Complete solutions to Exercise 16(f)

1. The probability is $\frac{1}{5}$ for each x value, hence:

x	1	2	3	4	5
$P(X=x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

By (16.26)

$$E(X)=\left(1\times\frac{1}{5}\right)+\left(2\times\frac{1}{5}\right)+\left(3\times\frac{1}{5}\right)+\left(4\times\frac{1}{5}\right)+\left(5\times\frac{1}{5}\right)$$
$$=3$$

Notice the symmetrical nature of the probability distribution, therefore the mean $E(X)=3$.

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2. By (16.32)

$$E(X^2)=(1^2\times 0.15)+(2^2\times 0.21)+(3^2\times 0.11)+(4^2\times 0.36)+(5^2\times 0.04)+(6^2\times 0.13)$$
$$=13.42$$

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3. By using your calculator the mean is $3V$ and S.D. is $1.095V$ (3 d.p.). Since the probability distribution is symmetrical the mean is clearly going to be 3.

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4. Easiest way to tackle this problem is to use your calculator. The mean $\mu=4.19V$ (2 d.p.) and S.D., $\sigma=1.94V$ (2 d.p.)

5. The mean should be close to $1V$ because the signal is concentrated near $1V$, that is higher probabilities are close to $1V$. Use calculator: mean $=1.11V$ (2 d.p.) and S.D. $=1.05V$ (2 d.p.)

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6. (i) Substituting the given x values into $P(X=x)=kx^2$ gives

$$P(X=1)=k$$
$$P(X=2)=4k$$
$$P(X=3)=9k$$
$$P(X=4)=16k$$

By (16.23), $16k+9k+4k+k=1$

$$30k=1 \text{ which gives } k=\frac{1}{30}$$

Substituting $k=\frac{1}{30}$ and the x values into $P(X=x)=kx^2$ gives

x	1	2	3	4
$P(X=x)$	$\frac{1}{30}$	$\frac{4}{30}$	$\frac{9}{30}$	$\frac{16}{30}$

By using a calculator we have:

(ii) $E(X)=3.33$ (2 d.p.) (iii) $S.D.=0.83$ (2.d.p.)

(iv) variance $=0.83^2=0.69$ (2 d.p.)

(16.26) $E(X)=x_1P(X=x_1)+x_2P(X=x_2)+\ldots+x_nP(X=x_n)$

(16.32) $E(X^2)=x_1^2P(X=x_1)+x_2^2P(X=x_2)+\ldots+x_n^2P(X=x_n)$

7. (i) Use calculator or

$$E(X) = (1\times 0.1) + (2\times 0.3) + (3\times 0.2) + (4\times 0.4)$$
$$= 2.9$$

(ii)

$$E\left(X^2\right) = \left(1^2\times 0.1\right) + \left(2^2\times 0.3\right) + \left(3^2\times 0.2\right) + \left(4^2\times 0.4\right)$$
$$= 9.5$$

(iii)

$$E\left(X^2 + X\right) \underbrace{=}_{\text{by (16.29)}} E\left(X^2\right) + E(X)$$
$$= 9.5 + 2.9 = 12.4$$

(iv)

$$E\left(5X^2 + 7X + 3\right) \underbrace{=}_{\text{by(16.29)}} 5E\left(X^2\right) + 7E(X) + 3$$
$$= (5\times 9.5) + (7\times 2.9) + 3 = 70.8$$

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8. (i)

$$E(k) = \sum_{all\ x} kP(X = x)$$
$$= k\underbrace{\sum_{all\ x} P(X = x)}_{=1 \text{ because of (16.23)}} = k$$

(ii)

$$E[af(X) + b] = \sum_{all\ x}[af(X) + b]P(X = x)$$
$$= \sum_{all\ x}[af(X)P(X = x) + b\ P(X = x)]$$
$$= \sum_{all\ x} af(X)P(X = x) + \sum_{all\ x} bP(X = x)$$
$$= a\sum_{all x} f(X)P(X = x) + \sum_{all x} bP(X = x)$$
$$= aE[f(x)] + E(b)$$
$$= aE[f(x)] + \underbrace{b}_{\text{by (i)}}$$

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(16.23) $p_1 + p_2 + \ldots + p_n = 1$

(16.29) $E[kf(X) + m] = kE[f(x)] + m$

9. We are given the following probability distribution:

x_j (number of goals scored)	0	1	2	3	4	5	6
$P(X = x_j)$	0.05	0.15	0.2	0.25	0.15	0.1	0.1

Mean number of goals $\mu = E(X)$ can be determined by using formula (16.26):

$$\mu = E(X) = x_1 P(X = x_1) + x_2 P(X = x_2) + \cdots + x_6 P(X = x_6) + x_7 P(X = x_7)$$
$$= (0\times 0.05) + (1\times 0.15) + (2\times 0.2) + (3\times 0.25) + (4\times 0.15) + (5\times 0.1) + (6\times 0.1)$$
$$= 3$$

The mean number of goals is $\mu = 3$.
The standard deviation is the square root of the variance. The variance formula is given by (16.31) $\sigma^2 = E(X^2) - \mu^2$

Above we found $\mu = 3$ which means we only need to find $E(X^2)$:

$$E(X^2) = x_1{}^2 P(X = x_1) + x_2{}^2 P(X = x_2) + \cdots + x_6{}^2 P(X = x_6) + x_7{}^2 P(X = x_7)$$
$$= (0^2\times 0.05) + (1^2\times 0.15) + (2^2\times 0.2) + (3^2\times 0.25) + (4^2\times 0.15) + (5^2\times 0.1) + (6^2\times 0.1)$$
$$= 11.7$$

Substituting $E(X^2) = 11.7$ and $\mu = 3$ into the above formula $\sigma^2 = E(X^2) - \mu^2$ gives

$$\sigma^2 = 11.7 - 3^2 = 2.7$$

The standard deviation is the square root of this $\sigma = \sqrt{2.7} = 1.643$ (3dp).

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10. Let *X* be the number of girls in a family with three children. We are given that

$$P(G) = \frac{1}{2} \text{ and } P(B) = \frac{1}{2}$$

This is binomial distribution with $n = 3$:

$$(p+q)^3 = p^3 + 3p^2q + 3pq^2 + q^3 \qquad (*)$$

Let *p* be the probability of having a girl, then $p = \frac{1}{2}$ and $q = \frac{1}{2}$. Substituting these into (*) yields

$$\left(\frac{1}{2}+\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3$$
$$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$$

The probability distribution is given by

Number of girls - x_j	0	1	2	3
$P\left(X = x_j\right)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

The expected value is given by formula (16.26):

$$E\left(X\right) = x_1P\left(X = x_1\right) + x_2P\left(X = x_2\right) + x_3P\left(X = x_3\right) + x_4P\left(X = x_4\right)$$
$$= \left(0\times\frac{1}{8}\right) + \left(1\times\frac{3}{8}\right) + \left(2\times\frac{3}{8}\right) + \left(3\times\frac{1}{8}\right) = 1.5$$

This is an example where statistics is **not** particularly helpful. Clearly we **cannot** have 1.5 girls in a family.

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11. The probability distribution is given by:

Score x_j	2	3	4	5	6	7	8	9	10	11	12
$P\left(X = x_j\right)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Since the table is symmetrical about the score 7 therefore the expected value should be 7. We can confirm this expected value by using formula (16.26):

$$E\left(X\right) = x_1P\left(X = x_1\right) + x_2P\left(X = x_2\right) + \cdots + x_{10}P\left(X = x_{10}\right) + x_{11}P\left(X = x_{11}\right)$$
$$= \frac{1}{36}\left[2+6+12+20+30+42+40+36+30+22+12\right] = 7$$

The variance is given by the formula $\sigma^2 = E\left(X^2\right) - 7^2$. We need to evaluate $E\left(X^2\right)$:

$$E\left(X^2\right) = x_1^{\,2}P\left(X = x_1\right) + x_2^{\,2}P\left(X = x_2\right) + \cdots + x_{10}^{\,2}P\left(X = x_{10}\right) + x_{11}^{\,2}P\left(X = x_{11}\right)$$
$$= \frac{1}{36}\left[2^2 + \left(3^2\times 2\right) + \left(4^2\times 3\right) + \left(5^2\times 4\right) + \left(6^2\times 5\right) + \left(7^2\times 6\right) + \left(8^2\times 5\right) + \left(9^2\times 4\right) + \left(10^2\times 3\right) + \left(11^2\times 2\right) + 12^2\right]$$
$$= 54.833$$

Substituting $E\left(X^2\right) = 54.833$ into $\sigma^2 = E\left(X^2\right) - 7^2$ gives

$$\sigma^2 = 54.833 - 7^2 = 5.833 \text{ (3dp)}$$

12. Using (16.33) we have

$$P(X = x) = \frac{e^{-1.6}1.6^x}{x!}$$

(a) $P(X = 0) = \frac{e^{-1.6}1.6^0}{0!} = 0.20$ (b) $P(X = 1) = \frac{e^{-1.6}1.6^1}{1!} = 0.32$

(c)

$$P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= 0.2 + 0.32 + \frac{e^{-1.6}1.6^2}{2!} = 0.78$$

(d) $P(X = 10) = \frac{e^{-1.6}1.6^{10}}{10!} = 6.12 \times 10^{-6}$

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13. The mean is given by

$$\mu = np = 4096 \times (1 \times 10^{-3}) = 4.096$$

(a) Using (16.33)

$$P(X = 0) = \frac{e^{-4.096}(4.096)^0}{0!} = 0.0167$$

(b) The probability for more than 2 errors is given by

$$1 - [P(X = 0) + P(X = 1) + P(X = 2)] = 1 - [0.0167 + 0.068 + 0.14]$$

$$= 0.776$$

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(16.23) $p_1 + p_2 + \ldots + p_n = 1$

(16.33) $P(X = x) = e^{-\mu}(\mu)^x / x!$