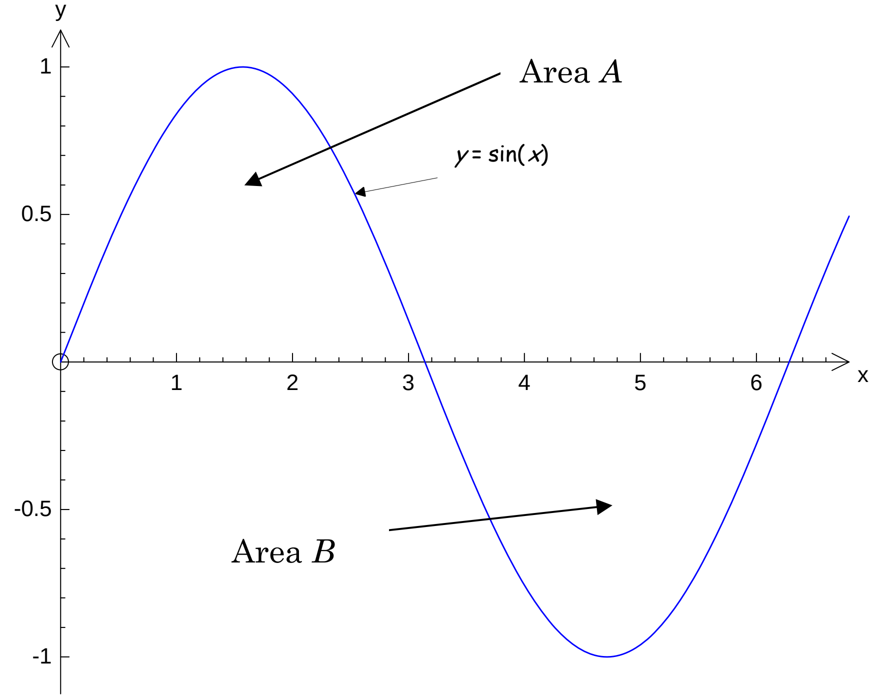
Complete solutions to Exercise 9(c)

1. We first make a sketch of $y=\sin(x)$:



By (9.6)

$$\text{Area } A=\int_0^{\pi}\sin(x)\,dx$$
$$=-\left[\cos(x)\right]_0^{\pi}$$
$$=-\left[\cos(\pi)-\cos(0)\right]=-[-2]=2$$

Similarly

$$\text{Area } B=-2$$

Thus

$$\text{total area}=2+2=4(\text{units})^2$$

2. We only need to consider one cycle, say between $t=0$ to $t=0.2$:
The mean value is the $\dfrac{\text{area}}{\text{interval}}$. The interval is 0.2 and the area of the triangle $=\frac{1}{2}(4\times 0.2)$: So

$$\text{mean value of } v=\frac{\frac{1}{2}(4\times 0.2)}{0.2}=2V$$

How do we find the R.M.S. value of v?
We need to find an equation for v. Since v is a straight line it is of the form

$$v=mt+c \qquad (*)$$

where m is the gradient and c is the v intercept. *What is the value of c?*

(9.6) $$\text{Area}=\int_a^b y\,dx$$

From graph

$$c = 4$$

What is the value of the gradient, m?

$$m = -\frac{4}{0.2} = -20$$

Hence substituting $m = -20$ and $c = 4$ into (*) gives

$$v = 4 - 20t$$

To find the R.M.S. value we use (9.8):

$$(R.M.S)^2 = \frac{1}{0.2}\int_0^{0.2} (4 - 20t)^2 dt$$

$$= 5\int_0^{0.2} \underbrace{4^2}_{\text{taking out the common factor}} (1 - 5t)^2 dt$$

$$= 5 \times 4^2 \int_0^{0.2} (1 - 5t)^2 dt$$

$$= 80\int_0^{0.2} \underbrace{(1 - 10t + 25t^2)}_{\text{expanding}} dt$$

$$= 80\left[t - \frac{10t^2}{2} + \frac{25t^3}{3}\right]_0^{0.2} \qquad \text{(Integrating)}$$

$$= 80\left\{\left[0.2 - (5 \times 0.2^2) + \left(\frac{25 \times 0.2^3}{3}\right)\right] - [0]\right\}$$

$$(R.M.S)^2 = 5.33$$

How do we find the R.M.S. value from this?

$$R.M.S = \sqrt{5.33} = 2.31\text{ V} \qquad \text{(2 d.p.)}$$

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3. Using (9.7) with $a = 0$, $b = \pi$, $y = i = I\sin(t)$ and $dx = dt$ we have:

$$\text{Mean value of } i = \frac{1}{\pi}\int_0^{\pi} I\sin(t)\,dt$$

$$= \frac{I}{\pi}\int_0^{\pi} \sin(t)\,dt \qquad \text{(Taking Out } I\text{)}$$

$$= \frac{I}{\pi}\left[-\cos(t)\right]_0^{\pi} \qquad \text{(Integrating)}$$

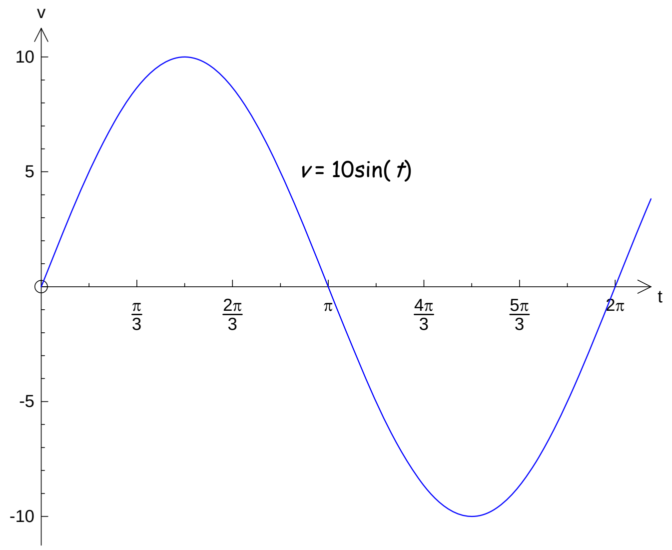
$$= -\frac{I}{\pi}\left(\underbrace{\cos(\pi)}_{=-1} - \underbrace{\cos(0)}_{=1}\right) = -\frac{I}{\pi}(-2)$$

$$\text{Mean value of } i = \frac{2I}{\pi} A$$

Using this result, the mean value of $\sin(t) = \frac{2}{\pi} A$ (substituting $I = 1$ into the above).

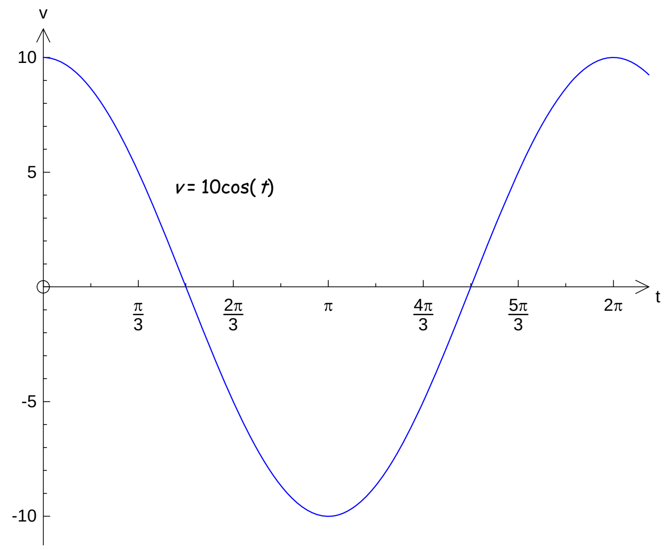
(9.8) $$(R.M.S.)^2 = \frac{1}{b-a}\int_a^b y^2 dx$$

4. (i) If we sketch $v = 10\sin(t)$ over the period of 0 to 2π then we have:



It can be clearly seen that the mean (average) value of $v = 10\sin(t)$ over 0 to 2π is $0V$.

(ii) Similarly for $v = 10\cos(t)$ we have:



So the mean value is $0V$.

5. Using (9.7) with $a = 0,\ b = 2\pi,\ y = v$ and $dx = d(\omega t)$ gives

$$\text{Mean value of } v = \frac{1}{2\pi}\int_0^{2\pi} V\sin(\omega t)d(\omega t)$$

$$= \frac{V}{2\pi}\int_0^{2\pi} \sin(\omega t)\,d(\omega t) \qquad (\text{Taking Out } V)$$

$$= \frac{V}{2\pi}\Big[-\cos(\omega t)\Big]_{\omega t=0}^{\omega t=2\pi} \qquad (\text{Integrating})$$

$$= -\frac{V}{2\pi}\Big(\underbrace{\cos(2\pi)}_{=1} - \underbrace{\cos(0)}_{=1}\Big) = -\frac{V}{2\pi}[0] = 0V$$

(9.7) Mean value of $y = \dfrac{1}{b-a}\displaystyle\int_a^b y\,dx$

6. Using (9.8) with $a=0,\ b=2\pi,\ y=i=I\cos(t)$ gives

$$\left(i_{R.M.S.}\right)^2=\frac{1}{2\pi}\int_0^{2\pi}\left(I\cos(t)\right)^2dt$$
$$=\frac{I^2}{2\pi}\int_0^{2\pi}\cos^2(t)dt \qquad (*)$$

How do we integrate $\cos^2(t)$ with respect to t?
Need to use

(4.67) $$\cos^2(t)=\frac{1}{2}\left[1+\cos(2t)\right]$$

The remaining evaluation is similar to **EXAMPLE 12**.

$$\int_0^{2\pi}\cos^2(t)dt=\int_0^{2\pi}\frac{1}{2}\left[1+\cos(2t)\right]dt$$
$$=\frac{1}{2}\int_0^{2\pi}\left[1+\cos(2t)\right]dt \qquad \left(\text{Taking Out } \frac{1}{2}\right)$$
$$=\frac{1}{2}\left[t+\frac{\sin(2t)}{2}\right]_0^{2\pi} \qquad (\text{Integrating})$$
$$=\frac{1}{2}\left\{\left[2\pi+\underbrace{\frac{\sin(2\times 2\pi)}{2}}_{=0}\right]-0\right\}$$
$$=\frac{1}{2}(2\pi)=\pi$$

Substituting $\int_0^{2\pi}\cos^2(t)dt=\pi$ into (*) gives

$$\left(i_{R.M.S.}\right)^2=\frac{I^2}{2\pi}(\pi)=\frac{I^2}{2}$$
$$i_{R.M.S.}=\sqrt{\frac{I^2}{2}}=\frac{\sqrt{I^2}}{\sqrt{2}}=\frac{I}{\sqrt{2}}$$

(9.7) Mean value of $y=\frac{1}{b-a}\int_a^b ydx$

(9.8) $$\left(R.M.S.\right)^2=\frac{1}{b-a}\int_a^b y^2dx$$

7. Using (9.7) we have

$$\text{Mean value of } v = \frac{1}{10-0}\int_0^{10} 10\left(1-e^{-0.1t}\right)dt$$

$$= \frac{10}{10}\int_0^{10}\left(1-e^{-0.1t}\right)dt$$

$$= \left[t - \frac{e^{-0.1t}}{\underbrace{-0.1}_{by\ (8.41)}}\right]_0^{10}$$

$$= \left[t + \underbrace{10}_{=-(-1/0.1)} e^{-0.1t}\right]_0^{10}$$

$$= \left(10 + 10e^{-(0.1\times 10)}\right) - \left(0 + 10\underbrace{e^0}_{=1}\right) = 10 + 10e^{-1} - 10 = 10e^{-1}$$

Mean value of $v = 10e^{-1} = 3.68V$ (2 d.p.)

To find the R.M.S. value we use (9.8):

$$(R.M.S.)^2 = \frac{1}{10}\int_0^{10}\left[10\left(1-e^{-0.1t}\right)\right]^2 dt$$

$$= 10\int_0^{10}\left[1 - 2e^{-0.1t} + \left(e^{-0.1t}\right)^2\right]dt \qquad \text{(Expanding)}$$

$$= 10\int_0^{10}\left[1 - 2e^{-0.1t} + e^{-0.2t}\right]dt$$

$$= 10\left[t - \underbrace{\frac{2e^{-0.1t}}{-0.1} + \frac{e^{-0.2t}}{-0.2}}_{by\ (8.41)}\right]_0^{10}$$

$$= 10\left\{\left[10 + 20e^{-(0.1\times 10)} - 5e^{-(0.2\times 10)}\right] - \left[0 + 20e^0 - 5e^0\right]\right\}$$

$$= 10\left\{\left[10 + 20e^{-1} - 5e^{-2}\right] - [15]\right\}$$

$$(R.M.S.)^2 = 16.809$$

So the R.M.S. value

$$R.M.S. = \sqrt{16.809} = 4.10V \quad \text{(2 d.p.)}$$

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8. The mean value is defined as the $\dfrac{\text{area}}{\text{interval}}$. The interval is 8 s and the area consists of a rectangle ($t = 0$ to $t = 3$), a trapezium ($t = 3$ to $t = 5$) and a triangle ($t = 5$ to $t = 8$).

Rectangle area $= 6 \times 3 = 18$ mA s

(8.41) $\int e^{kt+m}dt = e^{kt+m}/k$

(9.7) Mean value of $y = \dfrac{1}{b-a}\int_a^b y\,dx$

(9.8) $(R.M.S)^2 = \dfrac{1}{b-a}\int_a^b y^2 dx$

Trapezium area $=\frac{1}{2}(2)(6+10)=16$ mA s

Triangle area $=\frac{3\times(-10)}{2}=-15$ mA s

Total area $=18+16-15=19$ mA s

$$\text{Mean value } =\frac{19}{8}=2.38\text{ mA (3 s.f.)}$$

9. The mean value of v is evaluated by (9.7):

$$\text{Mean value of } v=\frac{1}{\pi}\int_0^\pi(\omega t)\sin(\omega t)d(\omega t) \qquad (\dagger)$$

How do we integrate this function?

Use integration by parts formula (8.45):

$$u=\omega t \qquad v'=\sin(\omega t)$$

$$u'=1 \qquad v=\int\sin(\omega t)\,d(\omega t)=-\cos(\omega t)$$

$$\int_0^\pi(\omega t)\sin(\omega t)\,d(\omega t)=-\left[\omega t\cos(\omega t)\right]_0^\pi+\int_0^\pi\cos(\omega t)d(\omega t)$$

$$=-\left[\pi\cos(\pi)-0\right]+\left[\sin(\omega t)\right]_0^\pi$$

$$=-\left[-\pi\right]+\underbrace{\left[\sin(\pi)-\sin(0)\right]}_{=0}=\pi$$

Substituting this into $(\dagger)$ gives:

$$\text{Mean value of } v=\frac{1}{\pi}(\pi)=1V$$

To find the $R.M.S.$ value we first obtain $(R.M.S.)^2$:

$$(R.M.S.)^2=\frac{1}{\pi}\int_0^\pi\left[(\omega t)\sin(\omega t)\right]^2d(\omega t)$$

$$=\frac{1}{\pi}\int_0^\pi(\omega t)^2\sin^2(\omega t)d(\omega t)$$

By using (4.68) we can rewrite $\sin^2(\omega t)$ as:

$$\sin^2(\omega t)=\frac{1}{2}\left[1-\cos(2\omega t)\right]$$

So we have

$$(R.M.S.)^2=\frac{1}{2\pi}\left[\int_0^\pi(\omega t)^2d(\omega t)-\int_0^\pi(\omega t)^2\cos(2\omega t)d(\omega t)\right] \qquad (*)$$

First integral on the right of (*) is straightforward but how do we find $\int_0^\pi(\omega t)^2\cos(2\omega t)d(\omega t)$?

Use integration by parts, (8.45):

$$u=(\omega t)^2 \qquad v'=\cos(2\omega t)$$

$$\frac{du}{d(\omega t)}=2(\omega t) \qquad v=\int\cos(2\omega t)\,d(\omega t)=\frac{\sin(2\omega t)}{2}$$

(8.45) $$\int uv'dt=uv-\int u'v\,dt$$

(9.7) $$\text{Mean value of } y=\frac{1}{b-a}\int_a^b y\,dx$$

Hence

$$\int_0^{\pi}(\omega t)^2\cos(2\omega t)\,d(\omega t)=\left[\frac{(\omega t)^2\sin(2\omega t)}{2}\right]_0^{\pi}-\int_0^{\pi}\frac{2(\omega t)\sin(2\omega t)}{2}\,d(\omega t)$$

$$=0-\int_0^{\pi}(\omega t)\sin(2\omega t)\,d(\omega t)$$

$$\int_0^{\pi}(\omega t)^2\cos(2\omega t)\,d(\omega t)=-\int_0^{\pi}(\omega t)\sin(2\omega t)\,d(\omega t) \qquad (**)$$

Use integration by parts again:

$$u=\omega t \qquad v'=\sin(2\omega t)$$

$$u'=1 \qquad v=\int\sin(2\omega t)\,d(\omega t)=-\frac{\cos(2\omega t)}{2}$$

Substituting into (**):

$$\int_0^{\pi}(\omega t)^2\cos(2\omega t)\,d(\omega t)=\left[\frac{\omega t\cos(2\omega t)}{2}\right]_0^{\pi}-\int_0^{\pi}\frac{\cos(2\omega t)}{2}\,d(\omega t)$$

$$=\frac{\pi}{2}-\left[\frac{\sin(2\omega t)}{4}\right]_0^{\pi}$$

$$=\frac{\pi}{2}$$

Evaluating the first integral of (*):

$$\int_0^{\pi}(\omega t)^2 d(\omega t)=\left[\frac{(\omega t)^3}{3}\right]_0^{\pi}=\frac{\pi^3}{3}$$

Substituting these evaluations into (*) gives:

$$(R.M.S.)^2=\frac{1}{2\pi}\left[\frac{\pi^3}{3}-\frac{\pi}{2}\right]$$

$$R.M.S.=\sqrt{\frac{1}{2\pi}\left[\frac{\pi^3}{3}-\frac{\pi}{2}\right]}=1.18V$$

The form factor:

$$f=\frac{1.18}{1}=1.18$$

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10. The mean value, M, of i between 0 to $\pi/2$ is given by:

$$M=\frac{1}{\pi/2}\int_0^{\pi/2}\sqrt{\cos(\omega t)}\,d(\omega t) \qquad (*)$$

To find the integral we use Simpson's rule with 4 equal integrals:

$$h=\frac{\pi/2}{4}=\frac{\pi}{8}$$

We establish a table of values:

ωt	0	$\pi/8$	$2\pi/8$	$3\pi/8$	$4\pi/8$
$\sqrt{\cos(\omega t)}$	1	0.961	0.841	0.619	0

Applying Simpson's rule

$$\int_0^{\pi/2}\sqrt{\cos(\omega t)}\,d(\omega t)\approx\frac{\pi/8}{3}\left[1+4(0.961+0.619)+2(0.841)+0\right]$$
$$=\frac{\pi/8}{3}[9.002]$$

Substituting into (*):

$$M=\frac{1}{\pi/2}\left(\frac{\pi/8}{3}[9.002]\right)=\frac{9.002}{12}=0.75V \qquad \text{(2 d.p.)}$$

11. Very similar to **EXAMPLE 14**. Mean force is $3.86kN$.

12. (i)

Shaded Area = [Area of rectangle]-[Area under $y=x^2$ between 0 and 1](*)

Area under $y=x^2$ between 0 and 1 = $\int_0^1 x^2dx$

$$\underbrace{=}_{\text{by (8.1)}}\left[\frac{x^3}{3}\right]_0^1=\frac{1}{3}$$

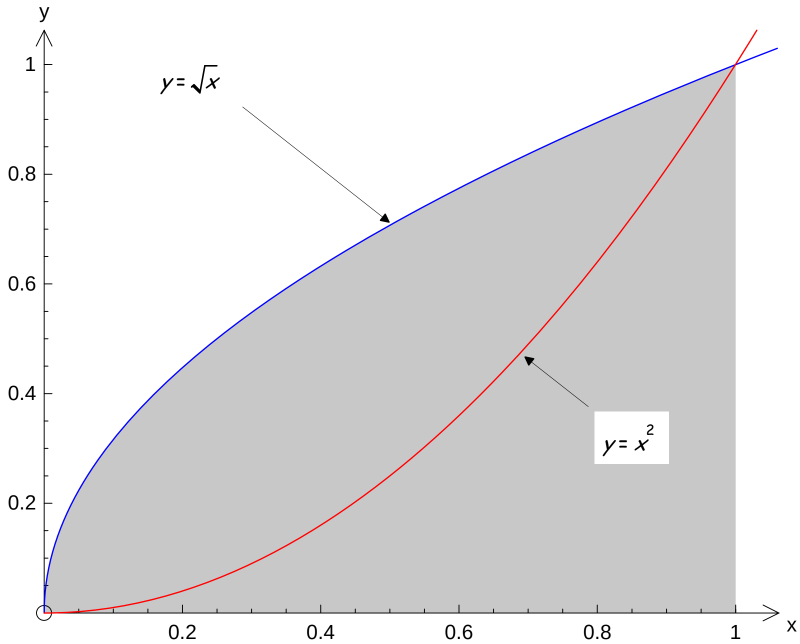
Replacing this in (*) gives

$$\text{Shaded Area} = \underbrace{1}_{\text{Area of rectangle}}-\frac{1}{3}=\frac{2}{3}$$

(ii) $\int_0^1 x^{1/2}dx \underbrace{=}_{by\ (8.1)}\left[\frac{x^{3/2}}{3/2}\right]_0^1$

$$=\frac{2}{3}\left[x^{3/2}\right]_0^1=\frac{2}{3}[1-0]=\frac{2}{3}$$

(iii)



(iv) Results are the same, $\frac{2}{3}$, because $x^{1/2}$ is the inverse function of x^2, that is $x^{1/2}$ reflects x^2 in the line $y=x$. Hence the above shaded area and the shaded area of part (i) are equal.