Complete Solutions to Exercise 2f

1. (a) Pascal's triangle with $n = 6$ gives the coefficients 1, 6, 15, 20, 15, 6 and 1. Hence

$$(x+1)^6 = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$$

(b) We have the same coefficients as part (a) but the index of a decreases by 1 and index of b increases by 1.

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

(c) *How do we expand* $(1-x)^6$ *?*

By applying (2.6) $(a+b)^n = C_n a^n + C_{n-1}a^{n-1}b + C_{n-2}a^{n-2}b^2 + \cdots + C_0 b^n$ with $a = 1$ and $b = -x$.

What are the coefficients C equal to?

By using Pascal's triangle for $n = 6$ which is 1, 6, 15, 20, 15, 6 and 1:

$$\begin{aligned}(1-x)^6 &= 1 + [6\times(-x)] + [15\times(-x)^2] + [20\times(-x)^3] + [15\times(-x)^4] \\ &\quad + [6\times(-x)^5] + (-x)^6 \\ &= 1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6 \qquad [\text{Simplifying}]\end{aligned}$$

2. (a) With $n = 4$ we have the coefficients 1, 4, 6, 4 and 1, (see Pascal's triangle chapter 2). Using (2.6) with $a = 5$, $b = x$ and $n = 4$ gives:

$$\begin{aligned}(5+x)^4 &= 5^4 + (4\times 5^3\times x) + (6\times 5^2\times x^2) + (4\times 5\times x^3) + x^4 \\ &= 625 + 500x + 150x^2 + 20x^3 + x^4 \\ &= x^4 + 20x^3 + 150x^2 + 500x + 625\end{aligned}$$

(b) With $n = 5$ we have the coefficients 1, 5, 10, 10, 5 and 1. Hence

$$\begin{aligned}(2+3x)^5 &= 2^5 + [5\times 2^4\times 3x] + [10\times 2^3\times (3x)^2] + [10\times 2^2\times (3x)^3] + [5\times 2\times (3x)^4] + (3x)^5 \\ &= 32 + [5\times 2^4\times 3]x + [10\times 2^3\times 3^2]x^2 + [10\times 2^2\times 3^3]x^3 + [5\times 2\times 3^4]x^4 + 3^5x^5 \\ &= 32 + 240x + 720x^2 + 1080x^3 + 810x^4 + 243x^5\end{aligned}$$

(c) The coefficients for $n = 6$ are 1, 6, 15, 20, 15, 6 and 1. Using (2.6) with $a = 4$ and $b = -3x$ gives:

$$\begin{aligned}(4-3x)^6 &= [4 + (-3x)]^6 \\ &= 4^6 + [6\times 4^5\times(-3x)] + [15\times 4^4\times(-3x)^2] + [20\times 4^3\times(-3x)^3] \\ &\quad + [15\times 4^2\times(-3x)^4] + [6\times 4\times(-3x)^5] + (-3x)^6 \\ &= 4096 + [6\times 4^5\times(-3)]x + [15\times 4^4\times(-3)^2]x^2 + [20\times 4^3\times(-3)^3]x^3 \\ &\quad + [15\times 4^2\times(-3)^4]x^4 + [6\times 4\times(-3)^5]x^5 + (-3)^6x^6 \\ &= 4096 - 18432x + 34560x^2 - 34560x^3 + 19440x^4 - 5832x^5 + 729x^6\end{aligned}$$

This can be rewritten as

$$(4-3x)^6 = 729x^6 - 5832x^5 + 19440x^4 - 34560x^3 + 34560x^2 - 18432x + 4096$$

(d) *How do we expand* $(2x-y)^4$ *?*

The coefficients for $n = 4$ is 1, 4, 6, 4 and 1. We use the formula

(2.6) $(a+b)^n = C_n a^n + C_{n-1}a^{n-1}b + C_{n-2}a^{n-2}b^2 + \cdots + C_0 b^n$

with $a = 2x$ and $b = -y$.

$$(2x-y)^4 = (2x)^4 + 4(2x)^3(-y) + 6(2x)^2(-y)^2 + 4(2x)(-y)^3 + (-y)^4$$
$$= 16x^4 + 4(-8x^3y) + 6(4x^2y^2) + 4(-2xy^3) + y^4$$
$$= 16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4$$

(e) This is more challenging because of the $\frac{1}{x}$ term inside the brackets. Using (2.6) with $n=5$ which gives the coefficients 1, 5, 10, 10, 5 and 1. We substitute $a=x$ and $b=\frac{1}{x}$ into the formula

(2.6) $\quad (a+b)^n = C_n a^n + C_{n-1}a^{n-1}b + C_{n-2}a^{n-2}b^2 + \cdots + C_0 b^n$

which gives

$$\left(x+\frac{1}{x}\right)^5 = x^5 + 5x^4\left(\frac{1}{x}\right) + 10x^3\left(\frac{1}{x}\right)^2 + 10x^2\left(\frac{1}{x}\right)^3 + 5x\left(\frac{1}{x}\right)^4 + \left(\frac{1}{x}\right)^5$$
$$= x^5 + 5x^3 + 10x^3\left(\frac{1}{x^2}\right) + 10x^2\left(\frac{1}{x^3}\right) + 5x\left(\frac{1}{x^4}\right) + \frac{1}{x^5}$$
$$= x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5} \qquad [\text{Cancelling}]$$

(f) Similarly we have

$$\left(x-\frac{1}{x}\right)^4 = x^4 + 4x^3\left(-\frac{1}{x}\right) + 6x^2\left(-\frac{1}{x}\right)^2 + 4x\left(-\frac{1}{x}\right)^3 + \left(-\frac{1}{x}\right)^4$$
$$= x^4 - 4x^2 + 6 - \frac{4}{x^2} + \frac{1}{x^4} \qquad [\text{Simplifying}]$$

(g) Using the (2.6) formula with $n=7$. Pascal's triangle for $n=7$ is 1, 7, 21, 35, 35, 21, 7 and 1. Of course these are the coefficients C's in formula (2.6):

(2.6) $\quad (a+b)^n = C_n a^n + C_{n-1}a^{n-1}b + C_{n-2}a^{n-2}b^2 + \cdots + C_0 b^n$

What are the values of a and b in this case?

$a=1$ and $b=x^2$. We have

$$\left(1+x^2\right)^7 = 1 + 7x^2 + 21\left(x^2\right)^2 + 35\left(x^2\right)^3 + 35\left(x^2\right)^4 + 21\left(x^2\right)^5 + 7\left(x^2\right)^6 + \left(x^2\right)^7$$
$$\underbrace{=}_{\substack{\text{Using the rules} \\ \text{of indices } (a^m)^n = a^{mn}}} 1 + 7x^2 + 21x^4 + 35x^6 + 35x^8 + 21x^{10} + 7x^{12} + x^{14}$$

3. Putting $n=10$ into $(1+x)^{n-1}$ gives $(1+x)^9$. Since Pascal's triangle, only goes up to 7, we have to extend this to 9:

$n=7$	1	7	21	35	35	21	7	1		
$n=8$	1	8	28	56	70	56	28	8	1	
$n=9$	1	9	36	84	126	126	84	36	9	1

The coefficients for an index of 9 is the last row:

$$(1+x)^9 = 1 + 9x + 36x^2 + 84x^3 + 126x^4 + 126x^5 + 84x^6 + 36x^7 + 9x^8 + x^9$$

Since Pascal's triangle is symmetrical we only need to determine the coefficients up to and including the term x^4.

The amplitudes are given by 1,9,36,84 and 126.

4. The coefficients for $n = 7$ are 1, 7, 21, 35, 35, 21, 7 and 1. Hence we have

$$\left(\frac{w}{4}-\frac{x}{3}\right)^7=\left(\frac{w}{4}\right)^7+7\left(\frac{w}{4}\right)^6\left(-\frac{x}{3}\right)+21\left(\frac{w}{4}\right)^5\left(-\frac{x}{3}\right)^2+35\left(\frac{w}{4}\right)^4\left(-\frac{x}{3}\right)^3+35\left(\frac{w}{4}\right)^3\left(-\frac{x}{3}\right)^4$$

$$+21\left(\frac{w}{4}\right)^2\left(-\frac{x}{3}\right)^5+7\left(\frac{w}{4}\right)\left(-\frac{x}{3}\right)^6+\left(-\frac{x}{3}\right)^7$$

$$=\frac{w^7}{4^7}+7\frac{w^6(-x)}{4^6 3}+21\frac{w^5(-x)^2}{4^5 3^2}+35\frac{w^4(-x)^3}{4^4 3^3}+35\frac{w^3(-x)^4}{4^3 3^4}$$

$$+21\frac{w^2(-x)^5}{4^2 3^5}+7\frac{w(-x)^6}{4(3)^6}+\frac{(-x)^7}{3^7}$$

$$=\frac{w^7}{16384}-\frac{7w^6x}{12288}+\frac{7w^5x^2}{3072}-\frac{35w^4x^3}{6912}+\frac{35w^3x^4}{5184}-\frac{7w^2x^5}{1296}+\frac{7wx^6}{2916}-\frac{x^7}{2187}$$