Teaching and Learning RESEARCH BRIEFING

Number 10

5-14 Mathematics in Scotland

Intensive Quantities: Why Primary School Mathematics Needs Them

Mathematics teaching focuses on extensive quantities (e.g. distance, volume or price). Intensive quantities (e.g. speed, density, value for money) are ignored, or treated in a piecemeal fashion. A survey with primary school children shows that this approach leads to enduring difficulties with intensive quantities, and undermines children's mastery of fractions. A teaching programme has been developed to remedy this, and it has been found to boost understanding of intensive quantities as well as fraction usage. The approach is compatible with current curricular demands, and extends them in a valuable way.

- Primary school children of all ages have difficulties with intensive quantities, showing that mastery does not develop without teaching. → Intensive quantities need to be taught at the primary school level, in the interests of comprehensive mastery of quantity.
- Children's difficulties with intensive quantities are primarily conceptual. Mastery of the arithmetical procedures is less significant. → Teaching needs to focus upon problem-solving strategy.
- Primary school pupils of all ages have difficulties using fractions to name intensive quantities. → The use of fractions to name intensive quantities needs to be taught. Children will not generalise their knowledge of fractions from extensive contexts to intensive ones.
- Two or three hours of teaching can boost children's understanding of intensive quantities and their ability to use fractions in intensive contexts. → Addressing the difficulties with intensive quantities does not require major upheavals to the primary school curriculum.

The research

This ESRC-TLRP Scottish Extension project addresses children's mastery of intensive quantities. Such quantities (e.g. speed, density, or monetary value) are based on the logic of co-variation where at least two variables are involved. Speed, for example, involves both the distance covered and the time taken to cover it, and is measured in units such as miles per hour. Intensive quantities contrast with extensive quantities such as distance, volume, or price, which are based on the logic of part–whole relations.

To see the difference between intensive and extensive quantities, consider adding orange squash from a jug that contains 20 decilitres to a second jug that contains 60 decilitres. You will have 80 decilitres of orange squash in total, since volume is an extensive quantity and the whole is the sum of the parts. By contrast, the concentration, and therefore the taste, of the orange squash is an intensive quantity. The concentration is directly proportional to the amount of orange concentrate, and inversely proportional to the amount of water. Thus two variables need to be considered, concentrate and water. If the orange squash in one jug is 20 per cent concentrate and the squash in the second jug is 60 per cent concentrate, mixing the two does not lead to 80 per cent concentration.

The distinction between intensive and extensive quantities is recognised elsewhere in the world, for instance in Japan, documented as a centre of excellence for pupil attainment in mathematics. However, the distinction has been neglected in the UK, including Scotland, where the present research was conducted. Teaching throughout the United Kingdom focuses upon extensive quantities only.

The neglect of intensive quantities is unfortunate. There is evidence that, in the absence of formal instruction, children often treat intensive quantities as if they were extensive. This leads to confusion with fundamental concepts in science and mathematics. For instance, a common response to 'What happens if you mix a tub of hot water with another tub of hot water?' is that the water will become even hotter, a response which suggests that temperature, an intensive quantity, is thought of as extensive. In addition, intensive quantities require representation in fraction or ratio form. This means that they provide a natural context for supporting fraction or ratio reasoning. This form of reasoning has been singled out by the Scottish Executive as an area of concern, for example in the *National Statement for Improving Attainment in Numeracy.*

Base-line survey

The project started by documenting Scottish children's current understanding of intensive quantities, via a base-line survey. The survey involved 42 intensive quantity problems of two types. The first type involved a fixed ratio, for instance 'Yesterday, Billy made juice from 3 oranges and 1 pineapple. Today, he is using 2 pineapples. How many oranges will he need to make the juice taste the same?' The second used a variable ratio, for example 'Joe jumps 8 hurdles in 40 seconds, and Peter jumps 5 hurdles in 30 seconds. Is Joe running faster? Is Peter running faster? Are both boys running at the same speed?' The problems varied in topic and included taste, speed, crowding, monetary value, and strength of chemicals. Some required mathematical computation (easy or hard) while others could be solved non-computationally. Eight problems were accompanied by a request for explanations of how the solutions had been obtained. Eight further problems contained an invitation to name the fraction of mixtures that was contributed by specified constituents. For instance, told that a paint mixture contained two bottles of white paint and two bottles of blue paint, children were asked what fraction of the mixture was made of blue paint.

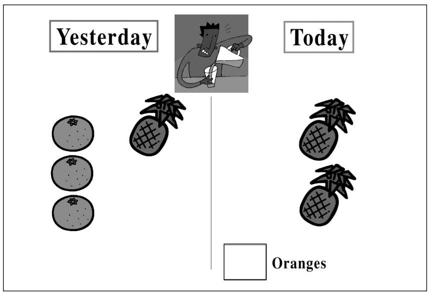

Figure 1

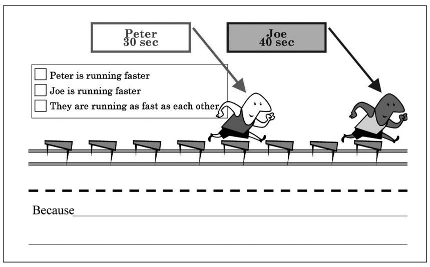

Figure 2

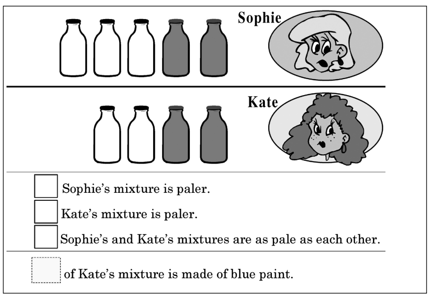

Figure 3

The problems were divided into two, equal-length sets, presented on successive days. The sets were displayed via PowerPoint to whole classes, and the children responded using answer booklets. The problems were presented to all of the P3, P4, P5, P6 and P7 children (N = 963, age range 7 to 12 years) in six schools who were in attendance throughout the visit. Responses were scored for accuracy of solution ('solution scores'), strategic insight as revealed in the explanations ('explanation scores'), and adequacy of fraction naming ('fraction scores').

Gender and school effects were negligible, but there was discernible improvement with age. Nevertheless, even the scores of the P7 children were considerably below the theoretical maximum. The pattern of results for solution scores suggested that with the youngest children, problem solving was largely non-computational, and the main difficulty was considering both of the relevant variables, such as orange concentrate and water, together. If only one variable was considered, it was usually the variable that was directly proportional to the outcome, for example orange concentrate rather than water when the question was about 'oranginess'. An exception occurred when the topic was speed. There, the inversely proportional variable (time) was more likely to be considered than the directly proportional variable (distance). With the older children, computational strategies were usually applied where required. However, the strategies frequently involved addition or subtraction when multiplication or division were needed. This led, for instance, to the belief that if you need 8 spoons of flour and 12 spoons of milk to make 8 pancakes, you will need 6 spoons of flour and 10 spoons of milk to make 6 pancakes.

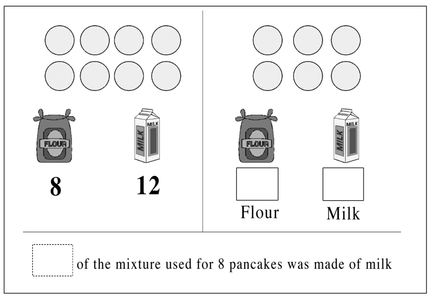

Figure 4

Even at P7, explanation scores were only around 50 per cent correct.

Explanation scores were the strongest predictors of solution scores, stronger even than age group. This suggests that strategic insight is crucial for problem solving with intensive quantities, and argues against teaching strategies that focus only on mathematical procedures. Although fraction scores also improved with age, the children had difficulties with fraction naming at all age levels. Even the easiest fractions, e.g. one-half/two-quarters, posed significant difficulties throughout the age range, with the majority of P7 children giving incorrect answers.

Intervention study

The base-line survey showed that in the absence of formal teaching, primary school children of all ages have difficulties with intensive quantities. The difficulties have adverse consequences for fractional representation. Thus, the results provided ample warrant for the second part of the project, which was an intervention study that contrasted alternative approaches to teaching.

The intervention revolved around four lessons. Lesson 1 helped children consider the variable that is inversely proportional to outcome, as well as the variable that is directly proportional. Lesson 2 was directed at the adoption of appropriate language to represent intensive quantities. Lesson 3 supported the use of adequate, multiplicative computational strategies. Lesson 4 encouraged the application of knowledge developed in previous lessons to new contexts.

There were four versions of the lessons. In two versions (RG and FG), the lessons involved whole-class teaching followed by group-based problem solving. In two versions (RI and FI), whole-class teaching was followed by individual problem solving. In two versions (RG and RI), the language used from Lesson 2 onwards involved ratio representation, as with one part orange concentrate to four parts water. In two versions (FG and FI), the language used was fractional, e.g. one-fifth orange to four-fifths water.

The intervention was conducted in seven primary schools, with 535 children participating throughout. Roughly 50 per cent were younger children (Summer P5 and Autumn P6, aged 9–10 years) and 50 per cent older (Summer P6 and Autumn P7, aged 10–11 years). Teachers' perceptions of mathematical ability (high, average or low) were recorded. Before participating in the lessons, the children were pre-tested using 18 base-line survey problems, to assess their initial understanding of intensive quantities.

Major implications

The base-line survey shows that in the absence of teaching, primary school children experience difficulties in reasoning about intensive quantities, and in naming even the simplest of fractions in intensive contexts, even when the same fractions are familiar from their work with extensive quantities. Since intensive quantities are relevant to both mathematics and science, these difficulties should be addressed through systematic instruction. This instruction should include consideration of key concepts, for example inversely and directly proportional variables, and the relevance of multiplicative as opposed to additive strategies.

The intervention study made modest demands upon classroom time, two to three hours in total. Yet, in that time, most pupils advanced their understanding of intensive quantities, and virtually all improved their ability to name fractions in intensive contexts. Given that intensive quantities provide a natural context for the teaching of fractions, this improved skill may also have positive consequences for fraction naming in extensive contexts. In any event, the intervention study profiles an approach to the teaching of intensive quantities which could readily be adopted in schools.

With four problems, the children were asked to write sentences explaining how they obtained their answers. With four further problems they were asked to name fractions. Problems were presented via PowerPoint to whole classes, as for the base-line survey. Responses were scored using the base-line survey scales so that each child obtained a solution score, an explanation score, and a fraction score.

The participating children were randomly assigned to the RG, RI, FG or FI conditions, or to control conditions to receive four lessons on the division of extensive quantities, using whole-class plus group (CG) or whole-class plus individual (CI) teaching. The lessons were presented on successive days, starting immediately after the pre-test. Each lasted around 25 minutes. The children worked in their normal (usually homogeneous ability) mathematics sets for the RG, FG and CG lessons. Immediate post-tests were administered one day after Lesson 4, and delayed post-tests about one month later. The post-tests followed the pre-test format, used the same number of problems, were of equivalent difficulty, and were scored in the same way.

There were no gender differences on any measure at pre-test or at either post-test. The older children obtained higher pre-test scores than the younger children on all three measures. The children with high mathematical ability obtained higher pre-test scores than the children with average mathematical ability, again on all three measures, and the children with average mathematical ability obtained higher pre-test scores than the children with low mathematical ability. Scores at immediate post-test were higher than scores at pre-test on all three measures, Solution scores at delayed post-test were higher than at immediate post-test, while immediate post-test gains in explanation and fraction scores were preserved at delayed post-test, but not improved upon.

The six conditions were equivalent at pre-test as regards solution scores, explanation scores and fraction scores. However, there were marked condition differences at immediate and delayed post-test, favouring one or more of the RG, RI, FG and FI conditions over the control conditions. Thus, even though the intervention lessons only lasted a couple of hours in total, they had significant and lasting benefits. The provision of group-based (RG or FG) as opposed to individual (RI or FI) problem solving made little difference to the outcome, although when differences occurred they favoured individual problem solving. It is recommended that whole-class plus individual problem solving be used when teaching intensive quantities.

The condition differences at immediate and delayed post-test as a function of pupil age and ability level showed that all the children, apart from younger children of low ability, benefited from the FI lessons. Teaching here led to marked improvement in fraction naming and strategic insight as revealed by explanation scores. The younger pupils of high ability and the older pupils of average and high ability also produced solution scores that were markedly better than the control pupils, but this time particularly if they had participated in the RI lessons. This suggests a staged approach to teaching, beginning with lessons based on the FI condition. When performance reaches a certain threshold, RI teaching is introduced.

Further information

Background information about intensive quantities (and about the parent project to the present research) can be obtained from Nunes, T, Desli, D & Bell, D (2003) The development of children's understanding of intensive quantities. *International Journal of Educational Research,* 39, 651–675.

Journal articles reporting the present research are currently in preparation. The project website (see below) provides further information about the results. Conference presentations on the base-line survey are also available:

- Howe, C, Jafri, S, Nunes, T & Bryant, P (2004) Intensive quantities: why they matter to Mathematics education. Paper presented at *TLRP Annual Conference,* Cardiff, November 2004.
- Howe, C, Nunes, T, Bryant, P & Jafri, S (2005) Children's developing understanding of intensive quantities: similarities and differences from extensive quantities. Poster presented at *Annual Conference of Society for Research into Child Development,* Atlanta, April 2005.

The warrant

Confidence in the conclusions can be based on the robustness of the empirical procedures, which comply with the highest scientific standards and were informed by the extensive experience of all three members of the project team. The sample of children was the largest ever used in research into intensive quantities. It covered a wide age range, and was fully representative of the socio-economic composition of Scotland. The intensive quantity problems respected all variables known to influence reasoning within the domain, and the problems were carefully piloted. They were presented in controlled circumstances, with counter-balanced orders of presentation.

The lessons for the intervention study were informed by the ESRC-TLRP Phase II parent project *The Role of Awareness in the Teaching and Learning of Literacy and Numeracy.* They were discussed with the project's Advisory Board, and with the teachers and head teachers in the participating schools. The lessons were carefully piloted. Field notes were taken during every lesson to ensure that they proceeded as intended.

Solution scores, explanation scores and fraction scores were independently computed for a 20 per cent sample of responses, by two judges. Inter-judge agreement was between 90 per cent and 99 per cent, demonstrating the high reliability of the three scales. All our conclusions are based on rigorous quantitative analysis, using inferential statistics (e.g. ANOVA and multiple regression). Reported differences were not simply statistically significant, but also showed substantial effect sizes.

The project team: Christine Howe, Terezinha Nunes and Peter Bryant

Contact:
Professor Christine Howe,
Department of Psychology,
University of Strathclyde,
40 George Street, Glasgow G1 1QE
c.j.howe@strath.ac.uk
Tel: +44 (0) 141 548 2575/4390

Website: http://www.strath.ac.uk/Departments/Psychology/DevEd/maths/mathsscotland.htm

June 2005

Teaching and Learning Research Programme

TLRP is the largest education research programme in the UK, and benefits from research teams and funding contributions from England, Northern Ireland, Scotland and Wales. Projects began in 2000 and will continue with dissemination and impact work extending through 2008/9.

Learning: TLRP's overarching aim is to improve outcomes for learners of all ages in teaching and learning contexts within the UK.

Outcomes: TLRP studies a broad range of learning outcomes. These include both the acquisition of skill, understanding, knowledge and qualifications and the development of attitudes, values and identities relevant to a learning society.

Lifecourse: TLRP supports research projects and related activities at many ages and stages in education, training and lifelong learning.

Enrichment: TLRP commits to user engagement at all stages of research. The Programme promotes research across disciplines, methodologies and sectors, and supports various forms of national and international co-operation and comparison.

Expertise: TLRP works to enhance capacity for all forms of research on teaching and learning, and for research-informed policy and practice.

Improvement: TLRP develops the knowledge base on teaching and learning and collaborates with users to transform this into effective policy and practice in the UK.

TLRP is managed by the Economic and Social Research Council research mission is to advance knowledge and to promote its use to enhance the quality of life, develop policy and practice and strengthen economic competitiveness. ESRC is guided by principles of quality, relevance and independence.