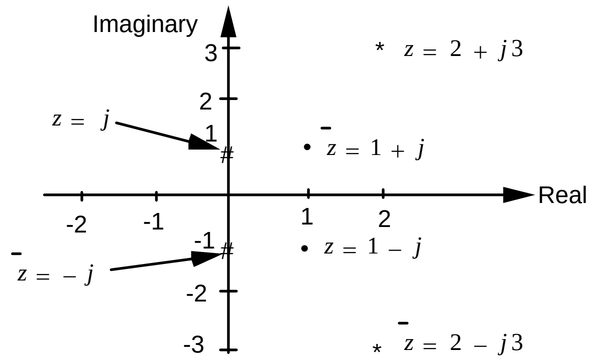
Complete solutions to Miscellaneous Exercise 10

1.



$\bar{z}$ is the reflection of z in the Real axis.

2. Writing each number into polar form.

(a) $1+j=\sqrt{2}\angle 45°$

$$(1+j)^{10}=\left(\sqrt{2}\angle 45°\right)^{10}$$

$$\underbrace{=}_{\text{by (10.19)}}\left(\sqrt{2}\right)^{10}\angle\left(10\times 45°\right)=32\angle 450°=j32$$

(b)

$$\left(-1-j\sqrt{3}\right)=2\angle\left(-120°\right)$$

$$\left(-1-j\sqrt{3}\right)^{7}=\left(2\angle\left(-120°\right)\right)^{7}$$

$$\underbrace{=}_{\text{by (10.19)}}2^{7}\angle\left(7\times\left(-120°\right)\right)$$

$$=2^{7}\angle\left(-840°\right)$$

$$=2^{7}\angle\left(-120°\right)=2^{6}\left(2\angle\left(-120°\right)\right)\underbrace{=}_{\text{from above}}2^{6}\left(-1-j\sqrt{3}\right)$$

(c)

$$\left(\cos(5\theta)+j\sin(5\theta)\right)\underbrace{=}_{by\ \text{(10.19)}}\left(\cos(\theta)+j\sin(\theta)\right)^{5}$$

$$\frac{\left(\cos(\theta)+j\sin(\theta)\right)^{5}}{\left(\cos(\theta)+j\sin(\theta)\right)}=\left(\cos(\theta)+j\sin(\theta)\right)^{4}\underbrace{=}_{by\ \text{(10.19)}}\cos(4\theta)+j\sin(4\theta)$$

3. (a) Equating real and imaginary parts of

$$x+jy=e^{-j\theta}\underbrace{=}_{\text{by (10.26)}}\cos(\theta)-j\sin(\theta)$$

gives

$$x=\cos(\theta)\quad\text{and}\quad y=-\sin(\theta)$$

(10.19) $\qquad z^{n}=r^{n}\angle n\theta$

(10.26) $\qquad e^{-j\theta}=\cos(\theta)-j\sin(\theta)$

(b) We have

$$\frac{1}{1+e^{j\theta}} = \frac{1}{1+\underbrace{\cos(\theta)+j\sin(\theta)}_{by\ (10.25)}} = \frac{1}{(1+\cos(\theta))+j\sin(\theta)}$$

$$\underbrace{=}_{\text{by (10.13)}} \frac{(1+\cos(\theta))-j\sin(\theta)}{(1+\cos(\theta))^2+\sin^2(\theta)} = \frac{(1+\cos(\theta))-j\sin(\theta)}{1+2\cos(\theta)+\underbrace{\cos^2(\theta)+\sin^2(\theta)}_{=1}}$$

$$x+jy = \frac{1+\cos(\theta)-j\sin(\theta)}{2(1+\cos(\theta))}$$

Equating real and imaginary parts gives

$$x = \frac{1+\cos(\theta)}{2(1+\cos(\theta))} = \frac{1}{2}, \quad y = -\frac{\sin(\theta)}{2(1+\cos(\theta))}$$

4. (a)

$$\sqrt{2}e^{j\pi/4} = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right)+j\sin\left(\frac{\pi}{4}\right)\right]$$

$$\underbrace{=}_{by\ TABLE\ 1} \sqrt{2}\left[\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}\right]$$

$$= \frac{\sqrt{2}}{\sqrt{2}}[1+j] = 1+j \qquad \left(\text{Cancelling } \sqrt{2}\text{'}s\right)$$

(b)

$$-e^{-j\pi/6} = -\left[\cos\left(\frac{\pi}{6}\right)-j\sin\left(\frac{\pi}{6}\right)\right]$$

$$\underbrace{=}_{\text{by TABLE 1}} -\left[\frac{\sqrt{3}}{2}-j\frac{1}{2}\right] = \frac{1}{2}\left[-\sqrt{3}+j\right]$$

5. Let $z = x+jy$ then $\bar{z} = x-jy$ (complex conjugate)

$$z\bar{z} = (x+jy)(x-jy) = x^2+y^2 = |x+jy|^2 = |z|^2$$

6. Putting each force into rectangular form via calculator or otherwise,

$$F_1 = 50\angle 30° = 43.30+j25$$

$$F_2 = 80\angle 60° = 40+j69.28$$

$$F_3 = 100\angle(-45°) = 70.71-j70.71$$

Adding the forces gives

$$F_1+F_2+F_3 = 154.01+j23.57$$

Also $155.8\angle 8.7° = 154.01+j23.57$ so we have

$$F = (F_1+F_2+F_3) - 155.8\angle 8.7° = 0$$

(10.13) $$\frac{a+jb}{c+jd} = \frac{(a+jb)(c-jd)}{c^2+d^2}$$

(10.19) $z^n = r^n\angle n\theta$

(10.25) $e^{j\theta} = \cos\theta + j\sin\theta$

7 .

$$I_R = \frac{240}{10+j10} = 12 - j12$$

$$I_Y = \frac{240\angle 120^\circ}{5+j8} = 11.94 + j22.46$$

$$I_B = \frac{240\angle 240^\circ}{2+j9} = -24.83 + j7.82$$

Adding these together gives

$$I_N = 12 - j12 + 11.94 + j22.46 - 24.83 + j7.82 = (-0.89 + j18.28)\ \text{A}$$

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8. We have

$$z = R + j\omega L + \frac{1}{j\omega C} = R + j\omega L - \frac{j}{\omega C} \quad \left(\text{because } \frac{1}{j} = -j\right)$$

$$= R + j\left(\omega L - \frac{1}{\omega C}\right)$$

Putting the imaginary part to zero

$$\text{Im}(z) = \omega L - \frac{1}{\omega C} = 0 \text{ gives } \omega^2 = \frac{1}{LC}$$

Hence square rooting both sides gives the required result, $\omega_r = \frac{1}{\sqrt{LC}}$

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9. Substituting $z_{oc} = 600e^{j\frac{\pi}{3}}$ and $z_{sc} = 400e^{-j\frac{\pi}{3}}$ into $\tanh(\gamma L) = \left(\frac{z_{sc}}{z_{oc}}\right)^{1/2}$ gives

$$\tanh(\gamma L) = \left(\frac{400e^{-j\frac{\pi}{3}}}{600e^{j\frac{\pi}{3}}}\right)^{1/2} \underbrace{=}_{\text{by (10.4)}} \left(\frac{2}{3}e^{-j\frac{\pi}{3}}e^{-j\frac{\pi}{3}}\right)^{1/2}$$

$$= \left(\frac{2}{3}e^{-j\frac{2\pi}{3}}\right)^{1/2}$$

$$\underbrace{=}_{\text{by (10.2)}} \left(\frac{2}{3}\right)^{1/2} e^{-j\frac{\pi}{3}}$$

$$= \sqrt{\frac{2}{3}}\underbrace{\left(\cos\left(\frac{\pi}{3}\right) - j\sin\left(\frac{\pi}{3}\right)\right)}_{(10.26)}$$

$$\underbrace{=}_{\text{using TABLE 1}} \frac{\sqrt{2}}{\sqrt{3}}\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{2}}{2\sqrt{3}}\left(1 - j\sqrt{3}\right) = \frac{1}{\sqrt{2}\sqrt{3}}\left(1 - j\sqrt{3}\right) = \frac{1}{\sqrt{6}}\left(1 - j\sqrt{3}\right)$$

(10.18) $\quad \frac{r\angle A}{q\angle B} = \frac{r}{q}\angle(A - B)$

(10.26) $\quad e^{-j\theta} = \cos(\theta) - j\sin(\theta)$

10. We solve the quadratic equation $s^2+4s+K=0$ by using the quadratic formula, (1.16):

$$s=\frac{-4\pm\sqrt{16-4K}}{2}=-2\pm\frac{\sqrt{4(4-K)}}{2}=-2\pm\sqrt{4-K}$$
$$=-2\pm\sqrt{(-1)(K-4)}$$
$$=-2\pm j\sqrt{(K-4)}$$

Equating real and imaginary parts of $\sigma\pm j\omega=-2\pm j\sqrt{K-4}$ gives

$$\sigma=-2 \text{ and } \omega=\sqrt{K-4}$$

Thus we have

$$\omega_n{}^2=\sigma^2+\omega^2=(-2)^2+\left(\sqrt{K-4}\right)^2=4+K-4=K$$

Thus $K=5^2=25$.

11. Substituting $\omega=10$ yields

$$\frac{\theta_0}{\theta_1}=\frac{1000}{200+j20+j^2100}$$
$$=\frac{1000}{100+j20}\underbrace{=}_{\substack{\text{dividing numerator}\\\text{and denominator}\\\text{by 20}}}\frac{50}{5+j}=\frac{50\angle 0°}{5.10\angle 11.31°}$$
$$\underbrace{=}_{\text{by (10.18)}}9.81\angle(-11.31°)$$

Gain $=9.81$ and phase $=-11.31°$

12. (a) We solve the quadratic equation $3z^2+2z+1=0$ by using the quadratic formula, (1.16):

$$z=\frac{-2\pm\sqrt{4-12}}{6}$$
$$=\frac{-1}{3}\pm\frac{\sqrt{-8}}{6}=\frac{-1}{3}\pm\frac{\sqrt{4\times(-2)}}{6}=-\frac{1}{3}\pm j\frac{\sqrt{2}}{3}$$

Evaluating the modulus

$$|z|=\sqrt{\frac{1}{9}+\frac{2}{9}}<1$$

The poles lie within the unit circle so the system is stable.

(b)

$$z^3+1=0 \text{ which gives } z^3=-1 \text{ so } z=-1$$

Thus $|z|=1$. A pole lies on the unit circle so the system is critically stable.

(1.16) $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

(10.18) $$\frac{r\angle A}{q\angle B}=\frac{r}{q}\angle(A-B)$$

13.(i) Expanding the denominator gives

$$(s+1+j)(s+1-j)+10 = s^2+2s+2+10$$

We have $s^2+2s+12=0$, solving this by applying (1.16)

$$s = \frac{-2\pm\sqrt{4-48}}{2} = -1\pm\frac{\sqrt{-44}}{2} = -1\pm j\sqrt{11}$$

Equating real and imaginary parts

$$\sigma = -1 \text{ and } \omega = \sqrt{11}$$

(ii)

$$\omega_n = \sqrt{(-1)^2+11} = \sqrt{12} \text{ rad/s}$$

(iii) $\zeta = \frac{|-1|}{\sqrt{12}} = 0.29$

(iv) $t_r = \frac{\pi}{2\sqrt{12}} = 0.45\ s$

14.(i) Evaluating the gain and phase for each ω by substituting the given values of ω:

$$\omega = 1;\quad T = \frac{10}{10+j} = \frac{10\angle 0^\circ}{10.05\angle 5.71^\circ} \underset{(10.18)}{=} 0.995\angle(-5.71^\circ)$$

Gain $= 0.995$ and phase $= -5.71^\circ$

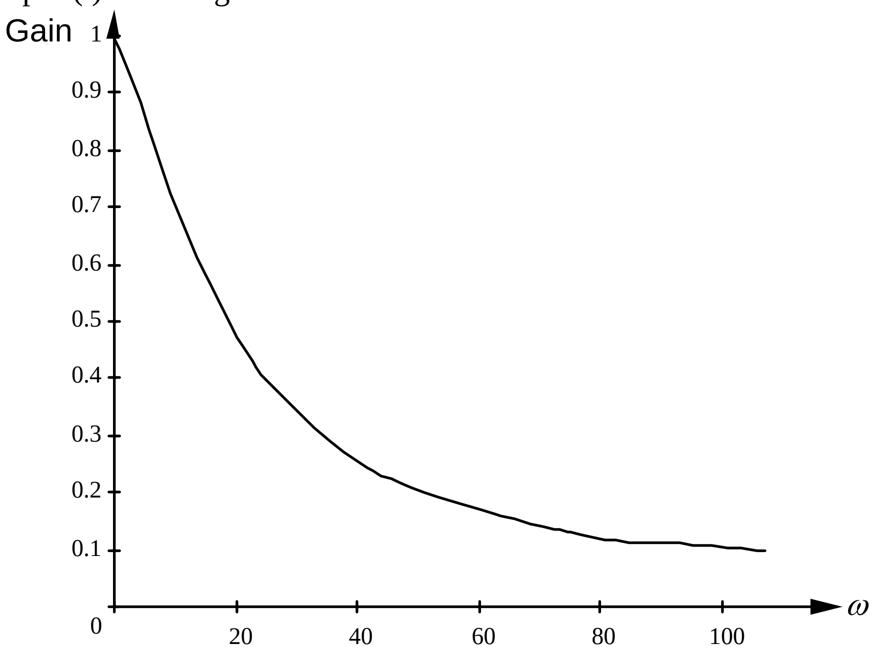
$$\omega = 10;\quad T = \frac{10}{10+j10} = 0.707\angle(-45^\circ)$$

$|T| = 0.707,\quad \arg(T) = -45^\circ$

$$\omega = 100;\quad T = \frac{10}{10+j100} = 0.0995\angle(-84.289^\circ)$$

$|T| = 0.0995,\quad \arg(T) = -84.289^\circ$

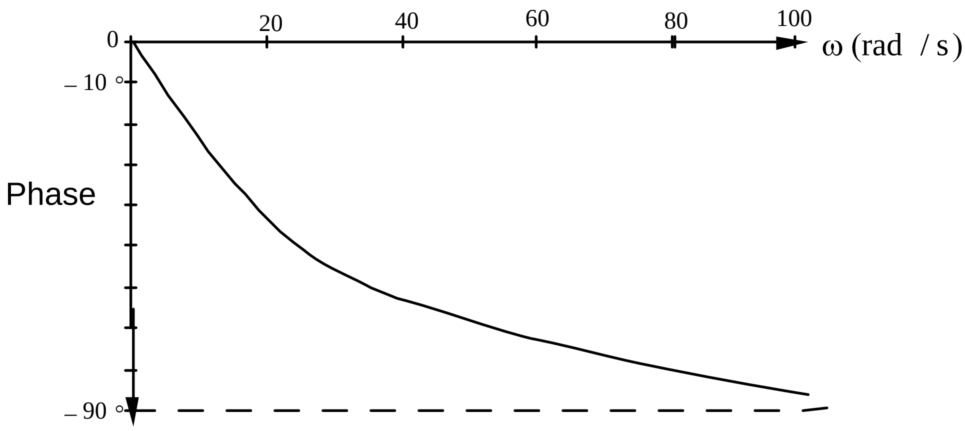
(ii) Notice from part(i) that the gain decreases as ω increases:



The phase increases as ω increases:

(1.16)

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$



15. Multiplying the numerator and denominator by X_c gives

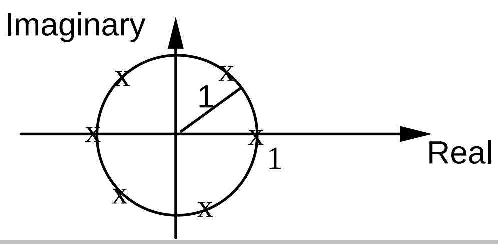
$$A = \frac{X_c - R}{X_c + R} = \frac{\frac{1}{j\omega C} - R}{\frac{1}{j\omega C} + R} = \frac{1 - j\omega CR}{1 + j\omega CR}$$

$$|A| = \frac{|1 - j\omega CR|}{|1 + j\omega CR|} \underbrace{=}_{\text{using (10.15)}} \frac{\sqrt{1 + (\omega CR)^2}}{\sqrt{1 + (\omega CR)^2}} = 1$$

16. The equation $1 - s^6 = 0$ gives $s^6 = 1$. One root is $1 = 1\angle 0°$, for the other roots, add $\left(\frac{360}{6}\right)^{\circ} = 60°$ to each angle:

$s_1 = 1\angle 0°$, $s_2 = 1\angle 60°$, $s_3 = 1\angle 120°$, $s_4 = 1\angle 180°$, $s_5 = 1\angle 240°$ and $s_6 = 1\angle 300°$

All the roots lie on the unit circle.



17. Multiplying the numerator and denominator of z by r:

$$z = \frac{jp\omega r}{1 + jkr} \underbrace{=}_{\text{by (10.13)}} \frac{jp\omega r(1 - jkr)}{1 + k^2r^2}$$

$$= \frac{jp\omega r - j^2 p\omega kr^2}{1 + k^2r^2}$$

$$= \frac{p\omega kr^2}{1 + k^2r^2} + j\frac{p\omega r}{1 + k^2r^2}$$

The real and imaginary parts of z are given by

(10.13) $$\frac{a + jb}{c + jd} = \frac{(a + jb)(c - jd)}{c^2 + d^2}$$

(10.15) $$|a + jb| = \sqrt{a^2 + b^2}$$

$$\text{Re}(z)=\frac{p\varpi kr^2}{1+k^2r^2},\quad \text{Im}(z)=\frac{p\varpi r}{1+k^2r^2}$$

18. By substituting $z = x + jy$ we have

$$w=ø+j\psi=3\left(x+jy\right)e^{j\frac{\pi}{3}}$$

$$=3\left(x+jy\right)\underbrace{\left[\cos\left(\frac{\pi}{3}\right)+j\sin\left(\frac{\pi}{3}\right)\right]}_{\text{by (10.25)}}$$

$$\underbrace{=}_{\text{using TABLE 1}} 3\left(x+jy\right)\left[\frac{1}{2}+j\frac{\sqrt{3}}{2}\right]$$

$$ø+j\psi=\frac{3}{2}\left(x+jx\sqrt{3}+jy\underbrace{=}_{\text{because } j^2=-1}\sqrt{3}y\right)$$

Equating real and imaginary parts gives

$$ø=\frac{3}{2}\left(x-\sqrt{3}y\right) \text{ and } \psi=\frac{3}{2}\left(\sqrt{3}x+y\right)$$

19. We apply the binomial theorem (2.6)

$$\left(\cos\left(3\theta\right)+j\sin\left(3\theta\right)\right)=\left(\cos\left(\theta\right)+j\sin\left(\theta\right)\right)^3$$

$$\underbrace{=}_{\text{by (2.6)}} \cos^3\left(\theta\right)+3\cos^2\left(\theta\right)j\sin\left(\theta\right)+3\cos\left(\theta\right)\left(j\sin\left(\theta\right)\right)^2+\left(j\sin\left(\theta\right)\right)^3$$

$$=\cos^3\left(\theta\right)+3\cos^2\left(\theta\right)j\sin\left(\theta\right)-3\cos\left(\theta\right)\sin^2\left(\theta\right)-j\sin^3\left(\theta\right)$$

$$=\cos^3\left(\theta\right)-3\cos\left(\theta\right)\sin^2\left(\theta\right)+j\left[3\cos^2\left(\theta\right)\sin\left(\theta\right)-\sin^3\left(\theta\right)\right]$$

Equating real parts gives

$$\cos\left(3\theta\right)=\cos^3\left(\theta\right)-3\cos\left(\theta\right)\sin^2\left(\theta\right)$$

$$=\cos^3\left(\theta\right)-3\cos\left(\theta\right)\left(1-\cos^2\left(\theta\right)\right)$$

$$=4\cos^3\left(\theta\right)-3\cos\left(\theta\right)$$

Equating imaginary parts gives

$$\sin\left(3\theta\right)=3\cos^2\left(\theta\right)\sin\left(\theta\right)-\sin^3\left(\theta\right)$$

$$=3\left(1-\sin^2\left(\theta\right)\right)\sin\left(\theta\right)-\sin^3\left(\theta\right)=3\sin\left(\theta\right)-4\sin^3\left(\theta\right)$$

20. (i) We have

$$z-\frac{1}{z}=z-z^{-1}$$

$$=\left(\cos\left(\theta\right)+j\sin\left(\theta\right)\right)-\underbrace{\left(\cos\left(\theta\right)-j\sin\left(\theta\right)\right)}_{\text{by (10.21)}}$$

$$=2j\sin\left(\theta\right)$$

(2.6) $\left(a+b\right)^n=C_na^n+C_{n-1}a^{n-1}b+C_{n-2}a^{n-2}b^2+\ldots+C_0b^n$

(10.19) $z^n=r^n\left(\cos\left(n\theta\right)+j\sin\left(n\theta\right)\right)$

(10.21) $z^{-n}=r^{-n}\left(\cos\left(n\theta\right)-j\sin\left(n\theta\right)\right)$

(ii) We have

$$z^3-\frac{1}{z^3}=z^3-z^{-3}$$

$$=\underbrace{\cos(3\theta)+j\sin(3\theta)}_{\text{by (10.19)}}-\underbrace{(\cos(3\theta)-j\sin(3\theta))}_{\text{by (10.21)}}$$

$$=2j\sin(3\theta)$$

(iii) Similar to part(ii) $\left(z^5-\frac{1}{z^5}\right)=2j\sin(5\theta)$

(iv) Applying the binomial theorem, (2.9), gives

$$\left(z-\frac{1}{z}\right)^5=z^5-5z^4\frac{1}{z}+10z^3\frac{1}{z^2}-10z^2\frac{1}{z^3}+5z\frac{1}{z^4}-\frac{1}{z^5}$$

$$\underbrace{=}_{\text{rearranging}}z^5-\frac{1}{z^5}-5z^3+5\frac{1}{z^3}+10z-10\frac{1}{z}$$

$$\left(z-\frac{1}{z}\right)^5=\left(z^5-\frac{1}{z^5}\right)-5\left(z^3-\frac{1}{z^3}\right)+10\left(z-\frac{1}{z}\right) \qquad (*)$$

(v) Substituting the Right Hand Side of parts (i) , (ii) and (iii) for the corresponding z's into (*) gives

$$(2j\sin(\theta))^5=(2j\sin(5\theta))-5(2j\sin(3\theta))+10(2j\sin(\theta))$$

$$(2j)^5\sin^5(\theta)=2j[\sin(5\theta)-5\sin(3\theta)+10\sin(\theta)]$$

$$\sin^5(\theta)=\underbrace{\frac{2j}{32j}}_{\substack{\text{because}\\(2j)^5=32j}}[\sin(5\theta)-5\sin(3\theta)+10\sin(\theta)]$$

$$=\frac{1}{16}[\sin(5\theta)-5\sin(3\theta)+10\sin(\theta)]$$

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21. Let $c=\cos(\theta)$ and $s=\sin(\theta)$, then by the Binomial theorem (2.6) we have

$$\cos(4\theta)+j\sin(4\theta)=(c+js)^4=c^4+4c^3(js)+6c^2(js)^2+4c(js)^3+(js)^4$$

$$=c^4+j4c^3s-6c^2s^2-j4cs^3+s^4$$

Equating real parts gives

$$\cos(4\theta)=\cos^4(\theta)-6\cos^2(\theta)\sin^2(\theta)+\sin^4(\theta)$$

Equating imaginary parts gives

$$\sin(4\theta)=4\cos^3(\theta)\sin(\theta)-4\cos(\theta)\sin^3(\theta)$$

(2.6) $(a+b)^n=C_na^n+C_{n-1}a^{n-1}b+C_{n-2}a^{n-2}b^2+\ldots+C_0b^n$

(10.19) $z^n=r^n(\cos(n\theta)+j\sin(n\theta))$

(10.21) $z^{-n}=r^{-n}(\cos(n\theta)-j\sin(n\theta))$