Complete solutions to Miscellaneous Exercise 15

1. We have

$$M = Kx + Px\left(1 - \frac{x}{L}\right)$$

$$\frac{\partial M}{\partial P} = x\left(1 - \frac{x}{L}\right) = x - \frac{x^2}{L}$$

$$\frac{\partial M}{\partial x} = K + P\left(1 - \frac{2x}{L}\right) = K + \frac{P}{L}(L - 2x)$$

2. We are given:

$$M = \frac{Kx}{L} + Pa\left(1 - \frac{x}{L}\right) - P(a - x)$$

$$\frac{\partial M}{\partial P} = a\left(1 - \frac{x}{L}\right) - (a - x) = a - \frac{ax}{L} - a + x = x - \frac{ax}{L}$$

$$\frac{\partial M}{\partial x} = \frac{K}{L} - \frac{Pa}{L} + P = \frac{K - Pa}{L} + P$$

3. We have

$$\Omega(x, y) = Ax^3y^2 + Bxy^2$$

$$\frac{\partial\Omega}{\partial x} = 3Ax^2y^2 + By^2,\ \frac{\partial^2\Omega}{\partial x^2} = 6Axy^2,\ \frac{\partial^3\Omega}{\partial x^3} = 6Ay^2,\ \frac{\partial^4\Omega}{\partial x^4} = 0$$

To find $\frac{\partial^4\Omega}{\partial y^4}$ we need

$$\frac{\partial\Omega}{\partial y} = 2Ax^3y + 2Bxy$$

$$\frac{\partial^2\Omega}{\partial y^2} = 2Ax^3 + 2Bx \qquad (*)$$

$$\frac{\partial^3\Omega}{\partial y^3} = 0 = \frac{\partial^4\Omega}{\partial y^4}$$

Also the mixed partial derivative

$$\frac{\partial^4\Omega}{\partial x^2\partial y^2} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\Omega}{\partial y^2}\right) = \frac{\partial^2}{\partial x^2}\underbrace{\left(2Ax^3 + 2Bx\right)}_{\text{by }(*)} = \frac{\partial}{\partial x}\left(6Ax^2 + 2B\right) = 12Ax$$

Substituting into $\frac{\partial^4\Omega}{\partial x^4} + 2\frac{\partial^4\Omega}{\partial x^2\partial y^2} + \frac{\partial^4\Omega}{\partial y^4} = \nabla^4\Omega$ gives

$$0 + 2(12Ax) + 0 = 24,\quad Ax \neq 0 \text{ unless } x = 0$$

Hence for $x \neq 0$, Ω is **not** an Airy stress function.

4. We have

$$\Delta P \approx \frac{\partial P}{\partial T}\Delta T + \frac{\partial P}{\partial V}\Delta V \qquad (*)$$

We are given $P = \frac{RT}{V}$. Finding the partial derivatives

$$\frac{\partial P}{\partial T} = \frac{R}{V} \quad \text{and} \quad \frac{\partial P}{\partial V} = -\frac{RT}{V^2}$$

The given percentage change can be written as

$$\Delta T = -0.013T \quad \text{and} \quad \Delta V = 0.002V$$

Substituting into (*) gives

$$\Delta P \approx \frac{R}{V}\left(-0.013T\right) - \frac{RT}{V^2}\left(0.002V\right)$$

$$= \frac{RT}{V}\left(-0.013 - 0.002\right) = P\left(-0.015\right)$$

The approximate change in P is a 1.5% decrease.

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5. By (15.8) we have

$$\Delta R \approx \frac{\partial R}{\partial L}\Delta L + \frac{\partial R}{\partial d}\Delta d \qquad (*)$$

$$R = \frac{cL}{d^2} = cLd^{-2}$$

$$\frac{\partial R}{\partial L} = \frac{c}{d^2} \qquad \frac{\partial R}{\partial d} = -\frac{2cL}{d^3}$$

$$\Delta L = \pm 0.03L, \quad \Delta d = \pm 0.02d$$

Substituting into (*)

$$\Delta R \approx \frac{c}{d^2}\left(\pm 0.03L\right) + \left(-\frac{2cL}{d^3}\right)\left(\pm 0.02d\right)$$

$$= \frac{cL}{d^2}\left(\pm 0.03\right) - \frac{cL}{d^2}\left(\pm 0.04\right)$$

$$= \frac{cL}{d^2}\left[\pm 0.03 - \left(\pm 0.04\right)\right] = R\left(0.07\right)$$

The largest error is 7%.

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6. By (15.8) we have

$$\Delta \sigma \approx \frac{\partial \sigma}{\partial r}\Delta r + \frac{\partial \sigma}{\partial l}\Delta l \qquad (*)$$

We are given

$$\sigma = \frac{kr^2}{l^2} = kr^2 l^{-2}$$

$$\frac{\partial \sigma}{\partial r} = \frac{2kr}{l^2} \quad \text{and} \quad \frac{\partial \sigma}{\partial l} = -\frac{2kr^2}{l^3}$$

From the question we have

$$\Delta r = 0.013r, \qquad \Delta l = -0.006l$$

Substituting into (*) gives

$$\Delta \sigma \approx \frac{2kr}{l^2}\left(0.013r\right) - \frac{2kr^2}{l^3}\left(-0.006l\right)$$

$$= \frac{2kr^2}{l^2}\left(0.013\right) + \frac{2kr^2}{l^2}\left(0.006\right)$$

$$= \frac{2kr^2}{l^2}\left(0.013 + 0.006\right) = 2\sigma\left(0.019\right) = 0.038\sigma$$

The approximate change in σ is an increase of 3.8%.

7. Rearranging V gives

$$V = V_0\left(1+\frac{r^3}{x^3}\right) = V_0 + V_0 r^3 x^{-3}$$

$$\frac{\partial V}{\partial x} = -3V_0 r^3 x^{-4} = -\frac{3V_0 r^3}{x^4}$$

Substituting these into a gives

$$a = V\frac{\partial V}{\partial x} = V_0\left(1+\frac{r^3}{x^3}\right)\left(-\frac{3V_0 r^3}{x^4}\right)$$

$$= -3V_0^2 r^3\left(1+\frac{r^3}{x^3}\right)\left(\frac{1}{x^4}\right)$$

For $a = 0$ we have

$$1+\frac{r^3}{x^3} = 0 \text{ which gives } x = -r$$

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8. We have

$$P = \frac{SA}{\alpha}\left[\frac{\sinh(\mu x)}{\mu\cosh(\mu L)} - x\right]$$

$$\frac{\partial P}{\partial x} = \frac{SA}{\alpha}\left[\frac{\mu\cosh(\mu x)}{\mu\cosh(\mu L)} - 1\right] = -\frac{SA}{\alpha}\left[1-\frac{\cosh(\mu x)}{\cosh(\mu L)}\right]$$

Substituting this into q gives

$$q = -\frac{1}{2}\frac{\partial P}{\partial x} = \frac{SA}{2\alpha}\left[1-\frac{\cosh(\mu x)}{\cosh(\mu L)}\right]$$

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9. Substituting $x = 0$ into $P = A\cosh(\mu x) + B\sinh(\mu x) + \alpha$ gives

$$P = A\cosh(0) + B\sinh(0) + \alpha$$

Putting $P = 0$ gives

$$0 = A + \alpha, \text{ hence } A = -\alpha$$

Also

$$\frac{\partial P}{\partial x} = \mu A\sinh(\mu x) + \mu B\cosh(\mu x)$$

Substituting $x = L$ and $\frac{\partial P}{\partial x} = 0$ gives

$$0 = \mu A\sinh(\mu L) + \mu B\cosh(\mu L)$$

From above we have $A = -\alpha$ so

$$\mu\alpha\sinh(\mu L) = \mu B\cosh(\mu L)$$

$$B = \frac{\alpha\sinh(\mu L)}{\cosh(\mu L)} = \alpha\tanh(\mu L)$$

Substituting $A = -\alpha$ and $B = \alpha\tanh(\mu L)$ into $P = A\cosh(\mu x) + B\sinh(\mu x) + \alpha$ yields

$$P = -\alpha\cosh(\mu x) + \alpha\tanh(\mu L)\sinh(\mu x) + \alpha$$

$$= \alpha\left[\tanh(\mu L)\sinh(\mu x) - \cosh(\mu x) + 1\right]$$

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10. We have

$$\Omega = r^2\left[A\cos(2\theta) + B\sin(2\theta)\right]$$
$$\frac{\partial\Omega}{\partial r} = 2r\left[A\cos(2\theta) + B\sin(2\theta)\right]$$

$$\frac{\partial^2\Omega}{\partial r^2} = 2\left[A\cos(2\theta) + B\sin(2\theta)\right]$$
$$\frac{\partial\Omega}{\partial\theta} = r^2\left[-2A\sin(2\theta) + 2B\cos(2\theta)\right]$$
$$\frac{\partial^2\Omega}{\partial\theta^2} = r^2\left[-4A\cos(2\theta) - 4B\sin(2\theta)\right] = -4r^2\left[A\cos(2\theta) + B\sin(2\theta)\right]$$

Also

$$\frac{\partial^2\Omega}{\partial r\partial\theta} = \frac{\partial}{\partial r}\left(\frac{\partial\Omega}{\partial\theta}\right)$$
$$= \frac{\partial}{\partial r}\left[r^2\left[-2A\sin(2\theta) + 2B\cos(2\theta)\right]\right]$$
$$= 2r\left[-2A\sin(2\theta) + 2B\cos(2\theta)\right]$$

Substituting the above into

$$\sigma_r = \frac{1}{r}\left(\frac{\partial\Omega}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2\Omega}{\partial\theta^2}\right)$$

gives

$$\sigma_r = \frac{1}{r}\left[2r\left[A\cos(2\theta) + B\sin(2\theta)\right]\right] + \frac{1}{r^2}\left[-4r^2\left[A\cos(2\theta) + B\sin(2\theta)\right]\right]$$
$$= 2A\cos(2\theta) + 2B\sin(2\theta) - 4A\cos(2\theta) - 4B\sin(2\theta)$$
$$\sigma_r = -2A\cos(2\theta) - 2B\sin(2\theta)$$

Substituting the above into

$$\sigma_\theta = \frac{\partial^2\Omega}{\partial r^2}$$

gives

$$\sigma_\theta = 2A\cos(2\theta) + 2B\sin(2\theta)$$

Substituting the above into

$$\tau_{r\theta} = \frac{1}{r^2}\left(\frac{\partial\Omega}{\partial\theta}\right) - \frac{1}{r}\left(\frac{\partial^2\Omega}{\partial r\partial\theta}\right)$$

gives

$$\tau_{r\theta} = \frac{1}{r^2}\left[r^2\left[-2A\sin(2\theta) + 2B\cos(2\theta)\right]\right] - \frac{1}{r}\left[2r\left[-2A\sin(2\theta) + 2B\cos(2\theta)\right]\right]$$
$$= -2A\sin(2\theta) + 2B\cos(2\theta) - 2\left[-2A\sin(2\theta) + 2B\cos(2\theta)\right]$$
$$= -2A\sin(2\theta) + 2B\cos(2\theta) + 4A\sin(2\theta) - 4B\cos(2\theta)$$
$$\tau_{r\theta} = 2A\sin(2\theta) - 2B\cos(2\theta)$$

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11. (a) We have

$$u(x,t) = e^{-kt}\sin(x)$$
$$\frac{\partial u}{\partial t} = -ke^{-kt}\sin(x)$$
$$\frac{\partial u}{\partial x} = e^{-kt}\cos(x)$$
$$\frac{\partial^2 u}{\partial x^2} = -e^{-kt}\sin(x)$$

Hence $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ so u satisfies the heat equation.

(b) We have

$$u(x,t) = e^{-kt}\cos(x)$$
$$\frac{\partial u}{\partial t} = -ke^{-kt}\cos(x)$$
$$\frac{\partial u}{\partial x} = -e^{-kt}\sin(x)$$
$$\frac{\partial^2 u}{\partial x^2} = -e^{-kt}\cos(x)$$

Hence

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

(c) We have

$$u(x,t) = e^{-kt}\left[\cos(x)+\sin(x)\right]$$
$$= e^{-kt}\cos(x) + e^{-kt}\sin(x)$$

Result follows from (a) + (b).

(d) We have

$$u(x,t) = e^{-\eta^2 kt}\left[\cos(\eta x)+\sin(\eta x)\right]$$
$$\frac{\partial u}{\partial t} = -\eta^2 ke^{-\eta^2 kt}\left[\cos(\eta x)+\sin(\eta x)\right]$$
$$\frac{\partial u}{\partial x} = e^{-\eta^2 kt}\left[-\eta\sin(\eta x)+\eta\cos(\eta x)\right]$$
$$\frac{\partial^2 u}{\partial x^2} = e^{-\eta^2 kt}\left[-\eta^2\cos(\eta x)-\eta^2\sin(\eta x)\right]$$
$$= -\eta^2 e^{-\eta^2 kt}\left[\cos(\eta x)+\sin(\eta x)\right]$$

Hence $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

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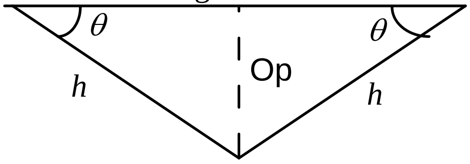
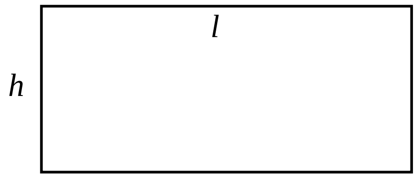
12. Consider the triangle





From (4.6)

$$Op = x\tan(\theta)$$

By (4.8) we have $h = \dfrac{x}{\cos(\theta)} = x\sec(\theta)$

$$\text{Area of triangle } = x\left[x\tan(\theta)\right] = x^2\tan(\theta)$$

$$\text{Area of the rectangle } = lx\sec(\theta)$$

Total surface area

$$A = (\text{Area of two triangles}) + (\text{Area of two rectangles})$$

$$A = 2x^2\tan(\theta) + 2lx\sec(\theta)$$

The volume is $1m^3$ so we have

$$1 = (\text{Area of triangle}) \times (l)$$

$$1 = x^2 l\tan(\theta)$$

$$l = \frac{1}{x^2\tan(\theta)} \qquad (\dagger)$$

Substituting for l into the equation for A:

$$A = 2x^2\tan(\theta) + \frac{2\sec(\theta)}{x\tan(\theta)} = 2x^2\tan(\theta) + \frac{2}{x\sin(\theta)}$$

For minimum surface area

$$\frac{\partial A}{\partial x} = 4x\tan(\theta) - \frac{2}{x^2\sin(\theta)}$$

$$\frac{\partial A}{\partial \theta} = 2x^2\sec^2(\theta) - \frac{2\cos(\theta)}{x\sin^2(\theta)}$$

We equate these to zero. From the first equation we have

$$4x\tan(\theta) - \frac{2}{x^2\sin(\theta)} = 0$$

$$x^3 = \frac{1}{2\sin(\theta)\tan(\theta)} = \frac{1}{2\sin(\theta)\left[\dfrac{\sin(\theta)}{\cos(\theta)}\right]} = \frac{\cos(\theta)}{2\sin^2(\theta)} \qquad (*)$$

(4.6) $\qquad \text{Opp} = \text{adj} \times \tan(\theta)$

(4.8) $\qquad \text{hyp} = \dfrac{\text{adj}}{\cos(\theta)}$

From the second equation we have

$$2x^2\sec^2(\theta)-\frac{2\cos(\theta)}{x\sin^2(\theta)}=0$$

$$x^3=\frac{\cos(\theta)}{\sin^2(\theta)\sec^2(\theta)}=\frac{\cos^3(\theta)}{\sin^2(\theta)} \qquad (**)$$

Equating (*) and (**) gives

$$\cos^2(\theta)=\frac{1}{2}, \quad \cos(\theta)=\frac{1}{\sqrt{2}}$$

$$\theta=\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=45°$$

Substituting into (*)

$$x^3=\left(\frac{1}{\sqrt{2}}\right)\div\left(2\left(\frac{1}{\sqrt{2}}\right)^2\right)=\frac{1}{\sqrt{2}}$$

$$x=\left(2^{-1/2}\right)^{1/3}=2^{-1/6}=0.891\text{ m}$$

Substituting in $l=\dfrac{1}{x^2\tan(\theta)}$ gives $l=\dfrac{1}{2^{-2/6}\tan(45°)}=2^{1/3}=1.260$ m

Check that these values give minimum by using MAPLE or any other symbolic manipulator.

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13. We have

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> P:=(R*T)/(V-b)*exp(-a/(R*V*T));
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$$P:=\frac{R\,T\,e^{\left(-\frac{a}{R\,V\,T}\right)}}{V-b}$$

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> f_eqn:=diff(P,V);
```

$$f\_eqn:=-\frac{R\,T\,e^{\left(-\frac{a}{R\,V\,T}\right)}}{(V-b)^2}+\frac{a\,e^{\left(-\frac{a}{R\,V\,T}\right)}}{(V-b)\,V^2}$$

```
> s_eqn:=diff(P,V,V);
```

$$s\_eqn:=2\frac{R\,T\,e^{\left(-\frac{a}{R\,V\,T}\right)}}{(V-b)^3}-\frac{2\,a\,e^{\left(-\frac{a}{R\,V\,T}\right)}}{(V-b)^2\,V^2}-\frac{2\,a\,e^{\left(-\frac{a}{R\,V\,T}\right)}}{(V-b)\,V^3}+\frac{a^2\,e^{\left(-\frac{a}{R\,V\,T}\right)}}{(V-b)\,V^4\,R\,T}$$

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> rt:=solve({f_eqn=0,s_eqn=0},{a,b});
```

$$rt:=\{b=\frac{1}{2}V,\ a=2\,R\,V\,T\}$$

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14. We have

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> P:=R*T/(V-b)-a/(V*(V+b)*T^0.5);
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$$P:=\frac{R\,T}{V-b}-\frac{a}{V\,(V+b)\,T^{0.5}}$$

```
> f_eqn:=diff(P,V);
```

$$f\_eqn := -\frac{R\,T}{(V-b)^2} + \frac{a}{V^2\,(V+b)\,T^{0.5}} + \frac{a}{V\,(V+b)^2\,T^{0.5}}$$

```
> s_eqn:=diff(P,V,V);
```

$$s\_eqn := 2\,\frac{R\,T}{(V-b)^3} - \frac{2\,a}{V^3\,(V+b)\,T^{0.5}} - \frac{2\,a}{V^2\,(V+b)^2\,T^{0.5}} - \frac{2\,a}{V\,(V+b)^3\,T^{0.5}}$$

```
> rt:={solve({f_eqn=0,s_eqn=0},{a,b})};
```

$$rt := \{\{a = (-0.1412203503 + 0.09453533367\,I)\,R\,T^{(3/2)}\,V,$$
$$b = (-1.629960525 - 1.091123636\,I)\,V\}, \{b = (-1.629960525 + 1.091123636\,I)\,V,$$
$$a = (-0.1412203503 - 0.09453533367\,I)\,R\,T^{(3/2)}\,V\},$$
$$\{a = 1.282440701\,R\,T^{(3/2)}\,V, b = 0.2599210499\,V\}\}$$

```
> %[3];
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$$\{a = 1.282440701\,R\,T^{(3/2)}\,V, b = 0.2599210499\,V\}$$