RT}{P} = V + \frac{K}{RT}$ to make P the subject gives $P = \frac{RT}{V + K/RT}$

Differentiating with respect to T by applying the quotient rule, (15.7), with

$$u = RT \qquad v = V + \frac{K}{RT}$$

$$\frac{\partial u}{\partial T} = R \qquad \frac{\partial v}{\partial T} = -\frac{K}{RT^2}$$

$$\frac{\partial P}{\partial T} = \frac{R\left(V + \frac{K}{RT}\right) - RT\left(-\frac{K}{RT^2}\right)}{\left(V + K/RT\right)^2}$$

$$= \frac{RV + K/T + K/T}{\left(V + K/RT\right)^2} = \frac{RV + 2K/T}{\left(V + K/RT\right)^2}$$

Since $c_p - c_v = T\left(\frac{\partial V}{\partial T}\right)\left(\frac{\partial P}{\partial T}\right)$, by substituting the above

$$c_p - c_v = T\left(\frac{R}{P} + \frac{K}{RT^2}\right)\left[\frac{RV + 2K/T}{\left(V + K/RT\right)^2}\right]$$

9. Rewriting the given equation as $P = RT(V-b)^{-1} - aV^{-2}$. Partially differentiating with respect to V gives

$$\frac{\partial P}{\partial V} = -RT\left(V-b\right)^{-2} + 2aV^{-3}$$

$$\frac{\partial^2 P}{\partial V^2} = 2RT\left(V-b\right)^{-3} - 6aV^{-4}$$

Equating these to zero and rearranging gives

$$\frac{2a}{V^3} = \frac{RT}{\left(V-b\right)^2} \qquad (\dagger)$$

$$2RT\left(V-b\right)^{-3} - 6aV^{-4} = 0 \qquad (\dagger\dagger)$$

(15.7) $$\frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{\frac{\partial u}{\partial y}v - u\frac{\partial v}{\partial y}}{v^2}$$

From $(\dagger\dagger)$ we have

$$\frac{6a}{V^4}=\frac{2RT}{(V-b)^3}$$

Transposing to make a the subject

$$a=\frac{RTV^4}{3(V-b)^3}$$

Substituting $a=\dfrac{RTV^4}{3(V-b)^3}$ into $(\dagger)$ gives

$$\frac{2RTV^4}{3(V-b)^3V^3}=\frac{RT}{(V-b)^2}$$

$$2V=3(V-b)=3V-3b$$

$$3b=3V-2V=V$$

$$b=\frac{V}{3}$$

At critical point $V=V_c$ therefore $b=\dfrac{V_c}{3}$.

Substituting $b=\dfrac{V}{3}$ into $a=\dfrac{RTV^4}{3(V-b)^3}$ gives

$$a=\frac{RTV^4}{3\left(\frac{2}{3}V\right)^3}=\frac{RTV^4}{2^3V^3/3^2}=\frac{RTV^4}{8V^3/9}=\frac{9RTV}{8}$$

Similarly $a=\dfrac{9RT_cV_c}{8}$

10. Let $f(x)=A\sin\left(\dfrac{\omega}{\alpha}x\right)+B\cos\left(\dfrac{\omega}{\alpha}x\right)$ and $g(t)=C\sin(\omega t)+D\cos(\omega t)$ then

$$u=u(x,t)=f(x)g(t)$$

For $\dfrac{\partial^2u}{\partial x^2}$ treat $g(t)$ as a constant

$$u=\left[A\sin\left(\frac{\omega}{\alpha}x\right)+B\cos\left(\frac{\omega}{\alpha}x\right)\right]g(t)$$

$$\frac{\partial u}{\partial x}=\left[\frac{A\omega}{\alpha}\cos\left(\frac{\omega}{\alpha}x\right)-\frac{B\omega}{\alpha}\sin\left(\frac{\omega}{\alpha}x\right)\right]g(t)$$

$$\frac{\partial^2u}{\partial x^2}=\left[-\frac{A\omega^2}{\alpha^2}\sin\left(\frac{\omega}{\alpha}x\right)-\frac{B\omega^2}{\alpha^2}\cos\left(\frac{\omega}{\alpha}x\right)\right]g(t)$$

$$=-\frac{\omega^2}{\alpha^2}\underbrace{\left[A\sin\left(\frac{\omega}{\alpha}x\right)+B\cos\left(\frac{\omega}{\alpha}x\right)\right]}_{=f(x)}g(t)$$

$$\frac{\partial^2u}{\partial x^2}=-\frac{\omega^2}{\alpha^2}f(x)g(t)\qquad(*)$$

For $\frac{\partial^2 u}{\partial t^2}$, treat $f(x)$ as a constant

$$u = f(x)\left[C\sin(\omega t) + D\cos(\omega t)\right]$$

$$\frac{\partial u}{\partial t} = f(x)\left[C\omega\cos(\omega t) - D\omega\sin(\omega t)\right]$$

$$\frac{\partial^2 u}{\partial t^2} = f(x)\left[-C\omega^2\sin(\omega t) - D\omega^2\cos(\omega t)\right]$$

$$= -\omega^2 f(x)\underbrace{\left[C\sin(\omega t) + D\cos(\omega t)\right]}_{=g(t)}$$

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 f(x)g(t) \qquad (**)$$

From (*) and (**) we have

$$\alpha^2\frac{\partial^2 u}{\partial x^2} = \alpha^2\left(-\frac{\omega^2}{\alpha^2}\right)f(x)g(t) = -\omega^2 f(x)g(t) = \frac{\partial^2 u}{\partial t^2}$$

Thus $u(x,t)$ satisfies the wave equation.

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11. (a) Let $f = f(x, y)$ then we have

$$f = x^3 + y^3 - 3xy$$

$$\frac{\partial f}{\partial x} = 3x^2 + 0 - 3y = 3x^2 - 3y$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(3x^2 - 3y\right) = 6x$$

Similarly we find $\frac{\partial^2 f}{\partial y^2}$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^3 + y^3 - 3xy\right) = 0 + 3y^2 - 3x = 3y^2 - 3x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(3y^2 - 3x\right) = 6y$$

Also

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(3y^2 - 3x\right) = -3$$

Substituting the above gives

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x\partial y}\right)^2 = 6x6y - (-3)^2 = 36xy - 9$$

(b) Similarly we have

$$f = 2xy + 2000x^{-1} + 2000y^{-1}$$

$$\frac{\partial f}{\partial x} = 2y - 2000x^{-2}$$

$$\frac{\partial^2 f}{\partial x^2} = 0 + 4000x^{-3} = 4000x^{-3}$$

Also

$$\frac{\partial f}{\partial y} = 2x + 0 - 2000y^{-2} = 2x - 2000y^{-2}$$

$$\frac{\partial^2 f}{\partial y^2} = 0 + 4000y^{-3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(2x - 2000y^{-2}\right) = 2 - 0 = 2$$

Substituting the above gives

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x dy}\right)^2 = \left(4000x^{-3}\right)\left(4000y^{-3}\right) - \left(2\right)^2$$

$$= \frac{4000^2}{x^3 y^3} - 4 = \frac{16 \times 10^6}{x^3 y^3} - 4$$