8. More about calculus in physics

This section is about physical quantities that change with time or change when a different quantity changes. Calculus is about the mathematics of rates of change (differentiation) and the cumulative effect of small changes (integration). In this section, we will be studying the mathematical processes of differentiation and integration and how we use them to form mathematical models of physical systems.

Part 1 Differentiation

Consider a physical variable y which depends on another physical variable x, for example the resistance of a metal wire depends on the temperature of the wire. Suppose y varies continuously with x as shown in Figure 1.

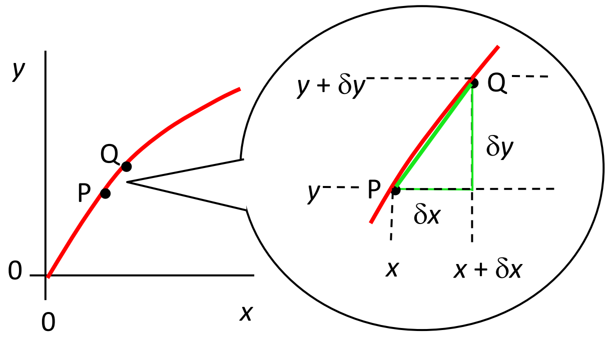

Figure 1

The rate of change of y with x at any point on the line in Figure 1 is given by the gradient of the line at that point. In Figure 1, the gradient decreases as x increases so the rate of change of y with x decreases as x increases. The gradient at any point is the tangent to the curve at that point.

Figure 1 shows a magnified view of two points P and Q that are close together. The coordinates of the two points are (x,y) for P and $(x + \delta x\,, y + \delta y)$ for Q where δx and δy are small changes of x and y. The gradient of the straight line PQ is equal to $\frac{\delta y}{\delta x}$. In the limit $\delta x \to 0$, Q $\to$P and the gradient of PQ becomes equal to the gradient of the tangent to the curve. Thus the rate of change of y with respect to x is equal to the limit of $\frac{\delta y}{\delta x}$ as $\delta x \to 0$. This is written as $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Where there is a mathematical equation for y in terms of x , the rate of change of y with respect to x can be determined by differentiating y with respect to x . Differentiation is a mathematical process with specific rules based on mathematical principles.

Differentiation of $y = x^2$

If $y = x^2$, then $y + \delta y = (x + \delta x)^2 = x^2 + 2x\delta x + \delta x^2$

so $\delta y = x^2 + 2x\delta x + \delta x^2 - y = 2x\delta x + \delta x^2$

Therefore $\frac{\delta y}{\delta x} = \frac{2x\delta x + \delta x^2}{\delta x} = 2x + \delta x$

Hence $\frac{dy}{dx} = \left(\frac{\delta y}{\delta x}\right)_{\delta x \to 0}$ = the limit of $2x + \delta x$ as $\delta x \to 0 = 2x$

Differentiation of $y = x^n$,

If $y = x^n$, then $y + \delta y = (x + \delta x)^n = x^n + nx^{n-1}\delta x + \frac{n(n-1)}{2}x^{n-2}\delta x^2$ + terms in higher powers of δx

so $\delta y = x^n + nx^{n-1}\delta x + \frac{n(n-1)}{2}x^{n-2}\delta x^2$ + + terms in higher powers of δx - y

$= x^n + nx^{n-1}\delta x + \frac{n(n-1)}{2}x^{n-2}\delta x^2 + \text{etc} - x^n$

$= nx^{n-1}\delta x + \frac{n(n-1)}{2}x^{n-2}\delta x^2 + \text{etc}$

Therefore $\frac{\delta y}{\delta x} = \frac{nx^{n-1}\delta x + \frac{n(n-1)}{2}x^{n-2}\delta x^2 + \text{ terms in higher powers of } \delta x}{\delta x}$

$= nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}\delta x +$ terms in higher powers of δx

Hence $\frac{dy}{dx} = \left(\frac{\delta y}{\delta x}\right)_{\delta x \to 0}$

= the limit as $\delta x \to 0$ of $nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}\delta x$ + terms in higher powers of δx

$= nx^{n-1}$

Notes ;

1.The expansion $(x + \delta x)^n$ generates a series of (n + 1) terms. The coefficient of the term in $x^{n-1}\delta x$ is always equal to n, as shown by the examples below.

$(x + \delta x)^2 = x^2 + 2x\delta x + \delta x^2$

$(x + \delta x)^3 = (x + \delta x)(x + \delta x)^2 = (x + \delta x)(x^2 + 2x\delta x + \delta x^2) = x^3 + 3x^2\delta x + 3x\delta x^2 + \delta x^2$

$(x + \delta x)^4 = (x + \delta x)(x + \delta x)^3 = (x + \delta x)(x^3 + 3x^2\delta x + 3x\delta x^2 + \delta x^2)$

$= x^4 + 4x^3\delta x + 6x^2\delta x^2 + 4x\delta x^3 + \delta x^4$

2.When x^n is subtracted from the expansion of $(x + \delta x)^n$ and the remainder is divided by δx, only the term n x^{n-1} remains in the limit $\delta x \to 0$

Differentiation of $y = e^{kx}$

As explained on p364 , for $y = e^x$, then $\frac{dy}{dx} = \mathbf{e}^x$

This is because the exponential function $e^x = 1 + x + x^2/2 + x^3/3{\times}2{\times}1 + x^4/4{\times}3{\times}2{\times}1 + \ldots$

Differentiating each term after the first term (1) in the above equation gives the preceding term so differentiating all the terms reproduces the same set of terms.

In other words, if y = $\mathbf{e}^{x}$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathbf{e}^{x}$

For $y = e^{kx}$, then let $z = kx$ *so* $y = e^z$

Therefore $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{dy}{dz} \times \frac{dz}{dx} = e^z \times k = k\, e^{kx}$

Differentiation of $y = \sin\theta$

As explained on p 310 and below, if $y = r \sin\theta$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = r\cos\theta$

Figure 2 - see Fig. 20.21 (p310 in textbook)

In the diagram, OP is a rotating phasor of length r which at time t is at angle θ to the x-axis. The y-coordinate of the tip of OP is therefore given by $y = r\sin\theta$

A short time δt later, OP is at angle $\theta + \delta\theta$ and the tip of OP has moved along an arc of length δs as shown in Figure 2. As a result, the y-coordinate has changed by δy where $\delta y = \delta s \cos\theta$.

Since $\delta s = r\,\delta\theta$ as shown in Figure 2, then $\delta y = r\cos\theta\ \delta\theta$

Hence $\frac{\mathrm{d}y}{\mathrm{d}\theta} = \left(\frac{\delta y}{\delta\theta}\right)_{\delta\theta \to 0} = \frac{(r\cos\theta)\,\delta\theta}{\delta\theta} = r\cos\theta$

Notes;

1.If $\theta = k\,x$ with $r = 1$ then $y = \sin kx$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x} = \cos\theta \times k = k\cos kx$

2. If $\theta = 2\pi f\,t$ with $r = 1$ then $y = \sin 2\pi f\,t$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \cos\theta \times 2\pi f = 2\pi f\ \cos 2\pi f\ t$

Differentiation of $y = \cos\theta$

In the right-angle triangle shown in Figure 3 , $\sin\theta = o/h$ and $\cos\theta = a/h$

Also, $\sin(90- \theta) = a/h$ and $\cos(90- \theta) = o/h$

Therefore $\sin\theta = \cos(90- \theta)$ and $\cos\theta = \sin(90- \theta)$

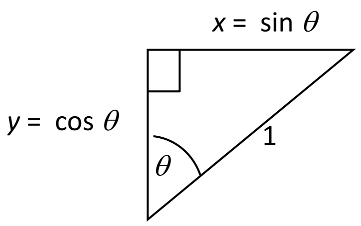

Figure 3 The right-angle triangle

If $y = \cos\theta$, then $\frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{\mathrm{d}(\sin(90-\theta))}{\mathrm{d}\theta} = -\frac{\mathrm{d}(\sin(90-\theta))}{\mathrm{d}(90-\theta)} = -\cos(90-\theta) = -\sin\theta$

Therefore $\frac{\mathrm{d}y}{\mathrm{d}\theta} = -\sin\theta$

Part 2 Integration

Consider a physical variable *y* which depends on another physical variable *x* and where their product is a physical quantity. For example, the tension *T* in a stretched rubber band depends on its extension *x* and the product $T\,\delta x$ is the work done δW to extend the rubber band by a small distance δx.

Figure 4 shows how the tension *T* in a rubber band varies with extension *x*.

- The area of a strip of width δx under the line of the graph represents the product $T\,\delta x$. Thus the area of the strip represents the work done δW to extend the rubber band by a small distance δx.
- The total area under the line of the graph from x_1 to x_2 is sum of all strips of width δx from x_1 to x_2. Thus the total area under the line of the graph from x_1 to x_2 represents the total work done to extend the rubber band from x_1 to x_2 .

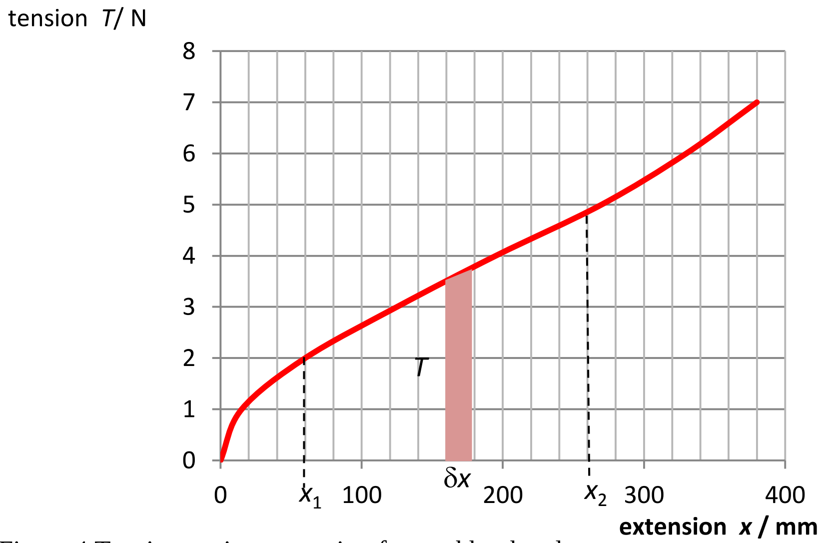

Figure 4 Tension against extension for a rubber band

The area under the line of a graph can be estimated or determined mathematically if the equation of the line is known.

1.**To estimate the area under the line**, count the number of graph grid squares under the line and multiplying that number by the physical quantity represented by 1 grid square. For example, in Figure 4, there are 33 grid squares under the line from x_1= 60 mm to x_2 = 260 mm and each square represents 20 mJ of work done (= 1 N × 20 mm). Therefore, the total work done to stretch the rubber band from extension x_1 to x_2 is 660 mJ.

The table below lists some further examples of graphs in which the area under the line has a physical significance.

y-axis	**x-axis**	**area**
force	distance	work done
velocity	time	distance travelled
force	time	change of momentum
current	time	charge flow
charge	potential	energy stored

2. To determine the area under the line mathematically, the equation for the line must be known. In other words, an equation for y in terms of x is needed. The area of a strip of width δx is therefore $y\ \delta x$. Let δS represent this area. The mathematical process of adding up all the strip areas from x_1 to x_2 is called **integration.** The mathematical sign for integration is the symbol $\int$.

Thus $\int_{x_1}^{x_2} y\ \mathrm{d}x$ represents the process of adding up all the strip areas from x_1 to x_2 . In other words, the total area from x_1 to x_2 = $\int_{x_1}^{x_2} dS = \int_{x_1}^{x_2} y\ \mathrm{d}x$.

Note Because $\delta S = y\ \delta x$ then $y = \frac{\mathrm{d}S}{\mathrm{d}x}$.

Therefore the total area from x_1 to x_2 $= \int_{x_1}^{x_2} y\ \mathrm{d}x = \int_{x_1}^{x_2} \frac{\mathrm{d}S}{\mathrm{d}x}\ \mathrm{d}x = \int_{x_1}^{x_2} \mathrm{d}S = S_{x_2} - S_{x_1}$

In Figure 1, $\int_{x_1}^{x_2} dW$ represents the process of adding up all the small changes of work done δW. So the total work done to extend the rubber band from x_1 to x_2 , $W = \int_{x_1}^{x_2} dW = \int_{x_1}^{x_2} T dx$. However, there isn't an equation for T in terms of x for a rubber band. So the estimation method must be used to find the work done.

However, for a stretched spring that obeys Hooke's law, $T = k\,x$ where k is the spring constant. So the work done to stretch a spring $W = \int_{x_1}^{x_2} dW = \int_{x_1}^{x_2} T dx = \int_{x_1}^{x_2} kx\ dx$.

Therefore $W = \int_{x_1}^{x_2} kx\ dx = k\int_{x_1}^{x_2} x\ dx$

Using the rules of differentiation from the previous section $\frac{d}{dx}\left(\frac{1}{2}x^2\right) = x$

so $\int_{x_1}^{x_2} x\,dx = \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{1}{2}x^2\right) dx = \left(\frac{1}{2}x_2{}^2\right) - \left(\frac{1}{2}x_1{}^2\right)$ as explained in the note above.

Therefore $W = \left(\frac{1}{2}kx_2{}^2\right) - \left(\frac{1}{2}kx_1{}^2\right)$

Some further examples of the use of integration in physics

1. Motion of an object moving with constant acceleration *a*

At any time, its rate of change of velocity $\frac{\mathrm{d}v}{\mathrm{d}t} = a$

Therefore if its initial velocity is *u* and its velocity at time *t* is ,

then $v - u = \int_{t_1=0}^{t_2=t} a\,dt = a\int_{t_1=0}^{t_2=t} dt = at$

Hence $\boldsymbol{v = u + at}$

2. Work done to escape from the surface of a spherical planet of radius *R*

Consider an object of mass *m* above the surface at distance *r* from the centre of a planet as shown in Figure 5.

Figure 5 Work done to escape from a planet – see Fig. 28.8 (b) (p446 in textbook)

Work must be done to move the object away from the planet . To increase its distance from the centre by δr, the amount of work that needs to be done $\delta W = F\,\delta r$ where *F* is the magnitude of the gravitational force on the object.

From Newton's Law of Gravitation (see topic 28.1) , $F = \frac{G\,Mm}{r^2}$ where *M* is the mass of the planet.

Hence $\delta W = F\,\delta r = \frac{G\,Mm}{r^2}\,\delta r$

Therefore the total work *W* that must be done to move the object from the surface to infinity is given by $\boldsymbol{W} = \int_{\boldsymbol{r=R}}^{\infty} \mathrm{d}W = \int_{\boldsymbol{r=R}}^{\infty} F\,\mathrm{d}r = \int_{\boldsymbol{r=R}}^{\infty} \frac{G\,Mm}{r^2}\,\mathrm{d}r$

Using the rules of differentiation from the previous section $\frac{d}{dr}\left(\frac{1}{r}\right) = \frac{d}{dr}\left(r^{-1}\right) = -\frac{1}{r^2}$

Therefore $\int_{\boldsymbol{r=R}}^{\infty} \frac{1}{r^2}\,\mathrm{d}r = \left[-r^{-1}\right]_{\boldsymbol{r=R}}^{\infty} = \frac{1}{R} =$

Hence $W = \int_{r=R}^{\infty} \frac{G\,Mm}{r^2}\,\mathrm{d}r = G\,Mm\int_{r=R}^{\infty} \frac{1}{r^2}\,\mathrm{d}r = \frac{G\,Mm}{R}$

Note; Because the potential energy of the object is zero at infinity and work done to escape from the surface $W = \frac{G\,Mm}{R}$, the potential energy of the object at the surface of the planet is equal to $-\frac{G\,Mm}{R}$. See p446 for a simplified version of the proof of this equation.

Part 3 Differential equations in physics

Differential equations are used in physics to form a mathematical model of a system. Such equations relate relevant physical variables and their rates of change to each other, as shown in the following examples;-

1.A sphere falling in a fluid (topic 4.2)

Figure 6 shows a spherical object of mass *m* falling in a fluid after being released at rest in the fluid. The object is acted on by the force of gravity (*mg*) downwards, the viscous drag of the fluid upwards and a constant upthrust *U* due to displacement of the fluid as explained in topic 8.8. The drag force *F* increases with speed *v* in accordance with the equation *F* = *kv* where k is a constant.

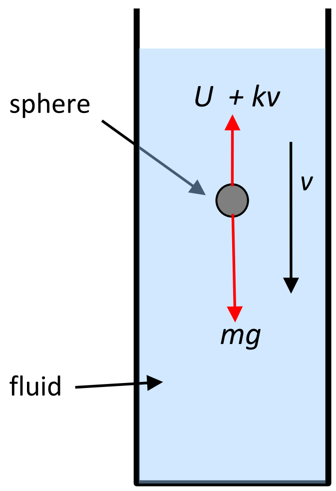

Figure 6 A sphere falling in a fluid

The resultant force on the object is therefore *mg* – *U* - *kv* . Therefore , the acceleration is not constant due to viscous drag and depends on velocity *v in* accordance with the equation *a* = *c* - *bv* where *c* = (*g* –*U*/*m*) and *b* = *k*/*m*

Therefore $\frac{\mathbf{d}\boldsymbol{v}}{\mathbf{d}\boldsymbol{t}} = \boldsymbol{c} - \boldsymbol{bv}$ which is an example of a **first-order differential equation**. Note that *c* and *b* are both constants in this situation

To solve this equation, substitute *y* for *c* - *bv* to give $\frac{\mathrm{d}y}{\mathrm{d}t} = -b\,y$.

Rearranging this equation gives $\frac{\mathrm{d}y}{y} = -b\ \mathrm{d}t$

Applying the process of integration to both sides gives $\int_{y_o}^{y} \frac{\mathrm{d}y}{y} = -b \int_{o}^{t} dt$

Hence $[\ln y]_{y_o}^{y} = -b\ [\ln t]_{o}^{t}$

Thus $\ln y - \ln y_o = -b\ t \quad so \quad \ln \left(\frac{y}{y_o}\right) = -bt$

Therefore $y = y_o e^{-bt}$ where y_o is the initial value of y.

To check this, differentiating both sides of the equation with respect to time gives

$\frac{\mathrm{d}y}{\mathrm{d}t} = -b\ y_o\ \mathrm{e}^{-bt} = -by$.

Therefore, since $y = c - bv$ then $y_o = c$ so the solution $y = y_o e^{-bt}$ becomes

$c - bv = c\ \mathrm{e}^{-bt}$.

Hence $\boldsymbol{v = c\ (1 - e^{-bt})\ /\ b}$ where $c = (g - U/m)$ and $b = k/m$

Figure 7 below shows how the speed v changes with time. Note that as time $t \rightarrow$ infinity, $e^{-bt} \rightarrow 0$ so speed $v \rightarrow c\ /\ b = (mg - U)\ /\ k$ which is therefore the terminal speed.

Figure 7 Terminal speed – see Fig. 4.10 (p58 in textbook)

2.An object undergoing simple harmonic motion.(topic 29.3)

Figure 8 shows an object of mass m on a spring at equilibrium and at an instant when it is oscillating vertically.

Figure 8 The oscillations of a loaded spring – see Fig 29.10 (a) and (b) (p459 in textbook)

When the object is displaced vertically from rest and then released, it oscillates vertically about its rest position. The tension T in the spring changes as the extension e of the spring changes in accordance with Hooke's Law $T = k\,e$

1. When the object is at rest as in Figure 8(a) , the tension in the spring at this position T_o = the weight of the mass (mg) . Hence $k\,e_o = mg$ where e_o is the extension of the spring from its natural length when it is at rest.

2. When the object is oscillating as in Figure 8(b),the resultant force F on it = $mg - T = T_o - T$ Since $T_o = k\,e_o$ and $T = k\,e$, then $T_o - T = k\,e_o - k\,e = -k\,(e - e_o)$

However, $e - e_o = s$, the displacement of the object from the rest position,

Hence the resultant force $F = T_o - T = -k\,(e - e_o) = -k\,s$

Therefore, the acceleration of the object $a = \frac{F}{m} = \frac{-ks}{m} = -w^2 s$ where □□□□□□ $\frac{k}{m}$

The acceleration is not constant and depends on displacement in accordance with the simple harmonic motion equation $a = -\omega^2 s$ where ω = the angular frequency of the oscillations.

Since $a = \frac{dv}{dt}$ and $v = \frac{ds}{dt}$, then $a = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$.

Hence the SHM equation above can be written as a **second- order differential equation**

$$\frac{d^2s}{dt^2} = -\omega^2 s \ .$$

This solution of this equation is $s = A \sin \omega t + B \cos \omega t$ where A and B are constants that depend on the initial values of displacement and velocity .

To check the solution above, since $\frac{d}{dt}(\sin \omega t) = \omega \cos \omega t$ and

$$\frac{d}{dt}(\cos \omega t) = -\omega \sin \omega t \quad \text{(see part 1)}$$

Differentiating both sides of the equation $s = A \sin \omega t + B \cos \omega t$ with respect to time t gives

$$\frac{ds}{dt} = A\omega \cos \omega t - B\omega \sin \omega t$$

Differentiating both sides of the equation for $\frac{ds}{dt}$ with respect to time t gives

$$\frac{d^2s}{dt^2} = -A\omega^2 \sin \omega t - B\omega^2 \cos \omega t$$

Hence $\frac{d^2s}{dt^2} = -\omega^2 s$

Note

1.The initial conditions determine the values of A and B. For example

- if $s = 0$ at $t = 0$ then B must be zero and therefore $s = A \sin \omega t$ where A is the maximum displacement (ie. the amplitude) of the oscillations. See Figure 9(a)
- if $\frac{ds}{dt} = 0$ at $t = 0$ then A must be zero and therefore $s = B \cos \omega t$ where B is the maximum displacement (ie. the amplitude) of the oscillations. See Figure 9(b)

Figure 9 Displacement – time graphs for an oscillating object (see Fig. 29.3 and 29.4, p454 in textbook)

(a) $s = A \sin \omega t$ (see Fig 29.3)
(b) $s = A \cos \omega t$ (see Fig 29.4 with the y-axis relabelled 'displacement' not 'velocity')

2. The above analysis assumes the drag force and the upthrust on the object are both negligible. If a drag force D is present and the upthrust is negligible, the resultant force F at

displacement s is given by the equation $F = -ks - D$. Therefore the acceleration is given by the equation $a = \frac{F}{m} = \frac{-ks-D}{m} = -\omega^2 s - (D/\text{m})$

For viscous drag , $D = kv$ where v is the velocity of the object and k is a constant.

Hence acceleration $a = -\omega^2 s - \left(\frac{k}{m}\right) v$

The variation of displacement s with time t may be determined using a spreadsheet or by solving the acceleration equation. The spreadsheet method is explained on p494-6 of the text book.

To solve the acceleration equation, substituting $\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$ for acceleration a and $\frac{\mathrm{d}s}{\mathrm{d}t}$ for velocity v gives the following second order differential equation

$\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = -\omega^2 s - \lambda \frac{\mathrm{d}s}{\mathrm{d}t}$ where $\lambda = \left(\frac{k}{m}\right)$

Rearranging this equation with all the terms on the left hand side gives

$$\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} - \lambda \frac{\mathrm{d}s}{\mathrm{d}t} - \omega^2 s = 0$$

For light damping , the viscous force is much less than the maximum restoring force (ie. λ is such that the λterm in the above equation is much less than the last term). In this situation , the solution of the above differential equation is $s = s_o e^{-\lambda t} \cos \omega t$.

Figure 10 shows a graph of the displacement against time for this solution. Notice that the effect of the viscous force is to make the amplitude smaller and smaller without affecting the time period. The above solution shows that the amplitude decreases exponentially.

Figure 10 Light damping – see Fig 29.14 (p462 in textbook)