Complete solutions to Miscellaneous Exercise 5

1. Remember m stands for milli and is 10^{-3} also μ is 10^{-6}. Substituting gives:

$$V = \left(20\times10^{-3}\right)\ln\left[\left(\frac{10\times10^{-3}}{1\times10^{-6}}\right)+1\right]$$

$$= \left(20\times10^{-3}\right)\ln\left(10001\right) = 0.18\ volts$$

---

2. We have

$$I = I_0\left[\exp\left(\frac{qV}{kT}\right)-1\right]$$

Dividing through by I_0 gives:

$$\frac{I}{I_0} = \exp\left(\frac{qV}{kT}\right)-1$$

Adding 1 to both sides:

$$\frac{I}{I_0}+1 = \exp\left(\frac{qV}{kT}\right)$$

How do we remove the exp on the Right Hand Side?
Take natural logs, ln, of both sides:

$$\ln\left(\frac{I}{I_0}+1\right) = \frac{qV}{kT} \qquad \text{[by (5.15)]}$$

Multiply both sides by $\dfrac{kT}{q}$:

$$\frac{kT}{q}\ln\left(\frac{I}{I_0}+1\right) = V$$

---

3. By cancelling the $k's$ we have:

$$\delta = \ln\left(\frac{e^{-\zeta\omega t_1}}{e^{-\zeta\omega t_2}}\right)$$

$$\underbrace{=}_{\text{by (5.2)}} \ln\left(e^{-\zeta\omega t_1-(-\zeta\omega t_2)}\right)$$

$$= \ln\left(e^{-\zeta\omega t_1+\zeta\omega t_2}\right) = \ln\left(e^{\zeta\omega t_2-\zeta\omega t_1}\right) \underbrace{=}_{\text{by (5.15)}} \zeta\omega t_2 - \zeta\omega t_1$$

$$\delta = \zeta\omega\left(t_2-t_1\right) \qquad \left(\text{factorizing } \zeta\omega\right)$$

---

4. Substituting $V_0 = 100$ into the original equation gives: $V = 100\left(1-e^{-t/\tau}\right)$

At $t=\tau$, $V = 100\left(1-e^{\frac{-\tau}{\tau}}\right) = 100\left(1-e^{-1}\right) = 63.2$ (by calculator)

At $t=2\tau$, $V = 100\left(1-e^{\frac{-2\tau}{\tau}}\right) = 100\left(1-e^{-2}\right) = 86.5$

(5.2) $\dfrac{a^m}{a^n} = a^{m-n}$

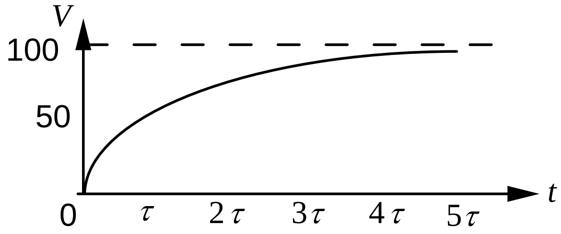
(5.15) $\ln\left(e^x\right) = x$

At $t = 3\tau$, $V = 100\left(1 - e^{\frac{-3\tau}{\tau}}\right) = 100\left(1 - e^{-3}\right) = 95.0$

At $t = 4\tau$, $V = 100\left(1 - e^{\frac{-4\tau}{\tau}}\right) = 100\left(1 - e^{-4}\right) = 98.2$

At $t = 5\tau$, $V = 100\left(1 - e^{\frac{-5\tau}{\tau}}\right) = 100\left(1 - e^{-5}\right) = 99.3$

Sketching the graph:



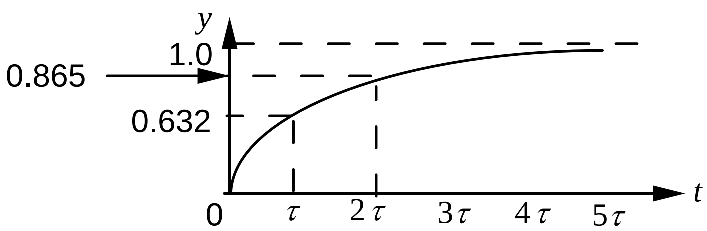
The percentage of V_0 after $t = \tau,\ 2\tau,\ 3\tau,\ 4\tau$ and 5τ is 63.2%, 86.5%, 95%, 98.2% and 99.3% respectively. [Note: The capacitor has a voltage V_0 (approximately) after 5 time constants (5τ)].

5. (i) $Y = \lim_{t\to\infty}\left(1 - e^{-\frac{t}{\tau}}\right) = 1 - \underbrace{0}_{\text{by } (5.11)} = 1$

(ii) Similar to solution 4: At $t = \tau,\ 2\tau,\ 3\tau,\ 4\tau$ and 5τ the value of y is $0.632, 0.865, 0.950, 0.982$ and 0.993 respectively.

(iii) 63.2%, 86.5%, 95%, 98.2% and 99.3%

(iv)



6. (i) The velocity v decreases as k increases.

(ii) As $t \to \infty$, $e^{-kt} \to 0$ so $v \to \dfrac{10}{k}$.

(iii)

(iv) Yes

(v) $k = 1, v \to 10m/s$, $k = 2, v \to 5m/s$ and $k = 5, v \to 2m/s$.

7. Very similar to solution 4 of EXERCISE 4(d) with $n = -0.2$ and $k = 0.058$

(5.11) $\lim_{x\to\infty}\left(e^{-x}\right) = 0$

8. Maple output is:

```
> y:=((H)/w)*(cosh((w*L)/(2*H))-1);
```

$$y := \frac{H\left(\cosh\left(\frac{w\,L}{2\,H}\right) - 1\right)}{w}$$

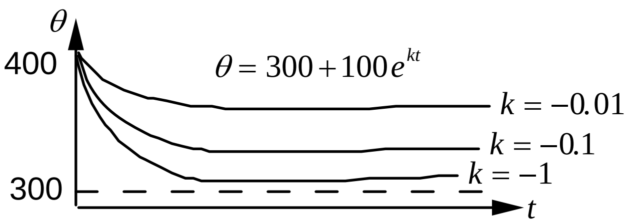
```
> t:=subs({L=270,w=0.16*10^3},y);
```

$$t := 0.006250000000\,H\left(\cosh\left(\frac{21600.00000}{H}\right) - 1\right)$$

```
> fsolve(t=55,H);
```

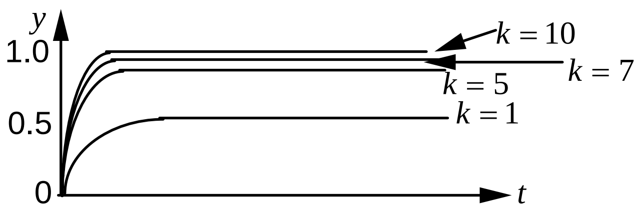
$$27863.51598$$

9.

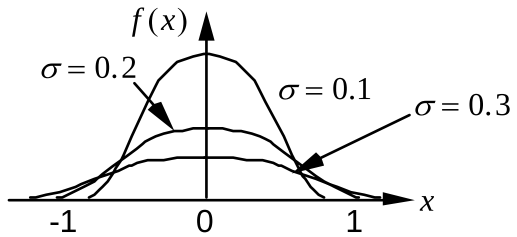


The temperature θ decreases more rapidly as k increases.
$\lim_{t\to\infty} \theta = 300$ in each case, because $e^{kt} \to 0$ as $t \to \infty$ for $k = -0.01,\ -0.1$ and -1.

10.



11.



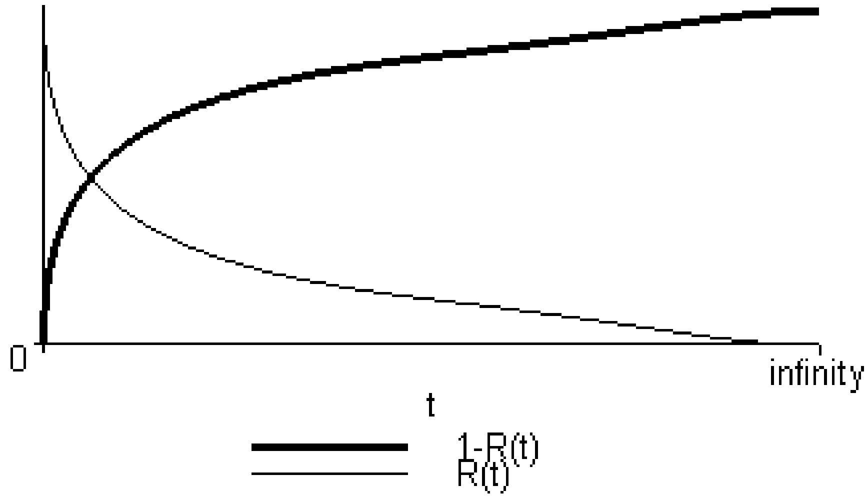
As σ increases the graph becomes more shallow.

12.(i)Maple Solution is:

```
> R:=t->exp(-sqrt(t));
```

$$R := t \to \mathbf{e}^{(-\sqrt{t})}$$

```
> plot({R(t),1-R(t)},t=0..infinity,color=[black,black],thickness=[3,1]);
```



(ii) As $t \to \infty$, $R(t) \to 0$ and $F(t) \to 1.$

13. Applying (5.24) and (5.25) gives:

$$\cosh(A)\cosh(B)+\sinh(A)\sinh(B)=\underbrace{\left(\frac{e^{A}+e^{-A}}{2}\right)}_{=\cosh A}\underbrace{\left(\frac{e^{B}+e^{-B}}{2}\right)}_{=\cosh B}+\underbrace{\left(\frac{e^{A}-e^{-A}}{2}\right)}_{=\sinh A}\underbrace{\left(\frac{e^{B}-e^{-B}}{2}\right)}_{=\sinh B}$$

$$=\frac{\left(e^{A}+e^{-A}\right)\left(e^{B}+e^{-B}\right)}{2\times 2}+\frac{\left(e^{A}-e^{-A}\right)\left(e^{B}-e^{-B}\right)}{2\times 2}$$

$$\underbrace{=}_{\substack{\text{using FOIL}\\ \text{on numerator}}}\frac{\left(e^{A}e^{B}+e^{A}e^{-B}+e^{-A}e^{B}+e^{-A}e^{-B}\right)}{4}+\frac{\left(e^{A}e^{B}-e^{A}e^{-B}-e^{-A}e^{B}+e^{-A}e^{-B}\right)}{4}$$

$$\underbrace{=}_{\text{by(5.1)}}\left(\frac{e^{A+B}+e^{A-B}+e^{-A+B}+e^{-A-B}}{4}\right)+\left(\frac{e^{A+B}-e^{A-B}-e^{-A+B}+e^{-A-B}}{4}\right)$$

$$=\frac{e^{A+B}+e^{A-B}+e^{-A+B}+e^{-A-B}+e^{A+B}-e^{A-B}-e^{-A+B}+e^{-A-B}}{4}$$

$$=\frac{2e^{A+B}+2e^{-A-B}}{4} \qquad \text{(Adding Numerator)}$$

$$=\frac{\cancel{2}\left(e^{A+B}+e^{-(A+B)}\right)}{\cancel{4}2}$$

$$=\frac{e^{A+B}+e^{-(A+B)}}{2}$$

$$\underbrace{=}_{\text{by (5.25)}}\cosh(A+B)$$

(5.1) $a^{m}a^{n}=a^{m+n}$

(5.24) $\sinh(x)=\dfrac{e^{x}-e^{-x}}{2}$

(5.25) $\cosh(x)=\dfrac{e^{x}+e^{-x}}{2}$

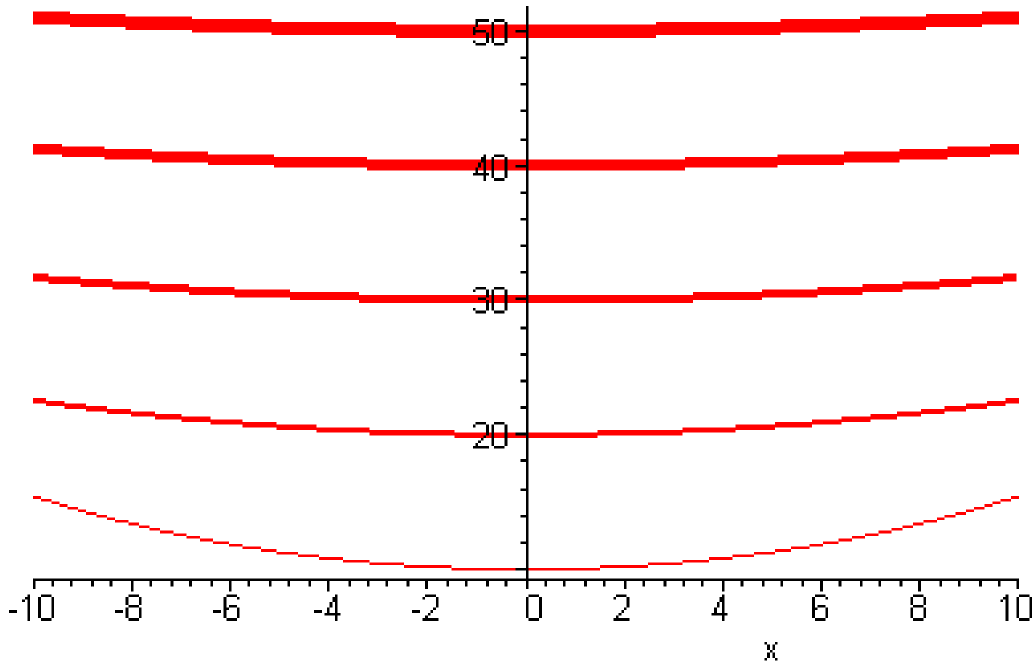
The MAPLE output for questions 14 and 15 is:

14.

```
> with(plots):
> y:=k->k*cosh(x/k);
```

$$y := k \to k\cosh\left(\frac{x}{k}\right)$$

```
> F1:=plot(y(10),x=-10..10,thickness=1):
> F2:=plot(y(20),x=-10..10,thickness=2):
> F3:=plot(y(30),x=-10..10,thickness=3):
> F4:=plot(y(40),x=-10..10,thickness=4):
> F5:=plot(y(50),x=-10..10,thickness=5):
> display({F1,F2,F3,F4,F5});
```

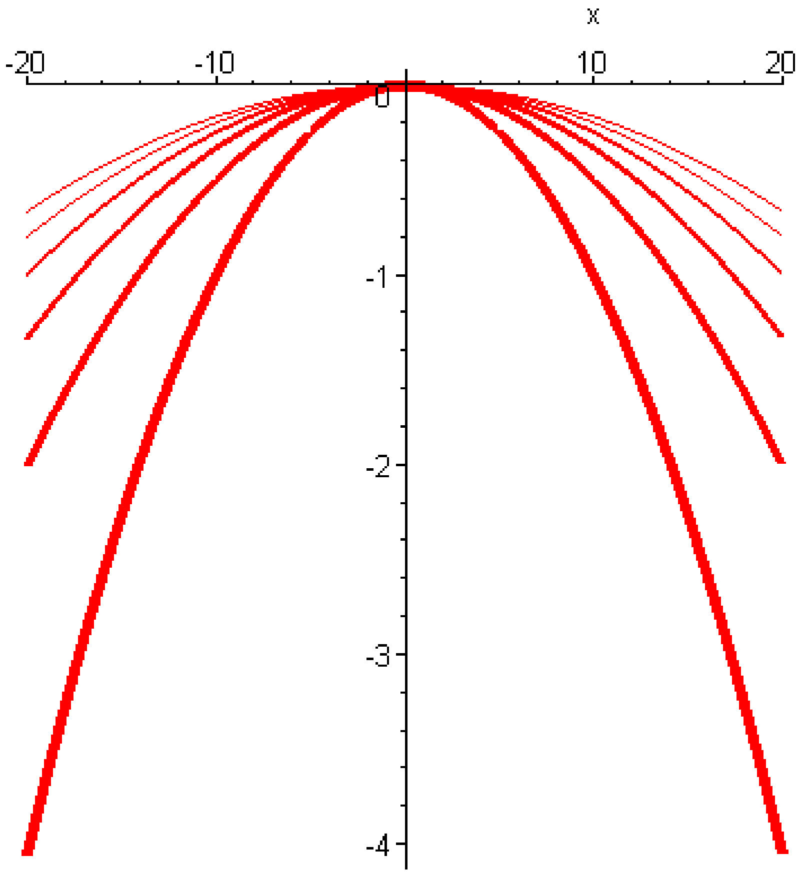


15.

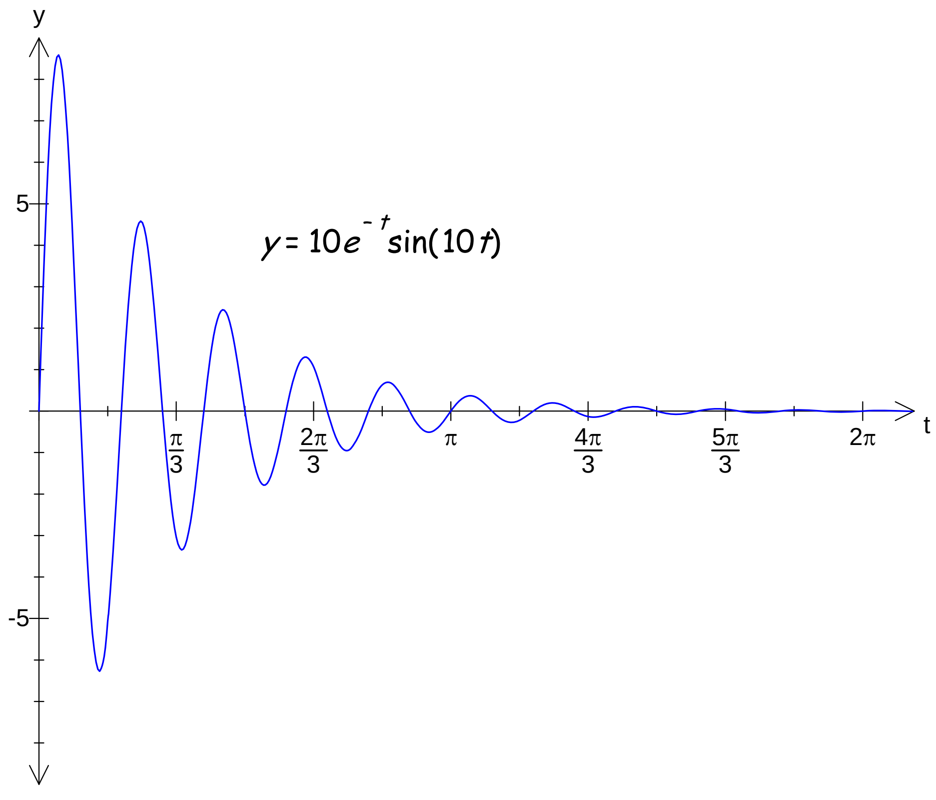
```
> y:=c->c-c*cosh(x/c);
```

$$y := c \to c - c\cosh\left(\frac{x}{c}\right)$$

```
> F1:=plot(y(50),x=-20..20,thickness=5):
> F2:=plot(y(100),x=-20..20,thickness=4):
> F3:=plot(y(150),x=-20..20,thickness=3):
> F4:=plot(y(200),x=-20..20,thickness=2):
> F5:=plot(y(250),x=-20..20,thickness=1):
> F6:=plot(y(300),x=-20..20,thickness=0):
> display({F1,F2,F3,F4,F5,F6});
```



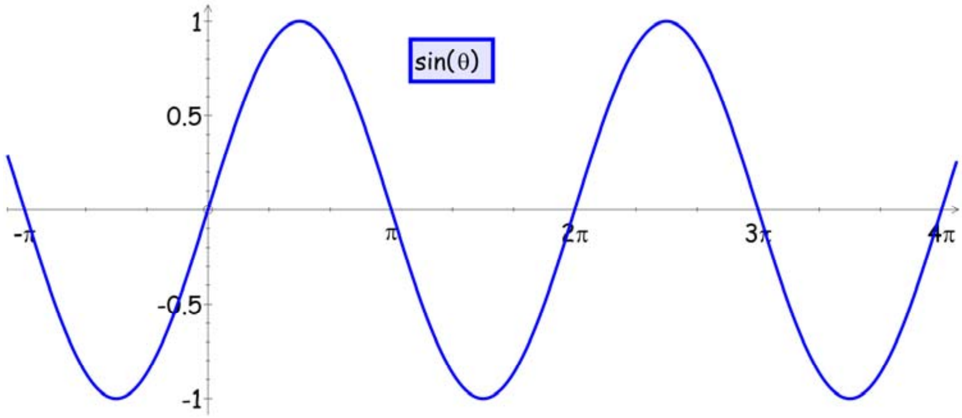
16. Need to find *t* values at the points where *y* is equal to zero.



The exponential function can never be zero therefore we must have the following:

$$10e^{-t}\sin(10t)=0 \quad \text{implies that } \sin(10t)=0$$

The sine graph is zero at whole number multiples of π:



We have

$$\sin(10t)=0 \quad \text{gives } 10t=0,\ \pi,\ 2\pi,\ 3\pi,\ 4\pi,\ \cdots,\ n\pi$$

Dividing through by 10 yields

$$t=0,\ \frac{\pi}{10},\ \frac{2\pi}{10},\ \frac{3\pi}{10},\ \frac{4\pi}{10},\ \cdots,\ \frac{n\pi}{10}$$

$$=0,\ \frac{\pi}{10},\ \frac{\pi}{5},\ \frac{3\pi}{10},\ \frac{2\pi}{5},\ \cdots,\ \frac{n\pi}{10} \qquad \left[\text{Simplifying the fraction}\right]$$