Complete solutions to Exercise 11(b)

1. Use (11.1) to find the determinants.

(a) $\det\begin{pmatrix}1 & 3\\ 5 & 7\end{pmatrix} = (1\times 7)-(5\times 3) = -8$ $\quad\det\begin{pmatrix}1 & 5\\ 3 & 7\end{pmatrix} = (1\times 7)-(3\times 5) = -8$

(b) $\det\begin{pmatrix}-1 & 2\\ 5 & 3\end{pmatrix} = (-1\times 3)-(5\times 2) = -13$ $\quad\det\begin{pmatrix}-1 & 5\\ 2 & 3\end{pmatrix} = (-1\times 3)-(2\times 5) = -13$

(c) $\mathbf{A} = \begin{pmatrix}-1 & -1\\ 0 & 0\end{pmatrix}$, $\mathbf{B} = \begin{pmatrix}-1 & 0\\ -1 & 0\end{pmatrix}$, $\det\mathbf{A} = \det\mathbf{B} = 0$

The matrix **A** is transposed (rows$\rightarrow$ columns) to give matrix **B**. The same numbers on each of the diagonals, so the determinant is the same, $\det\mathbf{A} = \det\mathbf{B}$.

2. By (11.1)

$$\det\begin{pmatrix}a & b\\ c & d\end{pmatrix} = ad-cb \qquad \det\begin{pmatrix}a & c\\ b & d\end{pmatrix} = ad-bc = ad-cb = \det\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

3. Use (11.1) to evaluate the determinants.

(a) (i) $\det\mathbf{A} = (1\times 6)-(5\times 3) = -9$ $\quad$ (ii) $\det\mathbf{B} = (3\times 5)-(-1\times 7) = 22$

(iii) $\det\mathbf{A}\times\det\mathbf{B} = -9\times 22 = -198$

(iv) $\mathbf{AB} = \begin{pmatrix}1 & 3\\ 5 & 6\end{pmatrix}\begin{pmatrix}3 & 7\\ -1 & 5\end{pmatrix} = \begin{pmatrix}0 & 22\\ 9 & 65\end{pmatrix}$

$$\det(\mathbf{AB}) = (0\times 65)-(9\times 22) = -198$$

(b) (i) $\det\mathbf{A} = (-1\times 1.5)-(10\times 170) = -1701.5$

(ii) $\det\mathbf{B} = [-30\times(-1.9)]-[-9.3\times 61] = 624.3$

(iii) $\det\mathbf{A}\times\det\mathbf{B} = -1701.5\times 624.3 = -1062246.45$

(iv) $\mathbf{AB} = \begin{pmatrix}-1 & 10\\ 170 & 1.5\end{pmatrix}\begin{pmatrix}-30 & -9.3\\ 61 & -1.9\end{pmatrix} = \begin{pmatrix}640 & -9.7\\ -5008.5 & -1583.85\end{pmatrix}$

$$\det(\mathbf{AB}) = [640\times(-1583.85)]-[(-9.7)\times(-5008.5)] = -1062246.45$$

(c) (i) $\det\mathbf{A} = [5\times(-5.6)]-[2.2\times(-3)] = -21.4$

(ii) $\det\mathbf{B} = [-7.1\times(-12.2)]-[-3.5\times(-2.1)] = 79.27$

(iii) $\det\mathbf{A}\times\det\mathbf{B} = -21.4\times 79.27 = -1696.378$

(iv) $\mathbf{AB} = \begin{pmatrix}5 & -3\\ 2.2 & -5.6\end{pmatrix}\begin{pmatrix}-7.1 & -2.1\\ -3.5 & -12.2\end{pmatrix} = \begin{pmatrix}-25 & 26.1\\ 3.98 & 63.7\end{pmatrix}$

$$\det(\mathbf{AB}) = (-25\times 63.7)-(3.98\times 26.1) = -1696.378$$

All the results satisfy $\det\mathbf{A}\det\mathbf{B} = \det(\mathbf{AB})$.

(11.1) $$\det\begin{pmatrix}a & b\\ c & d\end{pmatrix} = ad-cb$$

4. In each case $\det \mathbf{A} = 1$ so we use (11.3).
(a) Exchanging numbers 3 and 9 and placing a negative sign in front of the other numbers gives:

$$\mathbf{A}^{-1} = \begin{pmatrix} 3 & -2 \\ -13 & 9 \end{pmatrix}$$

(b) $\mathbf{A}^{-1} = \begin{pmatrix} 5 & -7 \\ -12 & 17 \end{pmatrix}$

(c) In this case $\mathbf{A}$ is the identity matrix, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and so $\mathbf{I}^{-1} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

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5. We first evaluate the determinant.
(a) By (11.1), $\det\begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix} = (5\times 1) - (3\times 4) = -7$. So using (11.4)

$$\mathbf{A}^{-1} = -\frac{1}{7}\begin{pmatrix} 1 & -4 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} -1/7 & 4/7 \\ 3/7 & -5/7 \end{pmatrix}$$

(b) We have $\det\begin{pmatrix} 3 & 6 \\ 7 & 8 \end{pmatrix} = (3\times 8) - (7\times 6) = 24 - 42 = -18$. Thus applying (11.4)

$$\mathbf{A}^{-1} = \begin{pmatrix} 3 & 6 \\ 7 & 8 \end{pmatrix}^{-1} = -\frac{1}{18}\begin{pmatrix} 8 & -6 \\ -7 & 3 \end{pmatrix} = \begin{pmatrix} -8/18 & 6/18 \\ 7/18 & -3/18 \end{pmatrix} = \begin{pmatrix} -4/9 & 1/3 \\ 7/18 & -1/6 \end{pmatrix}$$

(c) $\det \mathbf{A} = (7\times 2) - (14\times 1) = 0$. Since $\det \mathbf{A} = 0$, so $\mathbf{A}$ cannot have an inverse.

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6. Putting the equations into matrix form gives

$$\begin{pmatrix} 2 & 3 \\ 6 & 7 \end{pmatrix}\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

We need to find the inverse of $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 6 & 7 \end{pmatrix}$. By (11.4)

$$\begin{pmatrix} 2 & 3 \\ 6 & 7 \end{pmatrix}^{-1} = \frac{1}{(2\times 7) - (6\times 3)}\begin{pmatrix} 7 & -3 \\ -6 & 2 \end{pmatrix} = \frac{1}{-4}\begin{pmatrix} 7 & -3 \\ -6 & 2 \end{pmatrix}$$

Using (11.5) we have

$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} 7 & -3 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 10 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} 5 \\ -10 \end{pmatrix} = \begin{pmatrix} -5/4 \\ 10/4 \end{pmatrix}$$

$$i_1 = -1.25 \text{ A and } i_2 = 2.5 \text{ A}$$

(11.1)	$\det \mathbf{A} = (a\times d) - (b\times c)$
(11.4)	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
(11.5)	$\mathbf{u} = \mathbf{A}^{-1}\mathbf{b}$

7. These equations can be written as

$$30i_1 - 10i_2 = 12$$
$$-10i_1 + 35i_2 = 5$$

In matrix form we have

$$\begin{pmatrix} 30 & -10 \\ -10 & 35 \end{pmatrix}\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$$

Let $\mathbf{A} = \begin{pmatrix} 30 & -10 \\ -10 & 35 \end{pmatrix}$. Then by (11.4)

$$\begin{pmatrix} 30 & -10 \\ -10 & 35 \end{pmatrix}^{-1} = \frac{1}{(30\times 35)-(10\times 10)}\begin{pmatrix} 35 & 10 \\ 10 & 30 \end{pmatrix} = \frac{1}{950}\begin{pmatrix} 35 & 10 \\ 10 & 30 \end{pmatrix}$$

Using (11.5) gives

$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \frac{1}{950}\begin{pmatrix} 35 & 10 \\ 10 & 30 \end{pmatrix}\begin{pmatrix} 12 \\ 5 \end{pmatrix} = \frac{1}{950}\begin{pmatrix} 470 \\ 270 \end{pmatrix}$$

$$i_1 = \frac{470}{950} = \frac{47}{95}\text{ A and } \quad i_2 = \frac{270}{950} = \frac{27}{95}\text{ A}$$

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8. Similar to solutions 6 and 7.

(a) $x = 2$ and $y = 4$

(b) $x = -1$ and $y = 1$

(c) $x = \frac{1}{4}$ and $y = -\frac{1}{3}$

(11.1)	$\det \mathbf{A} = (a \times d) - (b \times c)$
(11.4)	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
(11.5)	$\mathbf{u} = \mathbf{A}^{-1}\mathbf{b}$