Complete solutions to Exercise 15(b)

1. We have $I=\frac{1}{12}bd^3$ where I is a function of b and d.

$$\frac{\partial I}{\partial b}=\frac{1}{12}d^3 \quad\text{and}\quad \frac{\partial I}{\partial d}=\frac{3}{12}bd^2$$

We need to find ΔI. By (15.8)

$$\Delta I\approx\frac{\partial I}{\partial b}\Delta b+\frac{\partial I}{\partial d}\Delta d$$
$$=\frac{1}{12}\left[d^3\Delta b+3bd^2\Delta d\right]$$

We are given $b=5,\ d=0.8,\ \Delta b=0.05$ and $\Delta d=0.008$. Substituting these gives

$$\Delta I\approx\frac{1}{12}\left[\left(0.8^3\times0.05\right)+\left(3\times5\times0.8^2\times0.008\right)\right]=0.00853$$

The approximate error in I is 0.00853m^4

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2. Given $\text{G}=\frac{\text{R}^4\theta}{\text{L}}$ we need to find ΔG. Note that G is a function of $R,\ Q$ and L. We have

$$\Delta G\approx\left(\frac{\partial G}{\partial R}\right)\Delta R+\left(\frac{\partial G}{\partial\theta}\right)\Delta\theta+\left(\frac{\partial G}{\partial L}\right)\Delta L$$
$$=\left(\frac{4R^3\theta}{L}\right)\Delta R+\left(\frac{R^4}{L}\right)\Delta\theta-\left(\frac{R^4\theta}{L^2}\right)\Delta L \qquad (*)$$

The percentage changes can be represented by

$$\Delta R=-0.005R,\quad \Delta\theta=0.02\theta \quad\text{and}\quad \Delta L=0.015L$$

Substituting these into (*) gives

$$\Delta G\approx\frac{4R^3\theta}{L}\left(-0.005R\right)+\frac{R^4}{L}\left(0.02\theta\right)-\frac{R^4\theta}{L^2}\left(0.015L\right)$$
$$=\frac{R^4\theta}{L}\left[\left\{4\times\left(-0.005\right)\right\}+0.02-0.015\right] \qquad (\text{factorizing})$$
$$=G\left[-0.015\right]$$

The approximate percentage change in G is a reduction of 1.5%.

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3. Q is function of A and h. By (15.8)

$$\Delta Q\approx\frac{\partial Q}{\partial A}\Delta A+\frac{\partial Q}{\partial h}\Delta h \qquad (*)$$

We can rewrite Q as

$$Q=\underbrace{\left(0.7\sqrt{2g}\right)}_{\text{constant}}A\sqrt{h}=\left(0.7\sqrt{2g}\right)Ah^{1/2}$$

$$\frac{\partial Q}{\partial A}=0.7\sqrt{2g}\sqrt{h} \quad\text{and}\quad \frac{\partial Q}{\partial h}=\frac{1}{2}\left(0.7\sqrt{2g}\right)Ah^{-1/2}=\frac{\left(0.7\sqrt{2g}\right)A}{2\sqrt{h}}$$

(15.8) $$\Delta z\approx\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y$$

Substituting these into (*) gives

$$\Delta Q \approx 0.7\sqrt{2g}\left[\sqrt{h}\Delta A + \frac{A}{2\sqrt{h}}\Delta h\right]$$

$$\underbrace{=}_{\substack{\text{substituting}\\ \text{the given values}}} 0.7\sqrt{2\times 9.81}\left[\sqrt{1.75}\left(0.2\times 10^{-3}\right) + \frac{2\times 10^{-3}}{2\times\sqrt{1.75}}\left(-0.25\right)\right]$$

$$= 2.34\times 10^{-4}$$

The discharge rate, Q, is increased by approximately $\left(2.34\times 10^{-4}\right)\text{m}^3/\text{s}$.

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4. We need to find Δf. Also f is a function of L and C. By (15.8)

$$\Delta f \approx \frac{\partial f}{\partial L}\Delta L + \frac{\partial f}{\partial C}\Delta C \qquad (*)$$

The percentage change can be represented by

$$\Delta L = \frac{-1.5}{100}\times L = -0.015L,\ \ \Delta C = -\frac{0.5}{100}\times C = -0.005C$$

Putting these into (*) gives

$$\Delta f \approx \frac{\partial f}{\partial L}\left(-0.015L\right) + \frac{\partial f}{\partial C}\left(-0.005C\right) \qquad (\dagger)$$

$$f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi}L^{-1/2}C^{-1/2}$$

$$\frac{\partial f}{\partial L} = -\frac{1}{2}.\frac{1}{2\pi}L^{-3/2}C^{-1/2} = -\frac{1}{4\pi L^{3/2}C^{1/2}} = -\frac{1}{4\pi L L^{1/2}C^{1/2}}$$

$$\frac{\partial f}{\partial L} = -\frac{1}{4\pi L\sqrt{LC}}$$

Similarly

$$\frac{\partial f}{\partial C} = -\frac{1}{4\pi C\sqrt{LC}}$$

Substituting $\frac{\partial f}{\partial L} = -\frac{1}{4\pi L\sqrt{LC}}$ and $\frac{\partial f}{\partial C} = -\frac{1}{4\pi C\sqrt{LC}}$ into ($\dagger$) gives

$$\Delta f \approx -\frac{1}{4\pi \cancel{L}\sqrt{LC}}\left(-0.015\cancel{L}\right) - \frac{1}{4\pi\cancel{C}\sqrt{LC}}\left(-0.005\cancel{C}\right)$$

$$= \frac{1}{2\pi\sqrt{LC}}\left(\frac{0.015}{2} + \frac{0.005}{2}\right)$$

$$= f\left(0.01\right) \qquad \left(\text{remember } \frac{1}{2\pi\sqrt{LC}} = f\right)$$

$$\Delta f \approx 0.01 f$$

The approximate percentage change in f is 1%.

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5. L is a function of $C,\ \rho,\ V$ and A that is $L = f\left(C,\ \rho,\ V,\ A\right)$. We have

$$\Delta L \approx \frac{\partial L}{\partial C}\Delta C + \frac{\partial L}{\partial \rho}\Delta\rho + \frac{\partial L}{\partial V}\Delta V + \frac{\partial L}{\partial A}\Delta A \qquad (\dagger)$$

The given percentages can be written as

$$\Delta C = \pm 0.01C,\ \ \Delta\rho = \pm 0.005\rho,\ \ \Delta V = \pm 0.006V \ \text{ and } \ \Delta A = \pm 0.001A$$

Need to find $\dfrac{\partial L}{\partial C}$, $\dfrac{\partial L}{\partial \rho}$, $\dfrac{\partial L}{\partial V}$ and $\dfrac{\partial L}{\partial A}$

$$L = \frac{C\rho V^2 A}{2}$$

$$\frac{\partial L}{\partial C} = \frac{\rho V^2 A}{2}, \quad \frac{\partial L}{\partial \rho} = \frac{CV^2 A}{2}, \quad \frac{\partial L}{\partial V} = \frac{2C\rho VA}{2} \quad \text{and} \quad \frac{\partial L}{\partial A} = \frac{C\rho V^2}{2}$$

Substituting all these into (†) and taking the positive change for largest error gives

$$\Delta L \approx \frac{\rho V^2 A}{2}(0.01C) + \frac{CV^2 A}{2}(0.005\rho) + \frac{2C\rho VA}{2}(0.006V) + \frac{C\rho V^2}{2}(0.001A)$$

$$= \frac{\rho V^2 AC}{2}(0.01) + \frac{\rho V^2 AC}{2}(0.005) + \frac{2\rho V^2 AC}{2}(0.006) + \frac{\rho V^2 AC}{2}(0.001)$$

$$\underbrace{=}_{\text{factorizing}} \frac{\rho V^2 AC}{2}\left[0.01 + 0.005 + (2 \times 0.006) + 0.001\right]$$

$$= L(0.028) = 0.028L$$

The largest percentage error in L is 2.8%.

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6. By transposing formula we have $\eta = \dfrac{p\pi d^4}{128QL}$. Hence η is a function of p, d, Q and L

$$\Delta\eta \approx \frac{\partial \eta}{\partial p}\Delta p + \frac{\partial \eta}{\partial d}\Delta d + \frac{\partial \eta}{\partial Q}\Delta Q + \frac{\partial \eta}{\partial L}\Delta L \qquad (\dagger)$$

The percentage errors can be written as

$$\Delta p = \pm\frac{1.1}{100}p = \pm 0.011p, \quad \Delta Q = \pm\frac{0.5}{100}Q = \pm 0.005Q,$$

$$\Delta L = \pm\frac{0.3}{100}L = \pm 0.003L \quad \text{and} \quad \Delta d = \pm\frac{0.1}{100}d = \pm 0.001d$$

Partially differentiating $\eta = \dfrac{p\pi d^4}{128QL}$ we have

$$\frac{\partial \eta}{\partial p} = \frac{\pi d^4}{128QL}$$

$$\frac{\partial \eta}{\partial d} = \frac{4p\pi d^3}{128QL}$$

$$\frac{\partial \eta}{\partial Q} = -\frac{p\pi d^4}{128Q^2 L}$$

$$\frac{\partial \eta}{\partial L} = -\frac{p\pi d^4}{128QL^2}$$

Substituting all these into $\Delta\eta \approx \dfrac{\partial \eta}{\partial \rho}\Delta\rho + \dfrac{\partial \eta}{\partial d}\Delta d + \dfrac{\partial \eta}{\partial Q}\Delta Q + \dfrac{\partial \eta}{\partial L}\Delta L$ gives

$$\Delta\eta \approx \frac{\pi d^4}{128QL}(\pm 0.011p) + \frac{4p\pi d^3}{128QL}(\pm 0.001d) - \frac{p\pi d^4}{128Q^2 L}(\pm 0.005Q) - \frac{p\pi d^4}{128QL^2}(\pm 0.003L)$$

For maximum error we let the first 2 terms on the right hand side be positive and last 2 terms be negative so that we have

$$\Delta\eta \approx \frac{0.011p\pi d^4}{128QL} + \frac{0.004p\pi d^4}{128QL} + \frac{0.005p\pi d^4}{128QL} + \frac{0.003p\pi d^4}{128QL}$$

$$= \frac{p\pi d^4}{128QL}\left[0.011 + 0.004 + 0.005 + 0.003\right]$$

$$= \eta\left[0.023\right]$$

$$\Delta\eta \approx 0.023\eta$$

The maximum percentage error is 2.3%.

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7. T is a function of P and V. Applying (15.9) to $T = \frac{PV}{R}$ gives

$$dT = \frac{\partial T}{\partial P}dP + \frac{\partial T}{\partial V}dV \qquad (*)$$

$$\frac{\partial T}{\partial P} = \frac{V}{R} \quad \text{and} \quad \frac{\partial T}{\partial V} = \frac{P}{R}$$

Hence (*) becomes

$$dT = \frac{V}{R}dP + \frac{P}{R}dV$$

$$= \frac{VdP + PdV}{R}$$
