Complete solutions to Exercise 6(g)

1. (a) $\frac{d}{dx}\left(x^2+y^2=4\right)$ gives $2x+2y\frac{dy}{dx}=0$

$$2y\frac{dy}{dx}=-2x$$

$$\frac{dy}{dx}=\frac{-2x}{2y}=-\frac{x}{y} \qquad \left(\text{provided } y\neq 0\right)$$

(b) $\frac{d}{dx}\left(x^3+y^3-2x=3\right)$ gives

$$3x^2+3y^2\frac{dy}{dx}-2=0$$

$$3y^2\frac{dy}{dx}=2-3x^2$$

$$\frac{dy}{dx}=\frac{2-3x^2}{3y^2} \qquad \left(\text{provided } y\neq 0\right)$$

(c) Differentiating $\frac{x^2}{4}+\frac{y^2}{16}=1$ with respect to x gives

$$\frac{2x}{4}+\frac{2y}{16}\frac{dy}{dx}=0$$

$$\frac{y}{8}\frac{dy}{dx}=-\frac{x}{2} \qquad \left[\text{Simplifying Fraction}\right]$$

$$\frac{dy}{dx}=-\frac{8x}{2y}=-\frac{4x}{y} \qquad \left(\text{provided } y\neq 0\right)$$

(d) Differentiating $x^2+y^2-4x-6y=-12$ with respect to x gives

$$2x+2y\frac{dy}{dx}-4-6\frac{dy}{dx}=0$$

$$\left(2y-6\right)\frac{dy}{dx}=4-2x$$

$$\frac{dy}{dx}=\frac{4-2x}{2y-6}=\frac{2\left(2-x\right)}{2\left(y-3\right)}=\frac{2-x}{y-3} \qquad \left(\text{provided } y-3\neq 0\right)$$

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2. (a) We need to find $\frac{d}{dx}\left(x^2+y^2-xy^2=5\right)$.

$$2x+2y\frac{dy}{dx}-\frac{d}{dx}\left(xy^2\right)=0 \qquad (\dagger)$$

How do we differentiate xy^2*?*

Use the product rule (6.31) because $xy^2=x\times y^2$:

$$u=x \qquad v=y^2$$

$$u'=1 \qquad v'=2y\frac{dy}{dx}$$

Substituting these into (6.31) gives

$$\frac{d}{dx}\left(xy^2\right)=1.y^2+x.2y\frac{dy}{dx}=y^2+2xy\frac{dy}{dx}$$

(6.31) $\qquad \left(uv\right)'=u'v+uv'$

Substituting this result into $(\dagger)$ gives

$$2x + 2y\frac{dy}{dx} - y^2 - 2xy\frac{dy}{dx} = 0$$

$$(2y - 2xy)\frac{dy}{dx} = y^2 - 2x \qquad \text{(Collecting like terms)}$$

$$\frac{dy}{dx} = \frac{y^2 - 2x}{2y - 2xy} \qquad (\text{Dividing by } 2y - 2xy \neq 0)$$

(b) To find $\frac{d}{dx}\left(\cos\left(x^2 + y\right) - 2xy = 0\right)$, we split this up into 2 halves:

$$\frac{d}{dx}\left[\cos\left(x^2 + y\right)\right] \underbrace{=}_{\text{by (6.20)}} -\sin\left(x^2 + y\right).\left[2x + \frac{dy}{dx}\right] = -\left[2x + \frac{dy}{dx}\right]\sin\left(x^2 + y\right)$$

$$\frac{d}{dx}(2xy) \underbrace{=}_{\text{by (6.31)}} 2y + 2x\frac{dy}{dx}$$

Collecting these gives

$$\frac{d}{dx}\left[\cos\left(x^2 + y\right) - 2xy = 0\right] = -\left(2x + \frac{dy}{dx}\right)\sin\left(x^2 + y\right) - \left(2y + 2x\frac{dy}{dx}\right) = 0$$

Combining the $\frac{dy}{dx}$ terms:

$$-\frac{dy}{dx}\left[\sin\left(x^2 + y\right) + 2x\right] = 2x\sin\left(x^2 + y\right) + 2y$$

$$\frac{dy}{dx} = -\frac{2x\sin\left(x^2 + y\right) + 2y}{2x + \sin\left(x^2 + y\right)} \qquad \left(\text{provided } 2x + \sin\left(x^2 + y\right) \neq 0\right)$$

(c) $\frac{d}{dx}\left[\ln(y) = \ln(x - 1) - \ln\left(x^2\right)\right]$ is found by using (6.18):

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x - 1}.1 - \frac{1}{x^2}.2x$$

$$\frac{dy}{dx} = y\left(\frac{1}{x - 1} - \frac{2}{x}\right) = y\left(\frac{x - 2(x - 1)}{x(x - 1)}\right) = y\left[\frac{2 - x}{x(x - 1)}\right] \quad \left(\text{provided } x(x - 1) \neq 0\right)$$

(d) We need to find $\frac{d}{dx}\left[\ln(xy) + 2y = 0\right]$. *What is* $\frac{d}{dx}\left[\ln(xy)\right]$ *equal to?*

Need to use (6.18)

$$\frac{d}{dx}\left[\ln(xy)\right] = \frac{1}{xy}.\frac{d}{dx}(xy)$$

$$= \frac{1}{xy}\underbrace{\left(1.y + x\frac{dy}{dx}\right)}_{\substack{\text{by (6.31) with} \\ u=x \text{ and } v=y}} = \frac{y}{xy} + \frac{x}{xy}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} \qquad [\text{Cancelling}]$$

(6.18) $\qquad \frac{d}{dx}\left[\ln(u)\right] = \frac{1}{u}\frac{du}{dx}$

(6.20) $\qquad \frac{d}{dx}\left[\cos(u)\right] = -\sin(u)\frac{du}{dx}$

(6.31) $\qquad (uv)' = u'v + uv'$

Also $\dfrac{d}{dx}[2y]=2\dfrac{dy}{dx}$. We have

$$\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}+2\frac{dy}{dx}=0$$

Collecting the $\dfrac{dy}{dx}$ terms:

$$\left(2+\frac{1}{y}\right)\frac{dy}{dx}=-\frac{1}{x}$$

$$\left(\frac{2y+1}{y}\right)\frac{dy}{dx}=-\frac{1}{x}$$

$$\frac{dy}{dx}=-\frac{y}{x(2y+1)} \quad \left(\text{provided } x(2y+1)\neq 0\right)$$

(e) We need to find $\dfrac{d}{dx}\left(e^{x+y}+\sin(x)=0\right)$.

Differentiating the power $x+y$ with respect to x gives $1+\dfrac{dy}{dx}$. Thus

$$\frac{d}{dx}\left(e^{x+y}\right)=e^{x+y}\left(1+\frac{dy}{dx}\right) \qquad \left[\text{By } \frac{d}{dx}\left(e^{u}\right)=e^{u}\frac{du}{dx}\right]$$

Hence differentiating gives:

$$\left(1+\frac{dy}{dx}\right)e^{x+y}+\cos(x)=0$$

$$\left(1+\frac{dy}{dx}\right)=-\frac{\cos(x)}{e^{x+y}}=-e^{-(x+y)}\cos(x)$$

$$\frac{dy}{dx}=-e^{-(x+y)}\cos(x)-1=-\left[1+e^{-(x+y)}\cos(x)\right]$$

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3. (i) For $a=4$ we have

$$x^3+y^3-12xy=0$$

Differentiating this gives

$$3x^2+3y^2\frac{dy}{dx}-12\underbrace{\left[(1)y+x\frac{dy}{dx}\right]}_{\text{By Product Rule}}=0$$

$$\left[3y^2-12x\right]\frac{dy}{dx}=12y-3x^2$$

$$\frac{dy}{dx}=\frac{12y-3x^2}{3y^2-12x}=\frac{3\left(4y-x^2\right)}{3\left(y^2-4x\right)}=\frac{4y-x^2}{y^2-4x}$$

(ii) Similarly we can show that for a general a we have

$$\frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}$$

(6.15) $$\frac{d}{dx}\left(ku^n\right)=nku^{n-1}\frac{du}{dx}$$

(6.22) $$\frac{d}{dt}\left[\sec(u)\right]=\sec(u)\tan(u)\frac{du}{dt}$$

4. Let $y=10^x$. Taking logs: $\ln(y)=\ln(10^x)\underbrace{=}_{\text{by (5.13)}} x\ln(10)$

Differentiating:

$$\frac{1}{y}\frac{dy}{dx}=\ln(10)$$

$$\frac{dy}{dx}=y\ln(10)=10^x\ln(10)\quad\left[\text{Substituting } y=10^x\right]$$

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5. (a) We have $y=x^{\sin(x)}$. Taking logs,

$$\ln(y)=\ln\left(x^{\sin(x)}\right)\qquad(*)$$

$$\underbrace{=}_{\text{by (5.13)}}\sin(x)\ln(x)$$

We use the product rule (6.31) to find $\frac{d}{dx}\left[\sin(x)\ln(x)\right]$.

$$u=\sin(x)\qquad v=\ln(x)$$

$$u'=\cos(x)\qquad v'=\frac{1}{x}$$

$$\frac{d}{dx}\left[\sin(x)\ln(x)\right]=\cos(x).\ln(x)+\sin(x).\frac{1}{x}$$

Hence differentiating (*):

$$\frac{1}{y}\frac{dy}{dx}=\cos(x).\ln(x)+\frac{\sin(x)}{x}$$

$$\frac{dy}{dx}=y\left[\cos(x).\ln(x)+\frac{\sin(x)}{x}\right]\quad(\text{Multiplying by } y)$$

$$=x^{\sin(x)}\left[\ln(x).\cos(x)+\frac{\sin(x)}{x}\right]\quad\left(\text{Substituting } y=x^{\sin(x)}\right)$$

(b) Taking logs of $y=\left(x^2+1\right)^{1/2}(3x+1)^{1/3}$ gives

$$\ln(y)=\ln\left[\left(x^2+1\right)^{1/2}(3x+1)^{1/3}\right]$$

$$\underbrace{=}_{\text{by (5.11)}}\ln\left(x^2+1\right)^{1/2}+\ln(3x+1)^{1/3}$$

$$\ln(y)=\frac{1}{2}\ln\left(x^2+1\right)+\frac{1}{3}\ln(3x+1)\quad[\text{By (5.13)}]$$

Differentiating this:

$$\frac{1}{y}\frac{dy}{dx}=\left(\frac{1}{2}\frac{2x}{x^2+1}\right)+\left(\frac{1}{3}\frac{3}{3x+1}\right)\quad\left[\text{By } \frac{d}{dx}\left[\ln(u)\right]=\frac{1}{u}\frac{du}{dx}\right]$$

$$=\frac{x}{x^2+1}+\frac{1}{3x+1}\qquad[\text{Simplifying}]$$

$$\frac{dy}{dx}=y\left(\frac{x}{x^2+1}+\frac{1}{3x+1}\right)=\left(x^2+1\right)^{1/2}(3x+1)^{1/3}\left[\frac{x}{x^2+1}+\frac{1}{3x+1}\right]$$

(5.11) $\ln(AB)=\ln(A)+\ln(B)$

(5.13) $\ln\left(A^n\right)=n\ln(A)$ (6.31) $(uv)'=u'v+uv'$

(c) Similarly to (b) we have

$$\ln(y)=\frac{1}{2}\ln\left(x^2+1\right)+\frac{1}{3}\ln(3x+1)-\frac{1}{5}\ln\left(x^4-2\right)$$

Differentiating gives

$$\frac{1}{y}\frac{dy}{dx}=\frac{x}{x^2+1}+\frac{1}{3x+1}-\frac{1}{5}\frac{4x^3}{x^4-2}$$

$$\frac{dy}{dx}=\frac{\left(x^2+1\right)^{1/2}(3x+1)^{1/3}}{\left(x^4-2\right)^{1/5}}\left[\frac{x}{x^2+1}+\frac{1}{3x+1}-\frac{4x^3}{5\left(x^4-2\right)}\right] \quad \text{(Substituting for } y\text{)}$$

(d) Taking logs of $y=e^{\left(x^x\right)}$ gives

$$\ln(y)=\ln\left(e^{x^x}\right)=x^x\underbrace{\ln(e)}_{=1}=x^x$$

By **EXAMPLE 35**, differentiating x^x gives $x^x\left[\ln(x)+1\right]$. Hence

$$\frac{1}{y}\frac{dy}{dx}=x^x\left[\ln(x)+1\right]$$

$$\frac{dy}{dx}=e^{\left(x^x\right)}x^x\left[\ln(x)+1\right]$$

(e) Taking logs of $y=x^{\left(x^x\right)}$ gives

$$\ln(y)=\ln\left(x^{x^x}\right)=x^x\ln(x)$$

How do we differentiate $\ln(y)=x^x\ln(x)$?

Use the product rule on $x^x\ln(x)$

$$u=x^x \qquad v=\ln(x)$$

$$u'=x^x\left[\ln(x)+1\right] \qquad v'=\frac{1}{x}$$

$$\frac{d}{dx}\left[x^x\ln(x)\right]=x^x\left[\ln(x)+1\right]\ln(x)+x^x\frac{1}{x}=x^x\left[\left[\ln(x)\right]^2+\ln(x)+\frac{1}{x}\right] \quad \text{[Factorizing]}$$

Hence we have

$$\frac{d}{dx}\left[\ln(y)\right]=x^x\left[\left[\ln(x)\right]^2+\ln(x)+\frac{1}{x}\right]$$

$$\frac{1}{y}.\frac{dy}{dx}=x^x\left[\frac{x\left[\ln(x)\right]^2+x\ln(x)+1}{x}\right]$$

$$\frac{dy}{dx}=yx^{x-1}\left[x\left[\ln(x)\right]^2+x\ln(x)+1\right]$$

$$=x^{\left(x^x\right)}x^{x-1}\left[x\left[\ln(x)\right]^2+x\ln(x)+1\right]=x^{\left(x^x\right)+x-1}\left[x\left[\ln(x)\right]^2+x\ln(x)+1\right]$$