Complete solutions to Exercise 2(d)

1. (a) $x^2-4x+3=(x-2)^2-4+3$

$$=(x-2)^2-1$$

(b) *How do we complete the square on* x^2+8x+9 *?*
Consider half the *x* coefficient, that is half of 8:

$$x^2+8x+9=\left(x+4\right)^2-16+9$$
$$=\left(x+4\right)^2-7$$

(c) Similarly we have

$$x^2-6x+8=\left(x-3\right)^2-9+8$$
$$=\left(x-3\right)^2-1$$

(d) Completing the square on the given quadratic we have

$$x^2-10x+2=\left(x-5\right)^2-25+2$$
$$=\left(x-5\right)^2-23$$

(e) $9+8x-x^2=9-(x^2-8x)$

$$=9-\left[(x-4)^2-\underbrace{16}_{4^2}\right]=25-(x-4)^2$$

(f) $x^2+7x+1=\left(x+\dfrac{7}{2}\right)^2-\left(\dfrac{7}{2}\right)^2+1=\left(x+\dfrac{7}{2}\right)^2-\dfrac{45}{4}$

(g) $2x^2+7x+1=2\left[x^2+\dfrac{7}{2}x+\dfrac{1}{2}\right]$

$$=2\left[\left(x+\frac{7}{4}\right)^2-\left(\frac{7}{4}\right)^2+\frac{1}{2}\right]$$
$$=2\left[\left(x+\frac{7}{4}\right)^2-\frac{41}{16}\right]$$
$$=2\left(x+\frac{7}{4}\right)^2-\frac{41}{8}$$

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2. Similar to **EXAMPLE 11**. All the hard work has been done in solution 1.
(a) From solution 1(a) we have

$$\left(x-2\right)^2-1=0$$
$$\left(x-2\right)^2=1,\ \ x-2=\pm 1 \text{ which gives } x=1,\ \ x=3$$

(b) By the above solution 1(b):

$$x^2+8x+9=\left(x+4\right)^2-7=0$$
$$\left(x+4\right)^2=7 \text{ implies that } x+4=\pm\sqrt{7}$$

This means that $x=-4\pm\sqrt{7}$ which is $x=-4-\sqrt{7}$, $x=-4+\sqrt{7}$

(c) By solution 1(c) we have $x^2-6x+8=\left(x-3\right)^2-1$. Equating this to zero yields

$$(x-3)^2-1=0$$

$$(x-3)^2=1 \quad \text{which means that} \quad x-3=\pm\sqrt{1}$$

We have $x=3\pm1=2,\ 4$.

(d) From solution to 1(d) we have $x^2-10x+2=(x-5)^2-23$. Equating this to zero yields:

$$(x-5)^2-23=0$$

$$(x-5)^2=23$$

$$x-5=\pm\sqrt{23} \quad \text{implies that} \quad x=5\pm\sqrt{23}$$

We have $x=5-\sqrt{23},\quad x=5+\sqrt{23}$.

(e) Similarly

$$25-(x-4)^2=0$$

$$(x-4)^2=25,\ x-4=\pm\sqrt{25}=\pm5$$

$$x=4\pm5 \ \text{which gives} \quad x=-1,\ 9$$

(f) We have

$$\left(x+\frac{7}{2}\right)^2-\frac{45}{4}=0$$

$$\left(x+\frac{7}{2}\right)^2=\frac{45}{4},\ x+\frac{7}{2}=\pm\sqrt{\frac{45}{4}}=\pm\frac{\sqrt{45}}{2}$$

$$x=-\frac{7}{2}\pm\frac{\sqrt{45}}{2}=\frac{-7\pm\sqrt{45}}{2}$$

(g) Very similar to the above. We have

$$2\left(x+\frac{7}{4}\right)^2-\frac{41}{8}=0$$

$$2\left(x+\frac{7}{4}\right)^2=\frac{41}{8},\ \left(x+\frac{7}{4}\right)^2=\frac{41}{16}$$

$$x+\frac{7}{4}=\pm\sqrt{\frac{41}{16}}=\pm\frac{\sqrt{41}}{4}$$

$$x=\frac{-7\pm\sqrt{41}}{4}$$

3. When $t=0,\ V=6$. To find minimum or maximum we need to complete the square.

$$V=t^2-5t+6$$

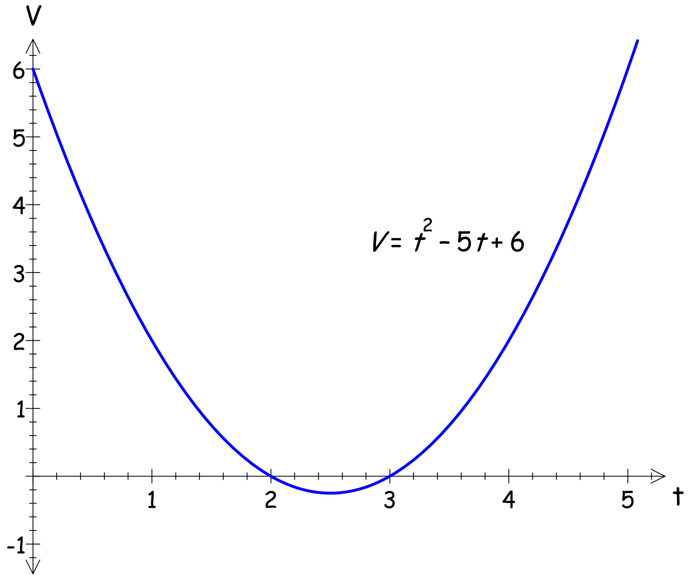
$$=\left(t-\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2+6$$

$$V=\left(t-\frac{5}{2}\right)^2-\frac{1}{4}$$

We have a minimum at $\mathrm{t}=\frac{5}{2}=2.5$ with $V=-\frac{1}{4}$. Need to find the values of t which gives $V=0$.

$$t^2 - 5t + 6 = 0$$
$$(t-3)(t-2) = 0$$
$$t = 3,\ 2$$

Hence the graph V cuts the t axis at $t = 2$, $t = 3$.



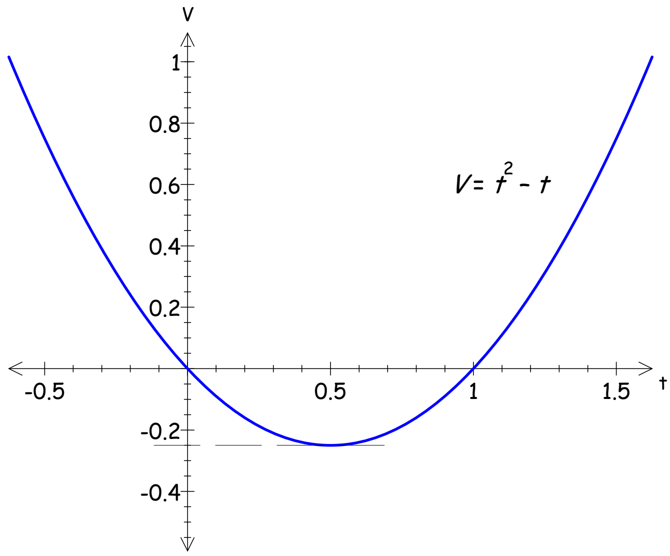
4. At $t = 0$, $V = 0$. Completing the square:

$$V = t^2 - t$$
$$= \left(t - \frac{1}{2}\right)^2 - \frac{1}{4}$$

This gives a minimum at $t = \frac{1}{2} = 0.5$ with $V = -\frac{1}{4}$. To find the values of t which gives zero voltage,

$$t^2 - t = 0$$
$$t(t-1) = 0 \text{ gives } t = 0, \quad t = 1$$

Combining all the information gives the graph:



5. Since there is a $-t^2$ term, we know it is a quadratic of the form $\cap$. Also $h = 12t - t^2$ cuts the t axis where $h = 0$:

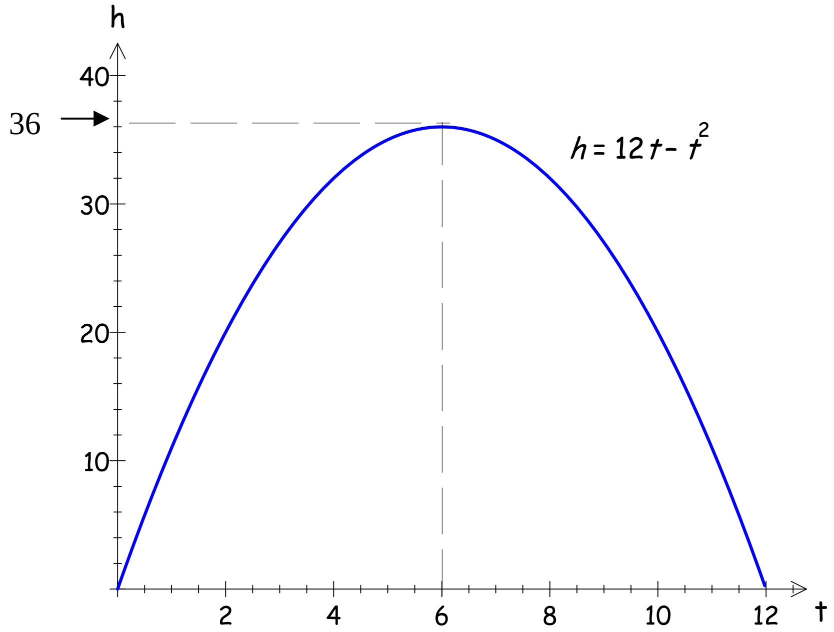
$$12t - t^2 = 0$$

$$t(12 - t) = 0 \text{ gives } t = 0, t = 12$$

The highest point is reached when $12t - t^2$ is a maximum. It can be found by completing the square:

$$\begin{aligned} 12t - t^2 &= -\left(t^2 - 12t\right) \\ &= -\left[\left(t-6\right)^2 - 36\right] \\ &= -36 - \left(t-6\right)^2 \end{aligned}$$

Hence $t = 6$ gives maximum height $h = 36$:



6. To find the maximum height we need to complete the square on *h*.

$$\begin{aligned} h = 200t - 4.9t^2 &= -(4.9t^2 - 200t) \\ &= -4.9\left(t^2 - \frac{200}{4.9}t\right) \\ &= -4.9\left(t^2 - 40.82t\right) \\ &= -4.9\left(t - \frac{40.82}{2}\right)^2 + \left[4.9 \times \left(\frac{40.82}{2}\right)^2\right] \end{aligned}$$

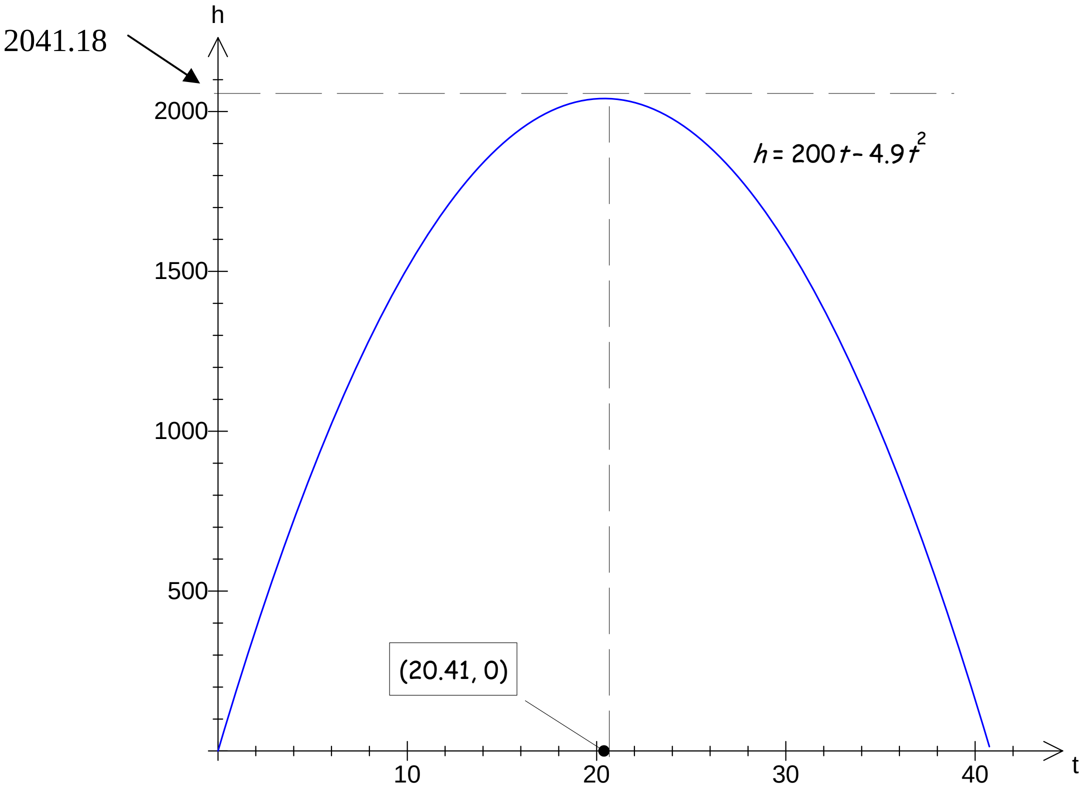
$$\text{h} = -4.9(\text{t} - 20.41)^2 + 2041.18$$

The maximum height reached is 2041.18m when $t = 20.41s$. The height, $\text{h} = 0$ when

$$200t - 4.9t^2 = 0$$

$$t(200 - 4.9t) = 0$$

$$t = 0, \quad t = \frac{200}{4.9} = 40.82$$



7. $M = 0$ when $Lx - x^2 = 0$. That is

$$x(L - x) = 0 \text{ gives } x = 0,\ x = L$$

To find maximum or minimum, complete the square on $Lx - x^2$;

$$Lx - x^2 = -(x^2 - Lx)$$
$$= -\left(x - \frac{L}{2}\right)^2 + \frac{L^2}{4}$$

Maximum occurs at $x = \frac{L}{2}$. To find the maximum value we substitute $x = \frac{L}{2}$ into

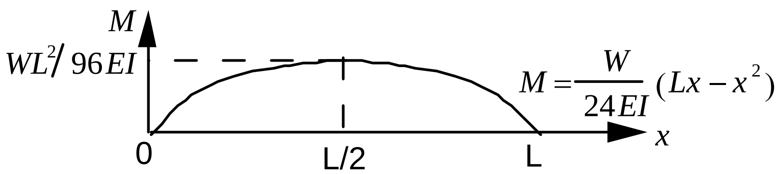
$$M = \frac{W}{24EI}\left(Lx - x^2\right)$$

$$M = \frac{W}{24EI}\left[L.\frac{L}{2} - \frac{L^2}{4}\right] = \frac{W}{24EI}\left[\frac{L^2}{2} - \frac{L^2}{4}\right]$$
$$= \frac{W}{24EI}\left(\frac{L^2}{4}\right)$$

Maximum value is

$$M = \frac{WL^2}{(24EI)\times 4} = \frac{WL^2}{96EI}$$

Hence the graph is



8. We divide the quadratic equation $ax^2+bx+c=0$ by a:

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

Completing the square:

$$\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}=0$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$$

$$=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}=\frac{b^2-4ac}{4a^2}$$

Taking the square roots of both sides:

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}=\pm\frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}}$$

$$=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

Making x the subject:

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$