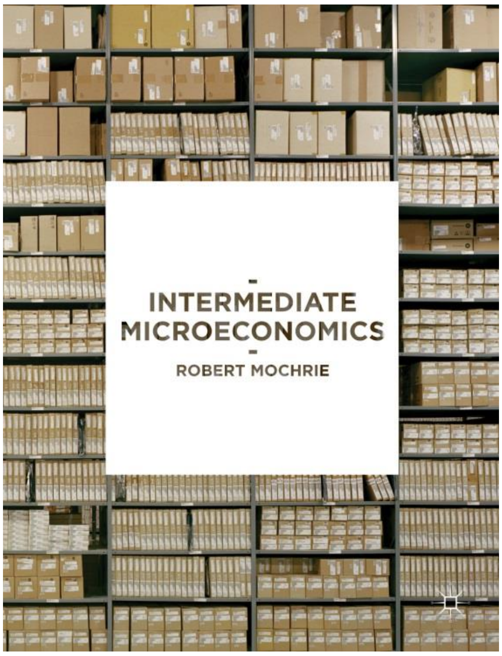
Solutions Manual: Part II

Resource allocation for people

Summary answers to the 'By yourself' questions



Chapter 3

X3.1 What are the opportunity costs of visiting a free public museum (such as the National Gallery or the British Museum if you are in the UK)?

Since there are no financial costs, the opportunity cost is mainly the time spent travelling plus the costs of travel that would not otherwise be incurred.

X3.2 What are the opportunity costs of completing a university degree?

The direct costs (in terms of fees) plus the cost of the effort spent working, which would otherwise have been available for leisure, plus the wages foregone from not being in employment (less the cost of effort required to obtain wages). We should not count the cost of financing consumption.

X3.3 What are the opportunity costs of reading this chapter?

Time and effort that could be used for other purposes.

X3.4 What is the opportunity cost of money?

Money is held to be spent on other goods and services – so while held as money, it is not financing consumption.

X3.5 Estimate the opportunity cost of a litre of milk in terms of litres of petrol.

Petrol is currently sold in the UK for about £1.12 per litre; milk is sold for about £0.45; the opportunity cost of 1 litre of petrol is approximately 2.5 litres of milk.

X3.6 For each of the following situations, sketch graphs showing the budget constraints and the affordable set.

a) **The price of a loaf of bread is £1.20; the price of cheese is £6.00 per kilogram; income is £30.**

The constraint is a downward sloping straight line. The affordable set is a triangle, bounded by the axes and the constraint. Showing consumption of bread (in loaves) on the horizontal axis and consumption of cheese (in kg) on the vertical axis, the intercept on the horizontal axis is (25, 0) and the intercept on the vertical axis is (0, 5).

b) **Prices are £1.50 and £9.00; income is £27.**

As for part a), but with the intercept on the horizontal axis at (18, 0) and on the vertical axis at (0, 3).

c) **Prices are £1.40 and £9.80; income is £39.20.**

As for part a), but with the intercept on the horizontal axis at (28, 0) and on the vertical axis at (0, 4).

X3.7 Consider these three cases:

(1) prices $p_b = 1.5$ and $p_c = 7.5$, with money available to finance consumption, $m = 30$;
(2) $p_b = 2$, $p_c = 12$, $m = 24$; and
(3) $p_b = 4$, $p_c = 16$, $m = 40$.

a) **Write down expressions for the acquisition cost, A, for bundle (b, c) and the budget constraint, C, for which $A(b, c) = m$.**

(1) Acquisition cost, A: $A(b, c) = 1.5b + 7.5c$; constraint, C: $A(b, c) = 30$; or $b + 5c = 20$.
(2) Acquisition cost, A: $A(b, c) = 2b + 12c$; constraint, C: $A(b, c) = 24$; or $b + 6c = 12$.
(3) Acquisition cost, A: $A(b, c) = 4b + 16c$; constraint, C: $A(b, c) = 40$; or $b + 4c = 10$.

b) **Sketch separate diagrams showing the budget set, B, and the budget constraint, C.**
Budget constraints downward sloping straight lines, and budget sets triangles bounded by the constraints and the axes.
(1) With consumption of bread measured on the horizontal axis and consumption of cheese measured on the horizontal axis, the intercepts of the constraint are (20, 0) and (0, 4).
(2) As in (1), but with the intercepts of the constraint at (12, 0) and (0, 2).
(3) As in (1), but with the intercepts of the constraint at (10, 0) and (0, 2.5).

c) **Confirm that that the budget sets in cases (2) and (3) are subsets of the budget set in case (1).**
The intercepts given by the budget constraint in case (2) lie closer to the origin than those in case (1); the budget constraint in case (2) therefore lies wholly within the affordable set for case (1). The same is true for the budget constraint in case (3).

X3.8 Show the effect on the budget sets in your diagrams of an increase in the money available to finance consumption, to: (a) $m_1 = 36$; (b) $m_1 = 40$; and (c) $m_1 = 60$.
The constraint shifts out, with the shape of the budget set remaining exactly the same, but with the constraint now intercepting the horizontal and vertical axes at (a) (24, 0) and (0, 4.8);
(b) (20, 0) and $\left(0, \frac{10}{3}\right)$*; and (c) (15, 0) and (0, 3.75).*

X3.9 Using the diagrams in X3.8, demonstrate that the acquisition function is convex.
A triangular affordable set is always convex. If we choose any two consumption bundles that are affordable then all consumption bundles that lie on the line segment joining them are also affordable, and indeed in the interior of the affordable set.
Choosing two consumption bundles on the boundary of the affordable set, then the line running between these bundles is the budget constraint, and all bundles on the constraint are just affordable.

X3.10 Suppose that a person with income m_1 only uses m_0 to finance consumption. What difficulties might this pose for calculating the opportunity cost of bread? How would your answer differ if income received today was also used to finance consumption tomorrow? What if goods other than bread and cheese could be chosen?
We define the opportunity cost of bread in terms of the amount of cheese that has to be given up in order to finance the consumption of more bread. Spending only m_0*, this person can increase consumption of both bread and cheese: there is no opportunity cost.*
If we allow spending to take place tomorrow as well as today, then the difference between income and expenditure will be savings, which can be used to finance expenditure tomorrow. The opportunity cost of expenditure today is then expenditure that has to be foregone tomorrow.
Lastly, if other goods are available, then if expenditure on these other goods is $m_1 – m_0$*, then all income is used to finance consumption, and the opportunity cost is properly defined.*

X3.11 In Figure 3.5b, we show a situation in which the price of a kilogram of cheese falls from p_c to $\frac{2}{3}p_c$. Calculate the effect:

a) **on someone who only buys bread;**
There is no change in the cost of the preferred bundle.

b) **on someone who only buys cheese;**
This person can now increase consumption of cheese by 50%.

c) **on the opportunity cost of bread.**
The opportunity cost of bread increases by 50%.

X3.12 In Figure 3.5c, we show a situation in which the money available to finance consumption, *m*, increases by 25%. Repeat X3.11.

a) Someone buying bread only can increase consumption by 25%.
b) Someone buying cheese only can increase consumption by 25%.
c) The opportunity price of bread does not change.

X3.13 Why might we consider that a price increase makes someone worse off, while a price fall makes someone better off?

With a price increase, consumption bundles that were previously affordable become unaffordable. If all money is being spent, then the consumption bundle that was originally purchased is one of those. Since well-being is derived from consumption in our model, the reduction in consumption after a price increase makes this person worse off.
Reversing the argument, with a price fall, consumption bundles that were previously unaffordable become affordable. If all money is being spent, then the consumption bundle that was originally purchased will lie in the interior of the new affordable set. Since well-being is derived from consumption in our model, the opportunity to increase consumption of all goods will make this person better off.

X3.14 Sketch graphs showing the budget constraints and the affordable set before and after prices and income change. State how the relative prices of bread and cheese change.

a) Initially, p_b = 1.20, p_c = 6.00, and *m* = 30. After price and income changes, p_b = 1.50, p_c = 6.00, and *m* = 36.

Measuring consumption of bread on the horizontal and consumption of cheese on the vertical axes, all budget constraints are downward sloping straight lines, with slope $-\frac{p_b}{p_c}$ equal to the opportunity cost of bread; intercept on the bread (horizontal) axis $\left(\frac{m}{p_b}, 0\right)$; and intercept on the cheese (vertical) axis $\left(0, \frac{m}{p_c}\right)$.

Here, initially, the relative price is 0.2, and the intercepts are at (25, 0) and (0, 5). After the price changes the relative price is 0.25 and the intercepts are at (24, 0) and (0, 6). The relative price of bread has increased.

b) Initially, p_b = 1.50, p_c = 9.00, and *m* = 27. After price and income changes, p_b = 1.80, p_c = 12.00, and *m* = 27.

Here, initially, the relative price is $\frac{1}{6}$, and the intercepts are at (18, 0) and (0, 3). After the price changes the relative price is 0.15 and the intercepts are at (15, 0) and (0, 2.25). The relative price of bread has fallen slightly (and the relative price of cheese has increased slightly.

c) Initially, p_b = 1.40, p_c = 9.80, and *m* = 39.20. After price and income changes, p_b = 1.20, p_c = 8.40, and *m* = 37.80.

Here, initially, the relative price is $\frac{1}{7}$, and the intercepts are at (28, 0) and (0, 4). After the price changes the relative price is $\frac{1}{7}$ and the intercepts are at (31.5, 0) and (0, 4.5). The relative price of bread is unchanged.

X3.15 Suppose that both the price of bread and the price of cheese, p_b and p_c, rise by 10%, and the amount of money available to finance consumption, *m*, also increases by 10%. Demonstrate that the budget set is the same before and after the price rise.

Initially, the budget constraint may be written $p_b b + p_c c = m$. After prices and money available to finance consumption, $1.1p_b b + 1.1p_c c = 1.1m$; but dividing through by the *common factor of 1.1, we obtain the equation of the original budget constraint. Since the boundary of the affordable set does not change, the whole affordable set also does not change.*

X3.16 Sketch diagrams showing the budget constraint for someone with the given endowments, given the opportunity cost of cheese:

a) Endowment: 12 loaves of bread plus 3 kg of cheese, where 1 kg of cheese can be traded for 3 loaves.

The budget constraint is a downward sloping straight line passing through the endowment point (12, 3) with slope $-\frac{1}{3}$. *Measuring consumption of bread on the horizontal axis and consumption of cheese on the vertical axis, the intercepts are at (21, 0) and (0, 7).*

b) Endowment: 20 loaves of bread plus 2 kg of cheese, where 1 kg of cheese can be traded for 8 loaves.

The budget constraint is a downward sloping straight line passing through the endowment point (20, 2) with slope $-\frac{1}{8}$. *Measuring consumption of bread on the horizontal axis and consumption of cheese on the vertical axis, the intercepts are at (36, 0) and (0, 4.5).*

c) Endowment: 18 loaves of bread plus 4 kg of cheese, where 1 kg of cheese can be traded for 6 loaves.

The budget constraint is a downward sloping straight line passing through the endowment point (18, 4) with slope $-\frac{1}{6}$. *Measuring consumption of bread on the horizontal axis and consumption of cheese on the vertical axis, the intercepts are at (42, 0) and (0, 7).*

X3.17 Sketch the budget constraint facing someone with an income of £84, who can buy up to 60 litres of petrol or 70 bottles of beer.

a. What are the prices of a litre of petrol and a bottle of beer?

The price of petrol, $p_r = \frac{m}{r} = \frac{84}{60}$ *= 1.4 (or £1.40); similarly the price of beer,* p_b *= 1.2 (or £1.20).*

b. Now suppose that anyone buying more than 40 l of petrol is given a 10% quantity discount. What is the discounted price of a litre of petrol?

£1.26.

i. How many bottles of beer can someone buying 40 l of petrol afford?

40 l of petrol costs £56, leaving £28 to buy beer. So this person can afford $23\frac{1}{3}$ *bottles of beer.*

ii. How much petrol can someone who buys no beer afford?

Buying no beer, this person might afford $66\frac{2}{3}$ *l of petrol at the discounted price of £1.26 per litre.*

iii. How many bottles of beer can someone buying 41 l of petrol afford?

41 l of petrol costs £51.66 at the discounted price. This leaves £32.34 to spend on beer, or 26.95 bottles.

c. Sketch a diagram showing the original budget constraint, and the new section of the budget constraint that reflects the discount. Why might it be very unusual for someone to buy 40 l of petrol?

Measuring consumption of petrol on the horizontal and consumption of beer on the vertical axis, the original budget constraint is a straight line, intercepting the horizontal (petrol) axis at 60 and the vertical (beer) axis at 70.

Allowing for the price discount the budget constraint becomes two line segments, one to the left of 40, and the other to the right of 40. The left hand segment is the original budget constraint. When consumption of petrol reaches 40l, the price discount is obtained, which means that the cost of petrol falls by £5.60 (from £56 to £50.40). This means that there would be £33.60 available to spend on beer, which pays for 28 bottles.

The second segment of the budget constraint therefore runs from (40, 28) to $\left(66\tfrac{2}{3}, 0\right)$.

X3.18 Reading books and going to a concert both require time and money. Suppose that a book costs £10 and takes six hours to read, while going to a concert costs £20 and takes 3 hours. For someone who is willing to spend £80 and 18 hours on these activities:

a) How many concerts can this person go to, without reading any books?

Able to afford to buy 4 tickets (but has time to go to 6 concerts).

b) How many books can this person read, without going to any concerts?

Able to buy up to 8 books, but only has time to read 3 of them.

c) Sketch the money and the time budget constraints for this person.

Measuring the consumption bundle with books on the horizontal and concert tickets on the vertical axis, the time budget runs from (3, 0) to (0, 6). The money budget runs from (8, 0) to (0, 4).

d) On your sketch, indicate the budget set.

Both time and money budgets have to be satisfied. The affordable set is therefore the area below both constraints.

X3.19 How would you interpret a negative price for a good? Sketch the budget constraint and budget set for consumption bundles with one good with a positive price and one with a negative price.

With a negative price people will only 'buy' the good if they are compensated for doing so. We might think of such a good as being harmful, or in some way bad, so that consumption reduces well-being.

The diagram might show the budget constraint as being upward sloping. Consumption of the good with a positive price is financed by the money received from accepting consumption of the good with a negative price. The affordable set therefore consists of all consumption bundles where the money received is no less than the expenditure.

Chapter 4

X4.1 Write out the weakly and strongly preferred sets for the alternatives A, B, C, D, E, and F, where A is weakly preferred to B, B is strongly preferred to C, which is at least as good as D, but better than E, and F is not (weakly) preferred to any other alternative.

We know from the above that B is strictly preferred to C, that D is strictly preferred to E and that E is strictly preferred to F. We cannot be certain that A is strictly preferred to B or that C is strictly preferred to D.

For the weakly preferred sets, we are certain that p(B) = {A, B}, p(D) = {A, B, C, D}, p(E) = {A, B, C, D, E}, and p(F) = {A, B, C, D, E, F}. p(A) = {A} if A is strictly preferred to B, but if this person is indifferent between A and B, p(A) = {A, B}.

For strictly preferred sets, we are certain that P(A) = ∅, P(C) = {A, B}, P(E) = {A, B, C, D}, and P(F) = {A, B, C, D, E}. P(B) = {A} if A is strictly preferred to B, but if this person is indifferent between A and B, P(A) = ∅. In the same way, P(D) = {A, B, C} if C is strictly preferred to D, but if this person is indifferent between C and D, P(D) = {A, B, C}.

X4.2 Three football fans have different preferences over the outcome of a game in which the team that they support, the Reds, play against their local rivals, the Blues.

- **i. Fan A simply cares about the outcome: a win is better than a draw, which is better than a defeat.**
- **ii. Fan B also cares about the goal difference, *d*, defined as the number of goals the Reds score, *r*, less the number of goals that the Blues score, *b*. [That is: *d* = *r* – *b*.]**
- **iii. Fan C cares about the number of goals that the Reds score, but also the goal difference, weighting them equally.**

a. For each fan, state the (weakly) preferred sets for:

i. a victory for the Reds, by one goal to nil;

For A, all victories are equally preferred, so the preferred set consists of all victories.

For B, all victories by a margin of one goal are equally preferred so the weakly preferred set also consists of all victories.

For C, an extra goal for the Reds is offset by two for the Blues, so that C is indifferent between 1 – 0, 2 – 2, 3 – 4, and so forth, so that 2r – b = 2. Any game with an outcome in which the Reds do better than that minimum level is strictly preferred.

ii. a draw in which each team scores two goals; and

For A, all draws are equally preferred, so the preferred set consists of all victories and draws.

For B, all draws are equally preferred so the weakly preferred set also consists of all victories and draws.

For C, an extra goal for the Reds is offset by two for the Blues, so that C is again indifferent between 1 – 0, 2 – 2, 3 – 4, and so forth, so that 2r – b = 2. Any game with an outcome in which the Reds do better than that minimum level is strictly preferred.

iii. a victory for the Blues by two goals to one.

For A, all defeats are equally preferred, so the preferred set consists of all games.

For B, all losses by a margin of one goal are equally preferred so the weakly preferred consists of all victories, all draws and all one goal defeats.

For C, an extra goal for the Reds is offset by two for the Blues, so that C is indifferent between 0 – 0, 1 – 2, 2 – 4, 3 – 6, and so forth, so that 2r = b. Any game with an outcome in which the Reds do better than that minimum level is strictly preferred.

b. Show that Fan C is indifferent between any pair of scores (r_0, b_0) and (r_1, b_1), for which $2(r_1 – r_0) = (b_1 – b_0)$.

$V(r_0, b_0) = r_0 + (r_0 – b_0) = 2r_0 – b_0$. $V(r_1, b_1) = r_1 + (r_1 – b_1) = 2r_1 – b_1$. In both cases, equal value is placed on goals scored by the Reds and the goal difference. For indifference between the outcomes, $V(r_0, b_0) = V(r_1, b_1)$, and the result follows.

c. Hence or otherwise, draw a diagram indicating clearly Fan C's (weakly) preferred set for a defeat by three goals to four.

For (r, b) = (3, 4), 2r – b = 2, so that the indifference set N = {(2, 0), (3, 4), (4, 6), . . .} The weakly preferred set P(3, 4) = {(r, b): $2r – b \geq 2$}

X4.3 Applying the strong preference operator, $\succ$, over the set of bundles C = {(*b*, *c*): *b, c* $\geq$ 0}, under what circumstances might the statements 'X $\succ$ Y' and 'Y $\succ$ X' both be false?

X $\succ$ Y means that X is strictly preferred to Y, while Y $\succ$ X means that Y is strictly preferred to X. Allowing for complete preferences, suppose that X and Y are considered equally good, so that the consumer is indifferent between the two consumption bundles; both statements are false.

X4.4 Show that if 'X $\succ$Y' is false, then '$X \preceq Y$' is true; while if '$X \succeq Y$' is true, then '$X \prec Y$' is false.

If X is not strictly preferred to Y, then either Y is strictly preferred to X or else X and Y are equally strongly preferred, so that Y is weakly preferred to X. Similarly if X is weakly preferred to Y, then either X is strictly preferred to Y or else X and Y are equally strongly preferred; and so Y is not strictly preferred to X.

X4.5 Consider the three consumption bundles, A: (b_A, c_A), B: (b_B, c_B) and C: (b_C, c_C), for which $A \succeq B$ and $B \succeq C$.

a) For there to be a unique ordering of the preferences, what must be true about the comparison of A and C?

A must be weakly preferred to C; $A \succeq C$

b) Sketch two diagrams, indicating the preferred sets of C and B, drawn in such a way that:

(1) in one A must lie in the preferred sets of both B and C; and

Draw the diagram so that the indifference curves through B and C never intersect, and with bundle B lying in the preferred set of C, and bundle A lying in the preferred set of B.

(2) in the other A lies in the preferred set of B, but not in the preferred set of C.

Draw the diagram so that the indifference curves through B and C intersect, but so that B lies in C's preferred set; then if A lies in the region between the indifference curves but on the other side of the indifference curve through C to B, then A is preferred to B, B to C and C to A.

X4.6 Show that if for consumption bundles X, Y and Z, $X \succeq Y$, $Y \succeq Z$, and $Z \succeq X$, preferences can only be transitive if all three bundles lie on the same indifference curve.

From these expressions, we see that X lies in the (weakly) preferred set of Y; Y in the preferred set of Z and Z in the preferred set of X. If preferences are transitive, then preferred sets will be nested. Suppose that Z is strictly preferred to X. Then Y must also be strictly preferred to X, and we have a contradiction. So Z must be ranked equally with both X and Y.

X4.7 Elena strictly prefers consumption bundle A to bundle B, is indifferent between B and C, strictly prefers C to D, and is indifferent between A and D. Confirm that Elena's preferences

violate transitivity; and that in a diagram the indifference curve passing through bundles A and D and the one passing through bundle B and C must intersect.
Completing successive pairwise comparisons, Elena strictly prefers A to B, and so A to C, and also A to D. But we are told that she is indifferent between A and D. This contradiction confirms that transitivity does not hold. In drawing a diagram, we require A and D to be on one indifference curve, and B and C both to be on a second indifference curve. If A lies in B's preferred set and C lies in D's preferred set, then there has to be an intersection of the indifference curves between the pairs of points on each indifference curve.

X4.8 Suppose Fedor claims that bundle X lies in the strictly preferred set of Y, but Y lies in the strictly preferred set of X. We persuade him firstly to exchange Y for X (and some money) and then X for Y (and some more money), so that he finishes with the original consumption bundle, but less money. What do you think would happen to someone with Fedor's preferences over time?
Fedor would systematically lose money. He would have to conclude that his preferences lead him to be exploited by other people, and avoid situations where this happens.

X4.9 Assume that Gabrielle faces much the same situation as Daniel, having a fixed amount of money, m, which she can use to buy a consumption bundle (b, c), consisting of quantities b of bread and c of cheese, with unit prices, p_b and p_c. Gabrielle's preferences are well behaved.

a) On a diagram, sketch Gabrielle's affordable set.
We show consumption of bread on the horizontal axis and consumption of cheese on the vertical axis. Gabrielle's affordable set is shown as a triangle with intersection on the horizontal axis at $\left(\frac{m}{p_b}, 0\right)$ *and intersection on the vertical axis at* $\left(0, \frac{m}{p_c}\right)$*. The boundary of the affordable set is the budget constraint, a line whose equation is* $p_b b + p_c c = m$*. The slope of the affordable set is the relative price,* $-\frac{p_b}{p_c}$*.*

b) Assume that Gabrielle chooses a consumption bundle M: (b_M, c_M), which lies on her budget constraint, but for which the marginal rate of substitution is greater than the relative price. Illustrate such a point on your diagram as the intersection of the budget constraint and an indifference curve.
The indifference curve is steeper than the budget constraint at M.

c) Confirm that Gabrielle can afford to buy consumption bundles in the preferred set, P(M).
Since the indifference curve through M is steeper than the budget constraint, there will be a region below and to the right of M that lies between the indifference curve and the budget constraint. In this area, consumption bundles are both affordable and preferred to M.

d) Confirm that Gabrielle can also buy consumption bundles on the indifference curve, I(M), for which the acquisition cost $A(b, c) < A(b_M, c_M)$.
This follows from the argument in part c).

e) Which do you think Gabrielle should do: maintain expenditure and buy a bundle that she prefers to M, as in (c); or reduce expenditure while buying a bundle in the set, I(M)?
We expect Gabrielle to maintain expenditure so that she acquires a consumption bundle in the preferred set of M.

X4.10 Suppose that Hanna faces an exactly similar problem to Gabrielle, except that she chooses a consumption bundle L:(b_L, c_L) that lies on an indifference curve for which the marginal rate of substitution is less than the relative price. Repeat X4.9.

We continue to show consumption of bread on the horizontal axis and consumption of cheese on the vertical axis. Hanna's affordable set is again a triangle with intersection on the horizontal axis at $\left(\frac{m}{p_b}, 0\right)$ *and intersection on the vertical axis at* $\left(0, \frac{m}{p_c}\right)$. *The boundary of her affordable set is the budget constraint, with equation,* $p_b b + p_c c = m$. *This is a straight line with slope* $-\frac{p_b}{p_c}$.

At point L on the diagram, the indifference curve is flatter than the budget constraint. This means that there is an area above and to the left of L in which consumption bundles are preferred to L, but they are also affordable. Hanna can reduce expenditure and buy a consumption bundle that she considers to be as good as L. We expect her, though, to maintain expenditure, choosing a consumption bundle that lies on the budget constraint and in the preferred set of L.

(It may be useful to draw sketches in these exercises using one colour of ink for indifference curves and another for budget constraints, and to shade preferred and affordable sets using pencil.)

X4.11 Sketch a diagram similar to Figure 4.7a, in which indifference curves take the form of nested closed loops. Choose a point within the most-preferred set (the smallest loop), and label it B. This is the bliss point.

a) Draw a budget constraint that touches one of the closed loops at consumption bundle K between the origin and the bliss point, B. Shade the preferred set P(K) and the affordable set bounded by the budget constraint that passes through K.

i) Identify clearly the intersection of the budget set and the preferred set P(K).

The affordable set should be drawn as a triangle, bounded by the axes and the budget constraint, with K a point on the constraint. The preferred set is the area bounded by the closed loop through K.

ii) What is the most-preferred, affordable bundle in this case?

Since K lies on the budget constraint and the indifference curve passing through K is convex, then K is the most-preferred, affordable consumption bundle.

iii) Does this consumer spend all the money available?

Yes.

b) Repeat exercise (a), but this time drawing a second budget constraint that touches the preferred set, P(K), further away from the origin than the bliss point, B.

i) Explain why the preferred set is the same as before, and that the affordable set has become larger.

The new consumption bundle just touches the indifference curve through K, so it has the same preferred set.

ii) Identify clearly the intersection of the new budget set and the preferred set P(K).

The whole of the preferred set lies within the affordable set, so the preferred set is the intersection of sets.

iii) What is the most-preferred, affordable bundle in this case?

The bliss point.

c) In this situation, does the consumer spend all the money available? Explain why this might occur, concentrating on any differences between parts (a) and (b).

When the bliss point is not in the affordable set, the consumer spends all the money available. Once the bliss point is affordable, it is always the most preferred bundle. In this case there is an ideal consumption bundle. Reaching this, the assumption of monotonicity has to be set aside.

X4.12 Sketch a diagram similar to Figure 4.7b, in which indifference curves take the form of nested open loops. From a point, A, on the upward-sloping segment of the outermost indifference curve, extend a straight line that intersects the indifference curves that bound the more preferred sets.

a) Explain why consumption bundle A can never be the most-preferred, affordable set.
Where the indifference curve is upward-sloping, it is always possible to reduce expenditure and obtain the same utility.

b) Draw a budget constraint that just touches the innermost indifference curve at point M.

i) Sketch the affordable set and the preferred set.
The affordable set is the triangle formed by the axes and the downward sloping line through M.

ii) Identify the intersection of the preferred set and the affordable set, and hence the most-preferred affordable bundle.
M is the unique point of intersection, and so the most-preferred, affordable bundle.

iii) Why might we always expect the most-preferred, affordable bundle to be found on the downward-sloping section of an indifference curve?
For the most-preferred, affordable consumption bundle, we require the marginal rate of substitution (the rate at which the consumer will give up consumption of one good for more of another) and the relative price (or opportunity cost) to be equal. By this definition, the opportunity cost is always the slope of the budget constraint, and with prices greater than zero, it will be strictly negative. This ensures that the MRS is also negative. Interpreting the MRS as the slope of the indifference curve through that consumption bundle, at the most-preferred, affordable consumption bundle the indifference curve will be downward sloping.

c) Why might we say that in the upper segment of these indifference curves, where the indifference curves are upward-sloping but flattening out, the consumer's appetite for cheese has been satiated?
At the point where the indifference curve becomes vertical, the MRS is no longer defined: no increase in the consumption of cheese can compensate for further reductions in consumption of bread. This means that there is satiation. In the region where the indifference curve is upward-sloping, but becoming flatter, increasing consumption of cheese requires additional consumption of bread for the consumer to feel indifferent to the change. Again, this is consistent with satiation of consumption of cheese.

X4.13 Sketch a diagram similar to Figure 4.7c, in which indifference curves take the form of nested curves that become increasingly steep.

a) Draw a budget constraint that just touches the indifference curve closest to the origin at point T: (b_T, c_T): $b_T, c_T > 0$. Sketch the affordable set bounded by the budget constraint and the preferred set of T, indicating clearly the intersection of the affordable and preferred sets.
Affordable set is the triangle formed by a downward sloping line through T and the two axes. Preferred set is the area above and bounded by the indifference curve.

b) Explain why T is not the most-preferred, affordable bundle in this situation.
With a point of intersection between the indifference curve and the budget constraint, we can find a region bounded by the indifference curve and the budget constraint in which all consumption bundles are affordable and preferred to T.

c) Indicate what you consider to be the most-preferred affordable bundle by the letter V, giving reasons for your choice.
V will be a point on one or other axis; defined so that it there intersects an indifference curve as far away from the origin as possible.

d) Draw another diagram with a set of indifference curves that have a similar shape to the ones that you have just used but that do not have such a steep slope at the intersection on the bread axis. Draw a budget constraint that intersects the indifference curve closest to the origin on the bread axis and that is steeper than the indifference curve.

i) At this intersection, which is larger: the marginal rate of substitution or the relative price?
The relative price is greater.

ii) If your budget constraint intersects other indifference curves in your map, at each intersection which is larger: the marginal rate of substitution or the relative price?
The relative price is always greater.

iii) What do you expect will be the most-preferred affordable consumption bundle in this case?
The intersection of the budget constraint with the cheese axis. Since MRS is less than the relative price, the consumer can always do better by substituting cheese from bread.

iv) Suppose we consider a society in which cheese has recently been invented, so that until now only bread has been consumed. Predict consumer responses to this new good. What would have to happen for consumers to start buying cheese?
Consumers will switch from bread only to cheese only; but they have to know about the new good.

X4.14 Sketch a diagram similar to Figure 4.7d, in which indifference curves take the form of upward-sloping, nested curves. Draw in a budget constraint, and indicate the affordable set.

a) Label the intersection of the budget set with the cheese axis, R. Sketch the preferred set of R (it may be necessary to add to the diagram an indifference curve passing through R).
Preferred set of R is the area above and to the left of the indifference curve through R.

b) Identify the intersection of the affordable set and the preferred set of R.
The intersection of sets consists of R only.

c) Show that the consumer will always prefer a bundle with less bread if the quantity of cheese is held constant. Hence or otherwise, explain our claim that in this situation, 'Bread is bad.'
Holding consumption of cheese constant, reducing the consumption of bread, we move to the left in the diagram, and so into the preferred set of the original consumption bundle. It follows that the less cheese that there is in the consumption bundle, the more strongly preferred it will be.

Chapter 5

X5.1 Using the concept of utility, evaluate the claim that we can increase total utility by transferring wealth from rich people to poor people. (You may find it useful to think of a rich miser, such as Ebenezer Scrooge in Charles Dickens's *A Christmas Carol*, and someone who is voluntarily poor, such as St Francis of Assisi, who renounced his family's wealth.)

The argument would rest on two assumptions: firstly that utility is increasing in total wealth, but at a decreasing rate; and secondly that every person's utility function is similar.

We treat utility functions as a representation of preferences. Such functions are unique up to an ordinal transformation. We cannot assume that the proposed restrictions on utility functions will hold. Indeed, it is possible that for a miser, utility will increase with wealth at an increasing rate. For someone who chooses voluntary poverty, it is not even clear that utility would increase with wealth.

X5.2 Suppose that my boss calls me into his office and offers to double my salary. Giving reasons, explain whether or not you agree with these statements:

a) 'I prefer having the higher salary to my existing salary.'

This simply means that I have not achieved my optimal income, and wish to earn more money.

b) 'My utility will double if I accept this offer.'

This would be consistent with my utility being equal to my salary. Were this the case, we would not need to use utility in place of income.

c) 'My utility will increase by at least 50 points if I accept this offer.'

This would only make sense if there were some common scale of utility that would allow for interpersonal comparisons.

X5.3 Confirm that Expression 5.3 is true, that is, if $U(Z_1) = U(Z_2)$, $V(Z_1) = V(Z_2)$.

From Expression 5.3, $U(Z_1) \geq U(Z_2) \Leftrightarrow V(Z_1) \geq V(Z_2)$.

If $U(Z_1) = U(Z_2)$, *then* $V(Z_1) \geq V(Z_2)$.

But we can also reverse terms in the statement. If $V(Z_1) = V(Z_2)$ *then* $U(Z_1) \geq U(Z_2)$.

If both of these statements are to be true, then $U(Z_1) = U(Z_2)$, $V(Z_1) = V(Z_2)$.

X5.4 Suppose that $v[U(Z)]$ is increasing in $U(Z)$, so that if $U(Z_1) \geq U(Z_2)$, $V(Z_1) \geq V(Z_2)$. Confirm that Expression 5.3 is true.

If $v[U(Z)]$ *is increasing in* $U(Z)$, *then* $u: V(Z) \rightarrow u[V(Z)]$ *is increasing in V. The conclusion follows in the same way as in the previous exercise.*

X5.5 Define three bundles: Z_1: (6, 4); Z_2: (3, 7); and Z_3: (5, 5). Frances' preference ordering is: $Z_2 \succ Z_3 \succ Z_1$. She assigns the bundles utility scores: $U(6, 4) = 10$; $U(3, 7) = 15$; and $U(5, 5) = 12$. Confirm that her ranking of the bundles remains the same after the transformations: (1) $v[U(Z)] = 1 + U(Z) + [U(Z)]^2$; and (2) $w[U(Z)] = [1 + U(Z)]^{0.5}$. (You will need a calculator to obtain values in the second transformation.)

Z	*(6, 4)*	*(3, 7)*	*(5, 5)*
U(Z)	*10*	*15*	*12*
V(Z)	*111*	*241*	*157*
W(Z)	*3.32*	*4*	*3.61*

In all three cases, Frances ranks the bundles in the same order.

X5.6 Suppose that we obtain the following utility values for bundles A, B, C, D and E:

Bundle Z	A	B	C	D	E
Utility, U(Z)	1	2	3	4	5

For each of the following rules:

a. Calculate the utility values for each bundle under the new rule.

b. Calculate the difference in the utility values between bundles A and B, B and C, C and D, and D and E.

c. Decide whether or not the relation appears to be monotonically increasing:

i. $v(U) = 1 + U$; ii. $v(U) = U^2$ iii. $v(U) = U^2 + U$

iv. $v(U) = U^2 - 4U$ v. $v(U) = U^{0.5}$ vi. $v(U) = U^{-1}$

vii. $v(U) = \ln U$

Bundle Z	*A*	*B*	*C*	*D*	*E*
Utility, U(Z)	*1*	*2*	*3*	*4*	*5*
i.	*2*	*3*	*4*	*5*	*6*
ii.	*1*	*4*	*9*	*16*	*25*
iii.	*2*	*6*	*12*	*20*	*30*
iv.	*-3*	*-4*	*-3*	*0*	*5*
v.	*1*	*1.41*	*1.73*	*2*	*2.24*
vi.	*1*	*0.5*	*0.33*	*0.25*	*0.2*
vii.	*0*	*0.69*	*1.10*	*1.39*	*1.61*

Except for transformations iv. and vi., all appear to be monotonically increasing.

X5.7 Being able to assign a larger utility number to a consumption bundle requires the underlying preferences to be transitive. Demonstrate that if for bundles A, B and C, U(A) ≥ U(B) and U(B) ≥ U(C), then $A \succeq C$.

If U(A) ≥ U(B) and U(B) ≥ U(C), then U(A) ≥ U(C). The result follows immediately from the definition of the utility function.

X5.8 Heidi currently intends to purchase and consume bundle X: (b_X, c_X), where b_X is the quantity of bread, and c_X the quantity of cheese in the bundle. She prefers bundle X_1: ($b_X + \delta b$, c_X). What do you conclude about Heidi's marginal utility of bread, MU_B? Why does she not intend to purchase bundle X_1?

We conclude that Heidi's marginal utility of bread is greater than zero and that she cannot afford to buy bundle X_1.

X5.9 Explain why, if Heidi's preferences are monotonically increasing, she would report marginal utilities, MU_B and MU_C, that would always be greater than zero.

Suppose otherwise. Then increasing consumption of one good, Heidi would report decreasing total utility from consumption, and that would indicate that she preferred not to increase consumption, so that her preferences would not be monotonically increasing.

X5.10 Suppose that Heidi reports that her marginal utility of bread, MU_B is constant, but that her marginal utility of cheese, MU_C, decreases with consumption.

a) Draw a diagram to show Heidi's current consumption bundle, X: (6, 3), and her current marginal rate of substitution, $MRS(6, 3) = -1$.

In the diagram, we show consumption of bread on the horizontal axis and consumption of cheese on the vertical axis. Indicating point X on the diagram, the indifference curve through that point has a slope of -1.

b) Suppose that Heidi reduces her consumption of bread and increases her consumption of cheese, while maintaining her current level of utility. Confirm that her MRS will increase.

Consuming less bread, but more cheese, Heidi moves up the indifference curve. With only the marginal utility of bread decreasing, the MRS, increases, and the indifference curve becomes steeper.

c) **Repeat (b), but showing that MRS will decrease following an increase in bread consumption and a decrease in cheese consumption.**
Consuming more bread, but less cheese, Heidi moves down the indifference curve. With well-behaved preferences, the indifference curve becomes flatter, and so its slope, the MRS, decreases.

d) **What do you conclude about the curvature of this indifference curve?**
The indifference curve has to be convex to the origin.

X5.11 Repeat X5.10, but for Isabel, who reports that her marginal utility of bread, MU_B, is decreasing, but that her marginal utility of cheese, MU_C, increases with consumption.
Consuming less bread, but more cheese, Isabel moves up the indifference curve. Thinking of successive small changes in utility resulting from a reduction in the consumption of bread, then for a specified reduction in utility, the reduction in consumption of bread becomes smaller. However, the associated increase in utility from consumption of cheese will become progressively larger. The indifference curve is therefore convex.

X5.12 Repeat X5.10, but for Jiang, who reports that both of her marginal utilities, MU_B and MU_C, decrease with consumption.
Consuming less bread, but more cheese, Jiang moves up the indifference curve. Thinking of successive small changes in utility resulting from a reduction in the consumption of bread, then for a specified reduction in utility, the reduction in consumption of bread becomes smaller. However, the associated increase in consumption of cheese will become larger. The indifference curve is convex.

X5.13 For the linear utility function given in Expression 5.8, assume that the quantity of cheese in the bundle remains constant.

a. **Show that as b increases by one unit from 1 to 2, utility U increases by w_b units.**
$U(b, c) = w_b b + w_c c$; $U(1, c) = w_b + w_c c$; $U(2, c) = 2w_b + w_c c$; $\Delta U = w_b$.

b. **Show that as b increases by ten units from 20 to 30, utility U increases by $10w_b$ units.**
$U(b, c) = w_b b + w_c c$; $U(20, c) = 20w_b + w_c c$; $U(30, c) = 30w_b + w_c c$; $\Delta U = 10w_b$.

c. **Show that as b increases by k units, from b_0 to $b_1 = b_0 + k$, utility U increases by kw_b units.**
$U(b, c) = w_b b + w_c c$; $U(b_0, c) = w_b b_0 + w_c c$; $U(b_1, c) = w_b(b_0 + k) + w_c c$; $\Delta U = kw_b$.

d. **Hence or otherwise, explain why the marginal utility of bread is w_b and the marginal utility of cheese is w_c.**
Increasing consumption of bread, the rate of increase of utility $\frac{\partial U}{\partial b} = w_b$*; and in the same way* $\frac{\partial U}{\partial c} = w_c$*. By definition, these partial derivatives are the marginal utilities of bread and cheese.*

X5.14 The marginal rate of substitution is the ratio of marginal utilities.

a. **Find an expression for the marginal rate of substitution, and confirm that it is the same for all consumption bundles.**
MRS = $-\frac{MU_b}{MU_c} = -\frac{w_b}{w_c}$*, and this is constant since the marginal utilities are both constant.*

b. Show that all indifference curves are straight lines.
For every consumption bundle, the marginal rate of substitution is illustrated graphically as the gradient of the indifference curve passing through the consumption bundle. There is a constant MRS, irrespective of levels of consumption. A curve with a constant gradient is a straight line.

c. Sketch a preference map showing at least three distinct indifference curves.
Measuring consumption of bread on the horizontal and consumption of cheese on the vertical axes, the indifference curves will be three downward sloping, parallel straight lines.

X5.15 Suppose that in Expression 5.10, v_b = 2 and v_c = 9.

a) Calculate the total utility obtained from the bundles X = (9, 2), Y = (18, 3), and Z = (45, 12).
U(9, 2) = 18; U(18, 3) = 27; U(45, 12) = 90

b) For each of these three bundles, what would be the increase in utility from adding to the consumption bundle:

i) one loaf of bread?
0, 0, 2

ii) 1 kg of cheese? [*Note:* Adding cheese is separate from, not consecutive to, adding bread.]
0, 9, 0

c) Beginning from bundle X, explain how utility changes as:

i) more and more bread is added to the consumption bundle (while the amount of cheese is held constant);
Utility remains constant.

ii) more and more cheese is added to the consumption bundle (while the amount of bread is held constant).
Utility remains constant.

d) Using your answers to the previous questions:

i) Sketch the indifference curve passing through X.

ii) Sketch a preference map showing the indifference curves on which bundles X, Y and Z lie.
Showing quantity of bread on the horizontal axis and quantity of cheese on the vertical axis, the indifference curve, U = 18, is L shaped, with its vertex at X.
The other two indifference curves are U = 27 and U = 90; both L shaped, with vertices at (13.5, 3) and (45, 10).

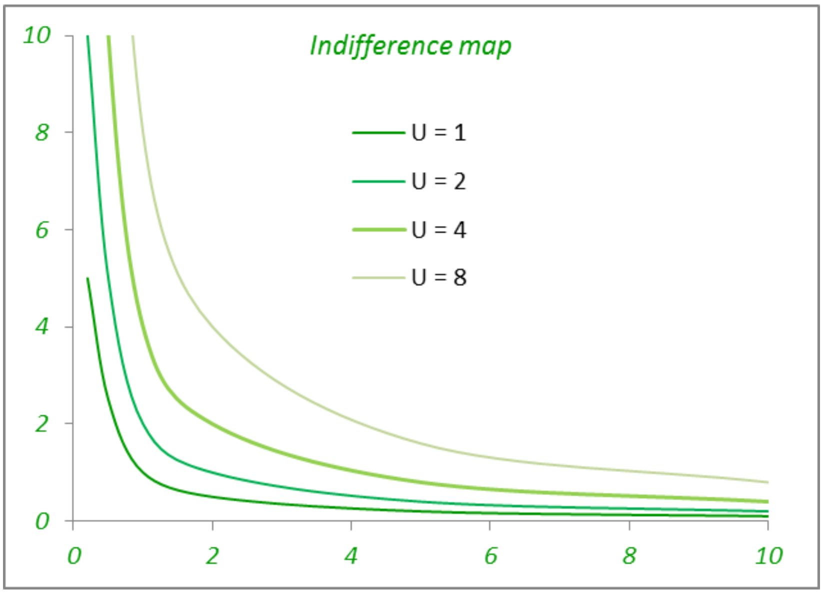
X5.16 The goal is to replicate Figure 5.5. Given the utility function, *U* = *bc*, we shall sketch the indifference curves for which *U* = 1, *U* = 2, *U* = 4 and *U* = 8.

a) Rearrange the expression *U* = *bc*, so that *c* is the subject.
c = U/b

b) Complete the following table:

b	0.2	0.5	1	2	5	10
c(*b*) [*U* = 1]	*5*	*2*	*1*	*0.5*	*0.2*	*0.1*
c(*b*) [*U* = 2]	*10*	*4*	*2*	*1*	*0.4*	*0.2*
c(*b*) [*U* = 4]	*20*	*8*	*4*	*2*	*0.8*	*0.4*
c(*b*) [*U* = 8]	*40*	*16*	*8*	*4*	*1.6*	*0.8*

c) Show each of these points on a diagram, only showing consumption bundles for which *b* and *c* are no more than 10.



d) Sketch the indifference curves by joining together the points identified on each curve.

X5.17 We now replicate Figure 5.5, but for the utility function, $U = b^{½} + c^{½}$.

a) Rearrange the expression, $U = b^{½} + c^{½}$, so that *c* is the subject.

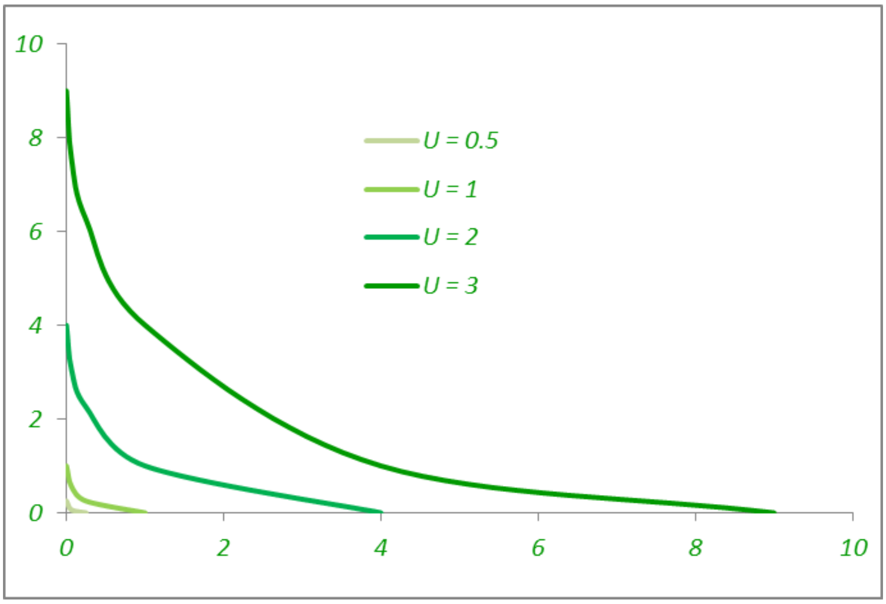
$c = (U - b^{0.5})^2$

b) Complete the following table:

b	0	0.0625	0.25	1	4	9
c(*b*) [*U* = 0.5]	0.25	0.0625	0			
c(*b*) [*U* = 1]	1	0.5625	0.25	0		
c(*b*) [*U* = 2]	4	3.0625	2.25	1	0	
c(*b*) [*U* = 3]	9	7.5625	6.25	4	1	0

c) Show each of these points on a diagram.

d) Sketch the indifference curves by joining together the points identified on each curve.



X5.18 For the utility function, $U(b,c)=\left(\frac{1}{b}+\frac{1}{c}\right)^{-1}$, confirm that:

a) The utility function can be rewritten $U(b,c)=\frac{bc}{b+c}$.

$$U(b,c)=\left(\frac{1}{b}+\frac{1}{c}\right)^{-1}=\left(\frac{b+c}{bc}\right)^{-1}=\frac{bc}{b+c}$$

b) The consumption bundle, (1, 1) lies on the indifference curve, $\frac{bc}{b+c}=0.5$.

Evaluating, bc = 1, b + c = 2, so $\frac{bc}{b+c}=0.5$.

c) The indifference curve can be expressed in explicit form, $c=\frac{b}{2b-1}$.

From part b), 2bc = b + c, so (2b – 1)c = b. The result follows by dividing through by 2b – 1.

d) If *b* = 0.5, then it is impossible to evaluate this expression for *c*.

If b = 0.5, then 2b – 1 = 0, so we cannot evaluate the fraction.

e) If *b* < 0.5, then *c* < 0; and we disregard the consumption bundle.

If b < 0.5, then 2b – 1 < 0. Since we require c > 0, this consumption bundle cannot be obtained.

f) As *b* → ∞, *c* → 0.5; which is to say, when *b* takes larger and larger values, *c* will become closer and closer to 0.5.

As b $\to \infty$, $c=\frac{b}{2b-1}=\frac{1}{2-\frac{1}{b}}\to\frac{1}{2}$ *since* $b^{-1}\to 0$.

X5.19 Economists frequently refer to utility as being an *ordinal* rather than a *cardinal* concept because it is only possible to interpret the ranking of utility values, not their absolute value.

a) Suppose that the amount of money that Arun has doubles. Would it be reasonable to suppose that his utility from consumption doubles?

No – were it to do so, then this would simply mean that Arun treated utility as identical to wealth.

b) Would it be reasonable to suppose that his utility from consumption increases but does not double?
Unless Arun is completely satiated, we would expect him to increase utility upon receiving more income. The statement is correct.

c) Critically assess the statement, 'Money is more valuable to the poor than to the rich, so we should redistribute income from the rich to the poor.'
This requires us to be able to make inter-personal comparisons of utility. It may be true that there is diminishing marginal utility of wealth. It is not necessarily the case.

X5.20 We have said that sufficient conditions for a utility function to be well behaved are that the marginal utilities are positive but decreasing. We can show that for the function $U(x, y) = x^{0.5} + y^{0.5}$, the marginal utility $MU_x(x, y) = 0.5x^{-0.5}$ and the marginal utility $MU_y(x, y) = 0.5y^{-0.5}$.

a) Confirm that the marginal rate of substitution $MRS = -(y/x)^{0.5}$.
$MRS = -\frac{MU_X}{MU_Y} = -\frac{0.5x^{-0.5}}{0.5y^{-0.5}} = -\left(\frac{y}{x}\right)^{0.5}$

b) Confirm that the consumption bundles (4, 0), (1, 1) and (0, 4) all generate utility 2.
$U(4, 0) = 4^{0.5} + 0^{0.5} = 2$; $U(1, 1) = 1^{0.5} + 1^{0.5} = 2$; $U(0, 4) = 4^{0.5} + 0^{0.5} = 2$

c) Confirm that the marginal rate of substitution for the three bundles in part (b) takes the values 0, –1 and undefined.
$MRS(4, 0) = -\left(\frac{0}{4}\right)^{0.5} = 0$; $MRS(1, 1) = -\left(\frac{1}{1}\right)^{0.5} = -1$ $MRS(0, 4) = -\left(\frac{4}{0}\right)^{0.5}$. *Since it is not possible to define a quotient where the divisor is zero, MRS is undefined.*

d) Repeat the exercise, sketching indifference curves that pass through the bundles (0.25, 0.25) and (4, 4).
$MU_X(0.25, 0.25) = MU_Y(0.25, 0.25) = 0.5*0.25^{-0.5} = 1$. So $MRS(0.25, 0.25) = -1$
$MU_X(4, 4) = MU_Y(4, 4) = 0.5*4^{-0.5} = 0.25$. So $MRS(4, 4) = -1$
The indifference curves have the shape of the curves found in X5.17. Vertical where they meet the vertical axis; horizontal where they meet the vertical axis and with slope -1 where x = y.

e) Confirm that the marginal rate of substitution on all three indifference curves that you have sketched have a slope of –0.5 where they meet the line $y = x/4$, but a slope of 2 where they meet the line $y = 4x$.
If $y = 0.25x$, $MRS(x, y) = -\left(\frac{y}{x}\right)^{0.5} = -\left(\frac{0.25x}{x}\right)^{0.5} = -0.5$. *If* $y = 4x$, $MRS(x, y) = -\left(\frac{y}{x}\right)^{0.5} = -\left(\frac{4x}{x}\right)^{0.5} = -2$.

X5.21 The utility function $U(s, t) = s^2t^2$ has associated marginal utilities $MU_s(s, t) = 2t^2s$ and $MU_t(s, t) = 2s^2t$.

a) Show that if $t = 1$, then MU_s is increasing in s; and if $s = 2$, MU_t is increasing in t.
If $t = 1$, $MU_s = 2s$; *as s increases, so does 2s. If* $s = 2$, $MU_t = 8t$; *as t increases so does 8t.*

b) Find the marginal rate of substitution for this utility function.
$MRS = -\frac{MU_s}{MU_t} = -\frac{2t^2s}{2s^2t} = -\frac{t}{s}$

c) **Confirm that the indifference curve associated with utility level 1 passes through the bundle (1, 1) and that there it has gradient *MRS* = –1.**
$U(1, 1) = 1^2.1^2 = 1$; $MRS(1, 1) = -\frac{1}{1} = -1$.

d) **Which, if either, of the following statements is true?**
i. **The marginal utility of good S is always decreasing for utility functions that represent well-behaved preferences.**
We have seen that this is not true.

ii. **The marginal rate of substitution is always decreasing for utility functions that represent well-behaved preferences.**
When preferences are well behaved, indifference curves are downward sloping and become flatter moving from left to right. Interpreting the MRS as the slope of the indifference curve, the statement is true.

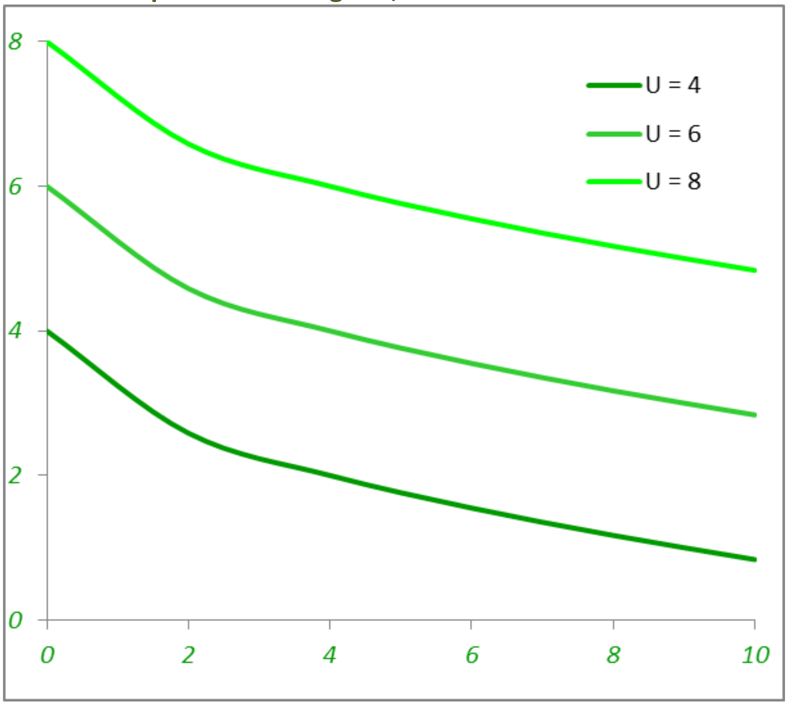
X5.22 Consider a rather different utility function from the ones that we have seen already:
$$U(b, c) = b^{½} + c.$$

a) **Obtain expressions for the indifference curves *U* = 4, *U* = 6 and *U* = 8 in terms of the variable *c*.**
$-c = U - b^{½}$

b) **Complete the following table:**

b	**0**	**2**	**4**	**6**	**8**	**10**
***c*(*b*) [*U* = 4]**	4	2.59	2	1.55	1.17	0.84
***c*(*b*) [*U* = 6]**	6	4.59	4	3.55	3.17	2.84
***c*(*b*) [*U* = 8]**	8	6.59	6	5.55	5.17	4.84

c) **Show these points on a diagram, and construct the indifference curves.**



d) Given that $MU_B = ½b^{-½}$ and $MU_C = 1$, calculate *MRS*. What do you conclude about the slope of the indifference curves, *U* = 4, *U* = 6 and *U* = 8 when *b* = 4? Can you generalize your answer for the slopes of any pair of indifference curves and any level of consumption, *b*?

MRS = $-\frac{MU_B}{MU_C} = -0.5b^{-0.5}$. *When b = 4, MRS = $-0.5*4^{-0.5}$ = -0.25. This is independent of the value of c. We note that all we need to know to calculate the marginal rate of substitution is the value of b. From this, we infer that every pair of indifference curves will have the same shape, the vertical distance between them constant moving along these.*

Chapter 6

X6.1 **Suppose that Juliet is currently planning to buy the bundle, Z_0: (b_0, c_0). She considers adding a quantity δb to the bundle, creating a new bundle, Z_δ: ($b_0 + \delta b$, c_0). Given her utility function U: $U(b, c) = bc$:**

a) **Calculate the values of $U(b_0, c_0)$ and $U(b_0 + \delta b, c_0)$.**
$U(b_0, c_0) = b_0c_0$; $U(b_0 + \delta b, c_0) = (b_0 + \delta b)c_0$

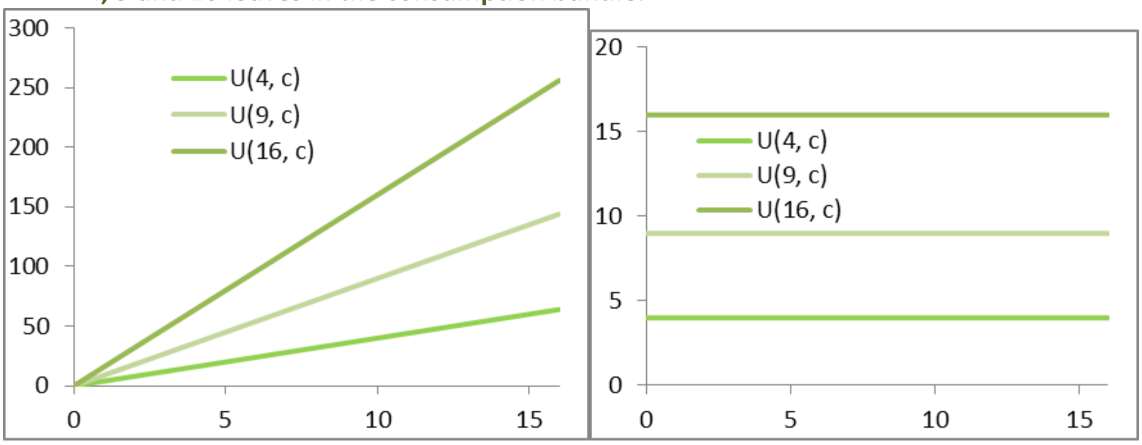
b) **Calculate the change in utility $\delta U = U(b_0 + \delta b, c_0) – U(b_0, c_0)$.**
$\delta U = c_0\delta b$

c) **Confirm that the rate of change of utility $MU_B(b_0, c_0) = c_0$.**
$MU_B = \frac{\delta U}{\delta b} = c_0$

X6.2 **Repeat X6.1, but assume that Juliet is now planning to buy the bundle, Z: (b, c), and thinking about the effect of increasing consumption of cheese by an amount δc.**
$U(b_0, c_0) = b_0c_0$; $U(b_0, c_0 + \delta c) = (c_0 + \delta c)b_0$; so $\delta U = b_0\delta c$; and $MU_C = \frac{\delta U}{\delta c} = b_0$

X6.3 **By applying the rules of (partial) differentiation, confirm that** $MU_B(b,c) = U_b(b,c) = \frac{\partial U}{\partial b} = c$ **and that** $MU_C(b,c) = U_c(b,c) = \frac{\partial U}{\partial c} = b$.
These results follow directly from the product rule for functions of two variables.

X6.4 **Sketch graphs showing the total utility, and the marginal utility, of cheese when there are 4, 9 and 16 loaves in the consumption bundle.**



X6.5 **Given his utility function, V: $V(b, c) = b^2c^2$, Karl is considering the value of the bundle, Z_0: (b_0, c_0). He considers adding a quantity δb to the bundle, which would create a new bundle, Z_δ: ($b_0 + \delta b$, c_0).**

a) **Calculate the values of $V(b_0, c_0)$ and $V(b_0 + \delta b, c_0)$.**
$V(b_0, c_0) = b_0^2c_0^2$ and $V(b_0 + \delta b, c_0) = (b_0 + \delta b)^2c_0$

b) **Show that the change in utility $\delta V = V(b_0 + \delta b, c_0) – V(b_0, c_0) = [2b_0.\delta b + (\delta b)^2]c_0^2$.**
$\delta V = V(b_0 + \delta b, c_0) – V(b_0, c_0) = (b_0 + \delta b)^2c_0 – b_0^2c_0^2 = [b_0^2 - 2b_0.\delta b + (\delta b)^2 - b_0^2]c_0^2 = [2b_0.\delta b + (\delta b)^2]c_0^2$.

c) **Confirm that the rate of change of utility $MU_B(b_0, c_0) = \lim_{\delta b \to 0} \frac{\delta V}{\delta b} = 2c_0^2b_0$.**

$\frac{\delta V}{\delta b} = [2b_0 + (\delta b)]c_0^2$ *As* $\delta b \to 0$, $\frac{\delta V}{\delta b} \to [2b_0]c_0^2$

X6.6 Repeat X6.5, but assume that Karl is now planning to buy the bundle Z: (b, c) and thinking about the effect of increasing consumption of cheese by an amount δc.

$V(b_0, c_0) = b_0^2c_0^2$ and $V(b_0, c_0 + \delta c) = (c_0 + \delta c)^2 b_0$.

Then $\delta V = V(b_0, c_0 + \delta c) - V(b_0, c_0) = (c_0 + \delta c)^2 b_0 - b_0^2c_0^2 = [c_0^2 - 2c_0.\delta c + (\delta c)^2 - c_0^2]b_0^2 = [2c_0.\delta c + (\delta c)^2]b_0^2$.

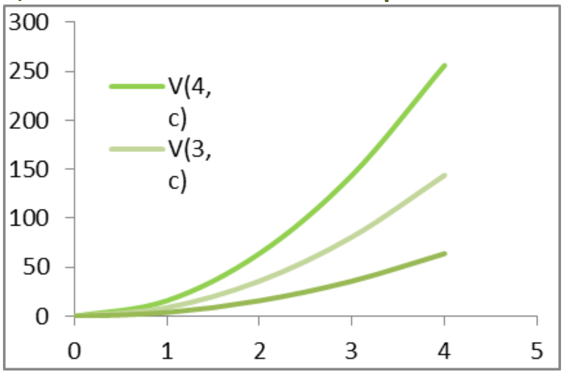
$\frac{\delta V}{\delta c} = [2c_0 + (\delta c)]b_0^2$ *As* $\delta c \to 0$, $\frac{\delta V}{\delta c} \to [2c_0]b_0^2$

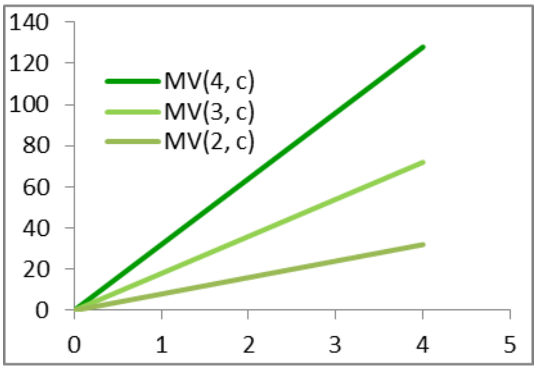
X6.7 By applying the rules of (partial) differentiation, confirm that:

$MU_B(b,c) = V_b(b,c) = \frac{\partial V}{\partial b} = 2c^2b$ **and that** $MU_C(b,c) = V_c(b,c) = \frac{\partial V}{\partial c} = 2b^2c$.

This follows immediately from application of the power rule of differentiation.

X6.8 Sketch graphs showing the total utility, and the marginal utility, of cheese when there are 2, 3 and 4 loaves in the consumption bundle.





X6.9 Applying the rules of differentiation, obtain the marginal utilities of bread and cheese for the following utility functions, U:

a. $U(b, c) = k_bb + k_cc$ **b.** $U(b, c) = b^{0.5}c^{0.5}$ **c.** $U(b, c) = b^{0.5} + c^{0.5}$

d. $U(b, c) = (b^{0.5} + c^{0.5})^2$ **e.** $U(b,c) = \frac{bc}{b+c}$

a. $MU_B = k_b$, $MU_C = k_c$ *b.* $MU_B = 0.5b^{-0.5}c^{0.5}$; $MU_C = 0.5b^{0.5}c^{-0.5}$

c. $MU_B = 0.5b^{-0.5}$, $MU_C = 0.5c^{-0.5}$ *d.* $MU_B = b^{-0.5}(b^{0.5} + c^{0.5})$, $MU_C = c^{-0.5}(b^{0.5} + c^{0.5})$

e. $MU_B = \left(\frac{c}{b+c}\right)^2$, $MU_C = \left(\frac{b}{b+c}\right)^2$

X6.10 Confirm that $U(b, c) > 0$, if $b, c > 0$; and that $\frac{dV}{dU} = 2U > 0$ if $U > 0$. Defining the marginal rate of substitution, $MRS(b,c) = -\frac{MU_B(b,c)}{MU_C(b,c)}$, show that for the functions U and V:

$$MRS(b,c) = -\frac{c}{b} \qquad [6.10]$$

$U(b, c) = bc > 0$ since $b, c > 0$. $V(b, c) = [U(b, c)]^2$, so by power rule, $\frac{dV}{dU} = 2U > 0$.

$MU_B = c$; $MU_C = b$; $MV_B = 2c^2b$; $MV_C = 2b^2c$; By substitution, we see that $MRS(b, c) = -\frac{c}{b}$

X6.11 Calculate the degree of homogeneity of the utility functions, U:

a. $U(b, c) = k_bb + k_cc$ **b.** $U(b, c) = b^{0.5}c^{0.5}$ **c.** $U(b, c) = b^{0.5} + c^{0.5}$

d. $U(b, c) = (b^{0.5} + c^{0.5})^2$ **e.** $U(b,c) = \frac{bc}{b+c}$.

A function, U, is homogeneous of degree r if $U(tb, tc) = t^rU(b, c)$.

a. $U(tb, tc) = k_btb + k_ctc = t(k_bb + k_cc) = tU(b, c)$. *So U is HOD 1.*

b. $U(tb, tc) = (tb)^{0.5}(tc)^{0.5} = t^{(0.5+0.5)}(b^{0.5}c^{0.5}) = t(b^{0.5}c^{0.5}) = tU(b, c)$. *So U is HOD 1.*

c. $U(tb, tc) = (tb)^{0.5} + (tc)^{0.5} = t^{0.5}(b^{0.5} + c^{0.5}) = t^{0.5}U(b, c)$. *So U is HOD 0.5.*

d. $U(tb, tc) = [(tb)^{0.5} + (tc)^{0.5}]^2 = (t^{0.5})^2 (b^{0.5} + c^{0.5})^2 = tU(b, c)$. *So U is HOD 1.*

e. $U(tb,tc)=\frac{(tb)(tc)}{tb+tc}=\frac{t^2(bc)}{t(b+c)}$ = tU(b, c). So U is HOD 1.

X6.12 Obtain expressions for MRS in all five cases above, where possible writing each as a function of the ratio $\frac{b}{c}$.

a. $MU_B = k_b$; $MU_C = k_c$; so MRS = $-\frac{MU_B}{MU_C}=-\frac{k_b}{k_c}$.

b. $MU_B = 0.5b^{-0.5}c^{0.5}$; $MU_C = 0.5b^{0.5}c^{-0.5}$; so MRS = $-\frac{MU_B}{MU_C}=-\frac{b^{-0.5}c^{0.5}}{b^{0.5}c^{-0.5}}=-\frac{c}{b}$.

c. $MU_B = 0.5b^{-0.5}$; $MU_C = 0.5c^{-0.5}$; so MRS = $-\frac{MU_B}{MU_C}=-\frac{b^{-0.5}}{c^{-0.5}}=-\left(\frac{c}{b}\right)^{0.5}$.

d. $MU_B = b^{-0.5}(b^{0.5}+c^{0.5})$; $MU_C = c^{-0.5}(b^{0.5}+c^{0.5})$; so MRS = $-\frac{MU_B}{MU_C}=-\frac{b^{-0.5}(b^{0.5}+c^{0.5})}{c^{-0.5}(b^{0.5}+c^{0.5})}=-\left(\frac{c}{b}\right)^{0.5}$

e. $MU_B = \frac{c}{b+c}-\frac{bc}{(b+c)^2}=\frac{c(b+c)-bc}{(b+c)^2}=\left(\frac{c}{b+c}\right)^2$; $MU_C = \frac{b}{b+c}-\frac{bc}{(b+c)^2}=\frac{b(b+c)-bc}{(b+c)^2}=\left(\frac{b}{b+c}\right)^2$; MRS $-\frac{MU_B}{MU_C}=-\left(\frac{c}{b}\right)^2$

X6.13 We have confirmed that the functions *U*: *U*(*b*, *c*) = *bc* and *V*: *V*(*b*, *c*) = b^2c^2 represent the same preferences.

a) Confirm that the function *W*: *W*(*b*, *c*) = $b^{0.5}c^{0.5}$ also represents these preferences.

Since W(b, c) = $[U(b, c)]^{0.5}$ and $\frac{dW}{dU}=0.5U^{-0.5}$ > 0, W is a monotonically increasing transformation of U; so W, U represent the same preferences.

b) Confirm that the partial derivative V_b is increasing in *b* while V_c is increasing in *c*, that U_b is independent of *b* and U_c independent of *c*, and that W_b is decreasing in *b* and W_c is decreasing in *c*.

$V = b^2c^2$; $\frac{\partial V}{\partial b}=2c^2b$; so $\frac{\partial^2V}{\partial b^2}=2c^2>0$. Similarly, $\frac{\partial V}{\partial c}=2b^2c$; so $\frac{\partial^2V}{\partial c^2}=2b^2>0$.

$U = bc$; $\frac{\partial U}{\partial b}=c$; so $\frac{\partial^2U}{\partial b^2}=0$. Similarly, $\frac{\partial U}{\partial c}=b$; so $\frac{\partial^2U}{\partial c^2}=0$.

$W = b^{0.5}c^{0.5}$; $\frac{\partial W}{\partial b}=0.5c^{0.5}b^{-0.5}$; so $\frac{\partial^2W}{\partial b^2}=-0.25c^{0.5}b^{-1.5}<0$. Similarly,.

$\frac{\partial W}{\partial c}=0.5b^{0.5}c^{-0.5}$; so $\frac{\partial^2W}{\partial c^2}=-0.25b^{0.5}c^{-1.5}<0$

X6.14 Even though it is often convenient to do so, why can we not simply assume that marginal utilities will always be decreasing?

We have seen from the example above that we can represent a single preference relation with utility functions that exhibit increasing, decreasing and constant marginal utilities.

X6.15 Confirm that the utility function in Expression 6.14 is homogeneous of degree 1, and that the marginal utilities, MU_B and MU_C, are decreasing.

A function, U, is homogeneous of degree r if U(tb, tc) = $t^rU(b, c)$.

For U: U(b, c) = $b^{0.25}c^{0.75}$, U(tb, tc) = $(tb)^{0.25}(tc)^{0.75}$ = $t^{(0.25+0.75)}b^{0.25}c^{0.75}$ = tU(b, c). So U is HOD 1.

$MU_B = 0.25b^{-0.75}c^{0.75}$; $MU_C = 0.75b^{0.25}c^{-0.25}$; $\frac{\partial MU_B}{\partial b}=-0.1875b^{-1.75}c^{0.75}$; $\frac{\partial MU_C}{\partial c}=-0.1875b^{0.25}c^{-1.25}$

X6.16 Suppose that a consumer has utility *U* = $b^{0.5}c^{0.5}$. For each of the following situations:

a. Obtain the marginal utilities MU_b and MU_c and the marginal rate of substitution.

$MU_B = 0.5b^{-0.5}c^{0.5}$; $MU_C = 0.5b^{0.5}c^{-0.5}$; MRS = $-\frac{c}{b}$.

b. For each of the following situations:

i. Obtain the relative price of the goods.

ii. Find the income expansion path.

iii. Find the most-preferred, affordable consumption bundle:

i. a consumer has an income *m* = 60, and faces prices p_b = 2 and p_c = 3;

ii. a consumer has an income m = 84 and faces prices p_b = 6 and p_c = 7;

iii. a consumer has an income m = 144 and faces prices p_b = 9 and p_c = 16.

i. a consumer has an income m = 60, and faces prices p_b = 2 and p_c = 3;

- *require MRS = $-\frac{c}{b} = -\frac{p_b}{p_c} = -\frac{2}{3}$, so that the income expansion path may be written 3c = 2b; and require the budget constraint 2b + 3c = 60 to be satisfied. Then substituting for 3c, 4b = 60, b = 15, c = 10.*

ii. a consumer has an income m = 84 and faces prices p_b = 6 and p_c = 7;

- *require MRS = $-\frac{c}{b} = -\frac{p_b}{p_c} = -\frac{6}{7}$, so that the income expansion path may be written 7c = 6b; and require the budget constraint 6b + 7c = 84 to be satisfied. Then substituting for 7c, 12b = 84, b = 7, c = 6.*

iii. a consumer has an income m = 144 and faces prices p_b = 9 and p_c = 16.

- *require MRS = $-\frac{c}{b} = -\frac{p_b}{p_c} = -\frac{9}{16}$, so that the income expansion path may be written 16c = 9b; and require the budget constraint 9b + 16c = 144 to be satisfied. Then substituting for 16c, 18b = 144, b = 8, c = 4.5.*

X6.17 Now suppose that the consumer has utility $U = b^{0.5} + c^{0.5}$. Repeat X6.16, but for the following situations:

$MU_B = 0.5b^{-0.5}$; $MU_C = 0.5c^{-0.5}$; MRS = $-\left(\frac{c}{b}\right)^{0.5}$.

a) A consumer has an income m = 60, and faces prices p_b = 2 and p_c = 3;

We require MRS = $-\left(\frac{c}{b}\right)^{0.5} = -\frac{p_b}{p_c} = -\frac{2}{3}$, so that the income expansion path may be written 9c = 4b; and require the budget constraint 2b + 3c = 60 to be satisfied. Then substituting for 3c, $\left(2+\frac{4}{3}\right)b = 60$; so $\frac{10}{3}b = 60$; and b = 18, c = 8.

b) A consumer has an income m = 100 and faces prices p_b = 4 and p_c = 6;

We require MRS = $-\left(\frac{c}{b}\right)^{0.5} = -\frac{p_b}{p_c} = -\frac{2}{3}$, so that the income expansion path may be written 9c = 4b; and require the budget constraint 2b + 3c = 50 to be satisfied. Then substituting for 3c, $\left(2+\frac{4}{3}\right)b = 50$; so $\frac{10}{3}b = 50$; and b = 15, c = $\frac{20}{3}$.

c) A consumer has an income m = 144 and faces prices p_b = 3 and p_c = 9.

We require MRS = $-\left(\frac{c}{b}\right)^{0.5} = -\frac{p_b}{p_c} = -\frac{1}{3}$, so that the income expansion path may be written 9c = b; and require the budget constraint 3b + 9c = 144 to be satisfied. Then substituting for 3b, 36c = 144, and c = 4, b = 36.

X6.18 Confirm that the utility functions, U: $U(b, c) = b^{0.25}c^{0.75}$ and V: $V(b, c) = c^3b$, represent the same preferences, by showing that: (1) V is a monotonically increasing transformation of U; and (2) $MRS(b, c)$ is the same when calculated using functions U and V.

It is easy to confirm that V = v(U): v(U) = U^4. Differentiating, $\frac{dV}{dU} = 4U^3$ > if U > 0.

Calculating marginal utilities, $MU_B = \frac{\partial U}{\partial b} = 0.25b^{-0.75}c^{0.75}$; $MU_C = \frac{\partial U}{\partial c} = 0.75b^{0.25}c^{-0.25}$; and $MV_B = \frac{\partial V}{\partial b} = c^3$; $MV_C = \frac{\partial V}{\partial c} = 3c^2b$. Calculating the ratios of these marginal products, MRS = $-\frac{MU_B}{MU_C} = -\frac{0.25b^{-0.75}c^{0.75}}{0.75b^{0.25}c^{-0.25}} = -\frac{c}{3b}$; but $-\frac{MV_B}{MV_C} = -\frac{c^3}{3c^2b} = -\frac{c}{3b}$; so that the marginal rate of substitution is the ratio of marginal utilities for the same function. This is sufficient to confirm that a single preference ordering is represented by both utility functions.

X6.19 We have stated that there are five assumptions necessary for preferences to be well behaved: (i) completeness; (ii) reflexivity; (iii) transitivity; (iv) monotonicity; and (v)

convexity. For each assumption, state why it is essential, and sketch a diagram showing indifference curves in which that assumption, and that assumption only, is violated.
Completeness – it has to be possible to compare every pair of consumption bundles. We can show this in a diagram in which there is a region through which no indifference curves pass.
Reflexivity – a consumption bundle has to belong to its own (weakly preferred set). In a diagram, the indifference curve of a consumption bundle specifically excludes that.
Transitivity – successive applications of the preference relation have to be consistent. This property would be violated if indifference curves were to cross.
Non-satiation – more of any good is always better. This property is inconsistent with upward-sloping indifference curves.
Convexity of preferences – linear combinations of any pair of consumption bundles in a preferred set also lie in the preferred set. This property is violated whenever indifference become steeper moving from left to right.

X6.20 Consider the following situation. Geoff's utility function is U: $U(b, c) = b^2 + c^2$ and he tries to use the rules that we have set out in this chapter to confirm that the utility-maximizing choice is the one that is predicted by the process in this chapter. Geoff reports that he tried the consumption bundle predicted, but found it much less satisfying than the one he chose without trying to use the rules.

a) Write down the equation of the indifference curve, $U = 1$.
$b^2 + c^2 = 1$

b) Show that the marginal rate of substitution $MRS = -b/c$.
$MRS = -\frac{MU_B}{MU_C} = -\frac{2b}{2c} = -\frac{b}{c}$.

c) Show that on the indifference curve, $\frac{d^2c}{db^2} < 0$, so that the indifference curve is concave.
We can write the indifference curve as $c = (1 - b^2)^{0.5}$. *Differentiating,* $\frac{dc}{db} = -b(1-b^2)^{-0.5} < 0$, *and so* $\frac{d^2c}{db^2} = -\left\{\frac{1}{(1-b^2)^{0.5}} + b\left(\frac{b}{(1-b^2)^{1.5}}\right)\right\} = -(1-b^2)^{-1.5} < 0$, *so that the indifference curve is concave.*

d) Sketch the indifference curve and explain why it should be that if $p_B < p_C$ then Geoff will spend all his money on good B.
The indifference curve is a quarter circle, centred on the origin, radius 1. If $p_B < p_C$, *then the constant acquisition cost line passing through (b, c) = (1, 0) lies below the indifference curve, and intersects the vertical axis below 1. The most preferred affordable consumption bundle is then (1, 0).*

e) Of the assumptions about preferences introduced in this chapter, which do not apply to Geoff's preferences?
Geoff's preferences are not convex.

X6.21 Sketch the following diagram, which represents Helga's preferences over goods B and C. We measure the quantity of good B on the horizontal axis and the quantity of good C on the vertical axis. Now draw an upward sloping straight line starting from the origin. Above the line, every indifference curve is vertical, while to the right of the line, every indifference curve is horizontal. (Every indifference curve is formed of two segments, one vertical, and one horizontal, which meet on the straight line that you have drawn.)

The indifference curves consist of a set of L shaped curves, each of which has its vertex on the upward-sloping line.

a) **Choose a consumption bundle to the right of the upward sloping straight line. Explain the effect on utility of increasing the quantity of increasing: (i) the quantity of good B in Helga's consumption bundle; and (ii) the quantity of good C.**
Increasing quantity of B has no effect on utility; increasing quantity of C does increase utility.

b) **Repeat part (a) for a consumption bundle that lies above the line.**
Here, increasing quantity of C has no effect on utility, whereas increasing quantity of B increases it.

c) **Now sketch a downward sloping line representing a constant acquisition cost given that Helga has an amount of money *m* to spend. Choose the line so that it just touches an indifference curve at its vertex. We have argued that where a constant acquisition cost line just touches an indifference curve, Helga cannot reallocate resources and increase the utility from consumption. Confirm that in an affordable consumption bundle it is impossible to increase either the quantity of good B or else the quantity of good C and increase utility.**
The affordability constraint has been drawn so that the preferred set is the region above and to the right of the point where it meets the indifference curve. The indifference curve is the boundary of the preferred set.

d) **Discuss whether or not the assumptions in X6.19 – (i) completeness; (ii) reflexivity; (iii) transitivity; (iv) monotonicity; and (v) convexity – are satisfied in your diagram. Hence explain whether or not you consider Helga's preference to be well behaved.**
All are satisfied.
Completeness: the utility function is defined for every consumption bundle.
Reflexivity: every consumption bundle lies in its own preferred set.
Transitivity: pairwise comparisons of utility are always consistent.
Non-satiation: an increase in consumption of a good never leads to a reduction in utility
Convexity: every linear combination of a pair of consumption bundles in a preferred set necessarily lies in the preferred set as well.

e) **Explain why we consider that Helga considers goods B and C to be perfect complements.**
These goods are perfect complements since there is an ideal ratio in which they should be consumed, and any deviation from that proportion cannot increase the utility derived from consumption.

X6.22 Ivan's preferences between goods B and C are such that they can be represented by the utility function $U(b, c) = 2b + c$.

a) **Confirm that Ivan obtains the same level of utility from the consumption bundles $(b, c) = (10, 0)$, and $(b, c) = (0, 20)$.**
$U(10, 0) = 20 = U(0, 20)$

b) **Sketch a diagram showing these two bundles and the indifference curve that they lie on.**
The bundles are at the end points of the line segment with equation $2b + c = 20$.

c) **Discuss whether or not Ivan's preferences appear to satisfy the assumptions of (i) completeness; (ii) reflexivity; (iii) transitivity; (iv) monotonicity; and (v) convexity. Does it seem to you that Ivan's preferences are well behaved?**

Since it is possible to evaluate the utility function for every consumption bundle; consumption bundles lie on the indifference curve passing through them; all indifference curves are parallel straight lines; lines further away from the origin represent a higher level of utility; and the boundary of each preferred set is a straight line; we can be satisfied that preferences are well behaved.

d) Now suppose that Ivan can buy units of good B at price $p_b = 4$, and units of good C at price $p_c = 2$. Sketch Ivan's budget constraint, given that the amount of money available for consumption $m = 40$.
Ivan's budget constraint is coincident with the indifference curve that we have already sketched.

e) From your diagram, what do you think is the best that we might say about Ivan's resource allocation?
Ivan will choose a consumption bundle on the indifference curve.

f) Now suppose that the price of good C increases slightly to 2.05. Sketch the new budget constraint, given that Ivan still has $m = 40$ to finance consumption. How does your answer to part (e) change?
The new budget constraint is slightly flatter than the original one, and lies below the indifference curve, meeting it on the horizontal axis – so the most-preferred, affordable consumption bundle contains only good B.

g) Repeat part (f), but now with the price of good B increasing to 4.005.
The new budget constraint is slightly steeper than the original one, and lies above the indifference curve, meeting it on the vertical axis – so the most-preferred, affordable consumption bundle contains only good C.

h) If we know that Ivan chooses a mixture of goods B and C, what can we say about the price ratio p_b/p_c?
The price ratio must be exactly 2.

i) Why do we consider that Ivan considers goods B and C to be perfect substitutes?
The marginal rate of substitution is constant; so that two units of B can always be substituted for a single unit of C.

X6.23 X6.21 and X6.22 explore special cases of preferences.

a) In X6.21, what assumption of 'good behaviour' is barely satisfied? In X6.22, which (different) assumption of 'good behaviour' is barely satisfied?
With perfect complements, non-satiation is only just satisfied; with perfect substitutes, it is convexity of preferences that is only just satisfied.

b) In diagrammatic terms, what is ruled out as contrary to the assumptions of good behaviour?
Non-satiation rules out upward-sloping segments for indifference curves; and convexity rules out indifference curves that become steeper, moving from left to right.

X6.24 In some textbook diagrams, indifference curves are drawn so that they are convex, but become upward-sloping at high values of consumption of one good. Explain what the upward-sloping component of the indifference curve means in terms of the assumptions of good behaviour.

The assumption of non-satiation is violated.

X6.25 Sometimes economists have argued for the existence of a bliss point, the most-preferred bundle. Sketch a diagram in which all the assumptions of good behaviour – except monotonicity – are satisfied, and there is still a bliss point. On your diagram, sketch a budget constraint that passes above and to the right of the bliss point. Explain how a consumer's behaviour will differ when there is a bliss point from the situation in which there are well-behaved preferences.

Since the bliss point is affordable, we can be certain that it will be purchased. The bliss point lies within the budget constraint, and so its acquisition cost is less than the sum available for consumption. In this case, the consumer has money available to finance further consumption.

X6.26 We have generally talked in terms of the most-preferred consumption bundle involving the purchase of both goods B and C. Suppose that Kaila spends all her money on good B, and none on good C. How might we reconcile this outcome with the fact that she has well-behaved preferences? [*Hint:* Suppose that the condition that Kaila's marginal rate of substitution is equal to the price ratio is satisfied only when all her money is spent on good B; and then consider what Kaila would do if that condition were never satisfied, so that for any consumption bundle the marginal rate of substitution is greater than the price ratio.]

It must be the case that Kaila's MRS is greater than the price ratio for all consumption bundles on the budget constraint. Indifference curves then cut through the budget constraint, with the utility achieved increasing as good B is substituted for good C. This continues until only good B is being consumed.

This also means that the conditions that we have given for utility maximization (MRS equal to the price ratio) can never be satisfied. This leads to the result that the most-preferred affordable consumption bundle cannot contain both goods.

Chapter 7

X7.1 Calculate Leena's demand for cheese when she has an amount of money, *m*, to finance consumption. Check that your answer is correct by calculating her spending on bread and cheese, and checking that her total spending is *m*. What fraction of her total expenditure *m* is on bread?

We know that 1.6b + 12c = m; and that b = 2.5c, so substituting for b, 16c = m, and $c = \frac{m}{16}$. *Then substituting back into the expenditure constraint, 1.6b + 12c =* $1.6\left(\frac{m}{6.4}\right) + 12\left(\frac{m}{16}\right) = m$*, and Leena spends all of her money in acquiring the consumption bundle.*

X7.2 Using the equations of the Engel curves, confirm that the income expansion path is *c* = 0.4*b*, as shown in Chapter 6. Confirm that expenditure on bread plus expenditure on cheese is always -equal to *m*. Explain what this result means.

Given that $(b^*, c^*) = \left(\frac{5m}{32}, \frac{2m}{32}\right)$*, we note that 2b* = 5c*, as required.*

X7.3 Explain why the following statements are false:

a) Where there are only two goods to consume, we expect both of them to be inferior.

With two inferior goods, demand for both decreases with income, and total expenditure falls. This would be inconsistent with preferences being well behaved.

b) Where there are only two goods to consume, we expect the demand for both to increase more rapidly than income.

Assuming that initially income exceeds expenditure, then increasing income, expenditure would increase more rapidly than income, which violates the consumer's expenditure constraint.

c) With only two goods, if demand for one good increases more quickly than income, then the other good must be inferior.

It must be true that as the expenditure share of one good increases, the expenditure share of the other one falls; but it is nonetheless possible for the expenditure share of a good to fall while total expenditure on it increases.

X7.4 Sketch a diagram showing an Engel curve that starts from the origin, but is upward-sloping and becomes steeper and steeper. Choose two or three points on the curve. Confirm that for each point, the slope of the tangent is greater than the slope of the line that joins the point to the origin. Using the definition of the income elasticity of demand, what do you conclude about its value for all points on the curve?

The tangents to the curve will certainly intersect the horizontal (income) axis to the right of the origin, and so are steeper. Defining the income elasticity of demand as the ratio of the slope of the tangent to the slope of the line connecting that point of tangency to the origin, we conclude that the income elasticity of demand is greater than one.

X7.5 Repeat X7.4, for a curve that is upward-sloping, but that becomes steadily flatter.

The tangents to the curve will certainly intersect the vertical (demand) axis above the origin, and so are flatter than the lines connecting the tangent points to the origin. Given the definition of the income elasticity of demand we conclude that the income elasticity of demand is positive, but less than than one.

X7.6 When does an Engel curve have elasticity equal to zero? [*Hint:* Use Expression 7.6.] At such a point, is the good normal or inferior?

For an income elasticity of demand equal to zero, demand neither increases nor decreases with income. The good is then neither normal nor inferior, but lies on the boundary between these two classes.

X7.7 Suppose that Omar reports that his utility, derived from consumption of a bundle of bread and cheese, is $U = b^{0.5}c^{0.5}$. For each of the following price pairs, (p_b, p_c):
i. $p_b = 2$ and $p_c = 3$; ii. $p_b = 6$ and $p_c = 7$; iii. $p_b = 9$ and $p_c = 16$

a) Obtain Omar's income expansion path, and his demands for bread and cheese.

Easy to check that for Omar, $MU_B = 0.5b^{-0.5}c^{0.5}$, $MU_C = 0.5b^{0.5}c^{-0.5}$; so that $MRS = -\frac{c}{b}$. For income expansion path, $MRS = -\frac{p_b}{p_c}$, so that his income expansion path has the equation $p_bb = p_cc$.

Then if $p_bb + p_cc = m$, we obtain demands $b = \frac{m}{2p_b}$; $c = \frac{m}{2p_c}$.

i. Expansion path $2b = 3c$; demands $b = \frac{m}{4}$; $c = \frac{m}{6}$

ii. Expansion path $6b = 7c$; demands $b = \frac{m}{12}$; $c = \frac{m}{14}$

iii. Expansion path $9b = 16c$; demands $b = \frac{m}{18}$; $c = \frac{m}{32}$

b) Illustrate his demands using Engel curves.

Engel curves will be straight line passing through the origin.

c) Calculate the expenditure shares of bread and cheese, and his income elasticity of demand for both goods.

Easy to confirm that for demands, b and c, as above, $p_bb = p_cc = 0.5m$, so expenditure shares, $s_b = \frac{p_bb}{m} = 0.5$; and $s_c = 0.5$.

X7.8 Now suppose that Philippa has utility $U = b^{0.5} + c^{0.5}$. Repeat X7.7, but for prices:
i. $p_b = 2$ and $p_c = 3$; ii. $p_b = 4$ and $p_c = 6$; iii. $p_b = 3$ and $p_c = 9$

Easy to check that for Philippa, $MU_B = 0.5b^{-0.5}$, $MU_C = 0.5c^{-0.5}$; so that $MRS = -\left(\frac{c}{b}\right)^{0.5}$. For income expansion path, $MRS = -\frac{p_b}{p_c}$, so that her income expansion path has the equation $p_b^2b = p_c^2c$.

Then if $p_bb + p_cc = m$, we can rewrite the constraint as $p_bb\left(1 + \frac{p_b}{p_c}\right) = m$, so that demand $b = \frac{m}{p_b}\left(\frac{p_c}{p_b + p_c}\right)$; and it follows that $c = \frac{m}{p_c}\left(\frac{p_b}{p_b + p_c}\right)$

Again, we see that the Engel curves are all straight lines passing through the origin, and that the expenditure shares are $s_b = \frac{p_c}{p_b + p_c}$; and $s_c = \frac{p_b}{p_b + p_c}$:

i. Expansion path $4b = 9c$; demands $b = \frac{3m}{10}$; $c = \frac{2m}{15}$; $s_b = 0.6$; $s_c = 0.4$

ii. Expansion path $16b = 36c$, or $4b = 9c$; demands $b = \frac{3m}{20}$; $c = \frac{m}{15}$; $s_b = 0.6$; $s_c = 0.4$

iii. Expansion path $9b = 81c$, or $b = 9c$; demands ; $b = \frac{m}{4}$; $c = \frac{m}{36}$; ; $s_b = 0.75$; $s_c = 0.25$

Again, we see that the Engel curves are all straight lines passing through the origin.

X7.9 Explain how reasonable you consider the HMRC allowance of £0.20 per mile cycled to be. [*Note:* 1 mile = 1.609 km.]

To be a sensible measure of expenditure, it should include all the costs associated with cycling one mile. If we take the replacement cost of a bicycle to be about £200, and assume that the bicycle is used for a short journey (of say five miles) in both directions on 100 days per year, for a total of 1,000 miles, then the cost of the bicycle can be recouped in a year. This seems a *much more generous allowance than approximately £0.50 for a car, where to recover replacement costs in one year would perhaps require in excess of 20,000 miles travel per year.*

X7.10 What evidence have we collected of car travel being a superior good? How would you assess the claim that public transport within cities tends to be an inferior good? Do you think that the same could be said of long-distance travel (for example trans-Atlantic air travel to North America)?

The people who tend to use cars appear to have characteristics of people with higher income. To determine whether or not this is correct, we would have to find some way of evaluating the spending patterns of people. We could simply ask them how much they spend on bus travel and their incomes, but this would not take account of other factors; and a better source of data would be through the compilation of expenditure diaries, which could then be analysed using statistical techniques.

With long-distance air travel, given the cost of a single journey, it seems likely that over a wide range of incomes, this will be a superior good.

X7.11 Other than differences in income, what factors might lead younger people to be more likely to use self-powered transport than older people? Why might the presence of such factors mean that we could easily overestimate the income elasticity of demand?

Younger people are likely to be fitter and so more capable of producing the effort needed to cycle. They might also have better eyesight and hearing and faster reactions (important for safety when cycling in traffic), and may have more friends who cycle. If we believe that people switch to other means of transport as their income increases, but do not take account of these other factors that mean that richer, older people are less likely to cycle, then we overestimate the (negative) effect of increasing income on demand, and so the income elasticity.

X7.12 Giving reasons, state whether you believe that air travel is a superior, a normal (but not superior), or an inferior good. Sketch an income offer curve showing how use of air and car travel might change as income increases.

We certainly expect air travel to be a normal good; and we also note that even on a single flight, there will often be very large differences in the prices charged in different cabins. People with high incomes might well also have jobs requiring them to travel by plane more frequently; all of this suggests that for at least some income levels, air travel is likely to be a superior good.

X7.13 We observe some people who drive everywhere, and others who have no car and only use public transport (ignoring walking and cycling). Suppose that there is a threshold income at which someone will buy a car, switching from public transport only to car use only. Sketch the income offer curve that illustrates this situation. [*Hint:* The income offer curve will not be continuous, but will jump when the switch is made.]

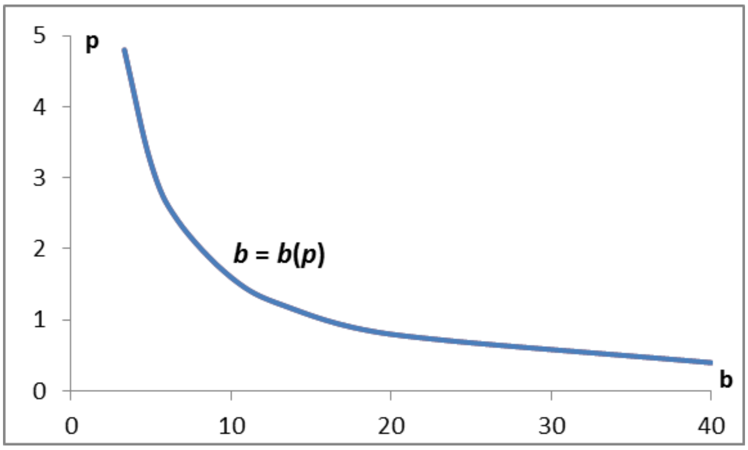
The income offer curve will have two arms, one along the public transport axis, which starts from the origin, and the other along the car axis, which will start from the level of demand at the switching income.

X7.14 If bread and cheese are normal goods but neither is superior, sketch the income offer curve for Salma, who consumes 2 kg of cheese and 9 loaves. [*Hint:* What very simple shape is defined as soon as we know two points on it?]

If both goods are normal, but neither is superior, then demand increases with income, but no faster than income. Unless demand for both goods is linear in income, one good would have to be superior. So Salma's income offer curve must be a straight line, starting from the origin, and passing through the consumption bundle (b, c) = (9, 2).

X7.15 Sketch the diagram that we have found, by using the following values of p_b and calculating the associated values of b: 0.4, 0.8, 1.2, 1.6, 2.4, 3.2, 4.8.

b(p)	40	20	13.33	10	6.67	5	3.33
p	0.4	0.8	1.2	1.6	2.4	3.2	4.8



X7.16 We have said that spending on bread stays the same as its price changes. What does this imply about spending on cheese? Given that the price of cheese is constant, calculate the effect of a change in the price of bread on the demand for cheese.

If spending on bread is constant, then spending on cheese will be constant. Since the price of cheese does not change, the demand for change remains constant as the price of bread changes; the effect of the change in the price of cheese is zero.

X7.17 Demonstrate that with unit price elasticity of demand for good X, so that $\varepsilon = -1$, the total amount of money that a consumer will spend on good X will remain constant as prices change.

Price elasticity of demand $\varepsilon = \frac{\partial X}{\partial p} \cdot \frac{p}{X} = -1$. *So* $\frac{\partial X}{\partial p} = -\frac{X}{p}$. *Writing expenditure E = pX,*

$\frac{dE}{dp} = X + p\frac{\partial X}{\partial p} = X - p\frac{X}{p} = 0$ *; so expenditure is constant.*

X7.18 For the demand function x: $x = 100p^{-0.5}$:

a) Calculate the quantity demanded for prices p = 0.25, 1, 4, 9 and 16, and sketch the demand curve.

p	*0.25*	*1*	*4*	*9*	*16*
$x = 100p^{-0.5}$	*200*	*100*	*50*	$\frac{100}{3}$	*25*

Demand curve is downward sloping, convex and does not intersect the axes.

b) Show that the elasticity of demand is –0.5. On your graph, show how the proportional change in price relates to the proportional change in demand.

$\varepsilon = \frac{\partial x}{\partial p} \cdot \frac{p}{x} = -50p^{-1.5} \cdot \frac{p}{100p^{-0.5}} = -0.5$

Following a change in price, the proportionate change in (the inverse of) demand is half of the proportionate change in price.

X7.19 For the demand function x: $x = 100p^{-2}$:

a) **Calculate the quantity demanded for prices p = 0.5, 1, 2, 3 and 4, and sketch the demand curve.**

p	0.5	1	2	3	4
$x = 100p^{-2}$	400	100	25	$\frac{100}{9}$	$\frac{100}{16}$

Demand curve is downward sloping, convex and does not intersect the axes. It is much flatter than the demand curve in the previous example.

b) **Show that the elasticity of demand is –2. On your graph, show how the proportional change in price relates to the proportional change in demand.**

$\varepsilon = \frac{\partial x}{\partial p} \cdot \frac{p}{x} = -200p^{-3} \cdot \frac{p}{100p^{-2}} = -2$

Following a change in price, the proportionate change in (the inverse of) demand is twice the proportionate change in price.

X7.20 Assume that a consumer's preferences are defined by the Cobb Douglas utility function, U: $U(x, y) = x^{\alpha}y^{1-\alpha}$, that the consumer has m = 60 to spend, and that the price of good x, p_x = 2, while the price of y, p_y, might vary.

a) **Show that the demand, $y = \frac{60(1-\alpha)}{p_y}$.**

The marginal utilities may be written $MU_X = \frac{\partial U}{\partial x} = \alpha x^{\alpha-1}y^{1-\alpha}$ and $MU_Y = \frac{\partial U}{\partial y} = (1-\alpha)x^{\alpha}y^{-\alpha}$, so $MRS = -\frac{MU_X}{MU_Y} = -\frac{\alpha}{1-\alpha} \cdot \frac{x^{\alpha-1}}{x^{\alpha}} \cdot \frac{y^{1-\alpha}}{y^{-\alpha}} = -\frac{\alpha y}{(1-\alpha)x}$. For the tangency condition to be satisfied, $MRS = -\frac{\alpha y}{(1-\alpha)x} = -\frac{p_x}{p_y} = -\frac{2}{p_y}$, so that $2(1-\alpha)x = \alpha p_y y$. Substituting for $2x$ in the budget constraint,

$2x + p_y y = 60$, $\left(\frac{\alpha}{1-\alpha} + 1\right)p_y y = 60$, so that $\frac{p_y y}{1-\alpha} = 60$. The result follows immediately.

b) **Confirm that the elasticity of demand ε_p = –1.**

Price elasticity of demand, $\varepsilon_p = \frac{\partial y}{\partial p} \cdot \frac{p}{y} = -60(1-\alpha)p_y^{-2} \cdot \frac{p_y}{60(1-\alpha)p_y^{-1}} = -1$

X7.21 Expenditure on a good is the product of demand and price. Demonstrate that expenditure on a good for which demand is inelastic increases with its price, while expenditure on a good for which demand is elastic falls with its price.

We start by assuming that the price elasticity of demand $\varepsilon = \frac{\partial X}{\partial p} \cdot \frac{p}{X} = -1$. So $\frac{\partial X}{\partial p} = -\frac{X}{p}$. Writing expenditure $E = pX$, $\frac{dE}{dp} = X + p\frac{\partial X}{\partial p} = X - p\frac{X}{p} = 0$; so expenditure is constant.

For inelastic demand, $\varepsilon = \frac{\partial X}{\partial p} \cdot \frac{p}{X} > -1$, so $\frac{\partial X}{\partial p} > -\frac{X}{p}$ and writing expenditure $E = pX$,

$\frac{dE}{dp} = X + p\frac{\partial X}{\partial p} > X - p\frac{X}{p} = 0$; so expenditure is increasing. A similar argument applies when demand is elastic, so that $\varepsilon < -1$.

X7.22 For the demand function x: $x(p) = 100 - 2p$:

a) **Calculate demand when price is zero, and the price when demand is zero, and sketch the demand curve.**

$x(0) = 100$; for $x(p) = 100 - 2p = 0$, $p = 50$; with quantity axis on horizontal and price on the vertical, demand curve will be a straight line connecting $(x, p) = (100, 0)$ to $(0, 50)$.

b) **Obtain the formula for the price elasticity of demand in this case. Confirm that the price elasticity of demand is zero when the good is free, undefined if demand is choked off by a high price, and minus one at the mid-point of the demand curve.**

$\varepsilon = \frac{\partial x}{\partial p} . \frac{p}{x} = -2 . \frac{p}{100-2p} = -\frac{2p}{100-2p}$ *; the good is free for p = 0, so that 2p = 0, and* $\varepsilon = -\frac{0}{100-0}$ *= 0; when p = 50, 100 – 2p = 0, so it is not possible to evaluate the price elasticity,* ε*; and when p = 25, 2p = 100 – 2p = 50, so that* ε *= -1.*

c) **Confirm that the derivative of the price elasticity of demand,** $\frac{d\varepsilon_p}{dp} = -\frac{100}{(p-50)^2}$ **< 0. Explain what this means in terms of the relationship between the price of the good and its elasticity of demand.**
Write $\varepsilon = 1 - \frac{100}{100-2p} = 1 + \frac{100}{2(p-50)} = 1 + \frac{50}{p-50}$. *Then applying the chain rule of differentiation,* $\frac{d\varepsilon}{dp} = -50(p-50)^{-2}$

X7.23 For the demand function *q* = 400 – 8*p*, repeat X7.22a and X7.22b, and then:
q(0) = 400; for q(p) = 400 – 8p = 0, p = 50; with quantity axis on horizontal and price on the vertical, demand curve will be a straight line connecting (q, p) = (400, 0) to (0, 50).
$\varepsilon = \frac{\partial q}{\partial p} . \frac{p}{q} = -8 . \frac{p}{400-8p} = -\frac{8p}{400-8p} = -\frac{p}{50-p}$ *; the good is free for p = 0, and* $\varepsilon = -\frac{0}{100-0}$ *= 0; when p = 50, 50 – p = 0, so it is not possible to evaluate the price elasticity,* ε*; and when p = 25, 8p = 400 – 8p = 200, so that* ε *= -1.*

a) **Show that when demand is elastic, this consumer reduces spending on the good if the price rises.**
Expenditure E = pq = (400 – 8p)p; so that $\frac{dE}{dp} = 400 - 16p$. *Expenditure increases if 400 – 16p > 0, or if p < 25; and decreases if p > 25.*
Elasticity of demand, $\varepsilon = \frac{\partial q}{\partial p} . \frac{p}{q} = -\frac{p}{50-p}$ *< -1 if p > 50 – p, or if p > 25; and 0 >* ε *> -1 if p < 25.*
So when demand is elastic, expenditure decreases as the price rises; and when demand is inelastic, expenditure increases as the price rises.

b) **Show that when demand is inelastic, this consumer increases spending on the good if the price rises.**
See above.

c) **At what price do you consider that the consumer will spend the most money?**
The consumer will spend the most money when ε *= -1.*

X7.24 Suppose that Thomas considers travel both by bus and by car simply to be necessary as he moves from one activity to another. Why might this suggest that bus and car travel will be close substitutes? Sketch a diagram showing the price expansion path. [*Hint:* Remember that if goods are perfect substitutes, there is only one price ratio at which both will be consumed.]
Bus and car travel provide similar services. As the price of public transport falls, there will be a price at which Thomas will switch mode of transport, choosing after that only to travel by car. The price expansion path will consist of a single point representing his demand for car travel when he uses only that, and a point showing his demand for bus travel when he uses that mode of transport.

X7.25 What reasons can you give for supposing that following a reduction in the price of public transport, the increase in bus usage might be greater than the reduction in car usage?
This simply requires people's demand for bus travel to have a non-zero price elasticity of demand. As well as switching from car to bus travel, there will be higher demand for bus travel – people travelling more often, or for longer distances – when the price falls.

X7.26 We have concentrated in this discussion on the price of travel. What other costs do you think we should also take into account? Suppose that instead of thinking about the prices in monetary terms, we concentrate on the time cost. How might we then interpret the price expansion path?

We have not thought about the cost of acquiring a car, which is a substantial sunk cost. In terms of time cost, we expect there to be substantial time savings from having a car. If we were to try to represent the price of travel in terms of financial payments and time used, then the prices that we would use for both modes of travel would be substantially higher. The price expansion path should show the effect on demands as these 'full prices' change.

X7.27 Could we ever observe an upward-sloping price offer path? Explain your answer.

Yes; if the two goods are complements, then as the price of one falls, we expect to see more of both being consumed.

Chapter 8

X8.1 Confirm that if p_c = 12, then Expression 8.2 may be written $b = 2.5c$. Show that for any value of p_c, Michael's expenditure on cheese will then always be three times his expenditure on bread.

When $p_c = 12$, $\frac{p_c}{24} = 0.5$, so $b = 2.5c$.

X8.2 Using Expression 8.2, calculate the quantity of bread, b_0, in the cheapest acceptable bundle, Z_0. [*Note:* This will be a function of p_c.]

If $c_0 = \left(\frac{3{,}072}{p_c}\right)^{0.25}$, then since $b = \frac{5}{24}p_c c$, $b_0 = \frac{5}{24}p_c\left(\frac{3{,}072}{p_c}\right)^{0.25} \approx 1.55 p_c^{\,0.75}$

X8.3 Given the Hicksian demands, $b^H(p_c)$ and $c^H(p_c)$, confirm that for any price, p_c:

a) the marginal rate of substitution, *MRS*, is equal to the price ratio $\frac{p_b}{p_c} = \frac{1.6}{p_c}$; and

$$MRS = -\frac{MU_B}{MU_C} = -\frac{\frac{\partial V}{\partial b}}{\frac{\partial V}{\partial c}} = -\frac{c^3}{3c^2 b} = -\frac{c}{3b} = -\frac{\left(\frac{3{,}072}{p_c}\right)^{0.25}}{\frac{5}{8}(3{,}072)^{0.25} p_c^{\,0.75}} = -\frac{8}{5p_c}$$

b) the cheapest, acceptable bundle, Z_0, always lies on the indifference curve $c^3b = 640$, which is the boundary of the acceptable set.

$V(b_0, c_0) = c_0^3 b_0 = \left(\frac{3{,}072}{p_c}\right)^{0.75} \cdot \frac{5}{24}p_c\left(\frac{3{,}072}{p_c}\right)^{0.25} = 640$

X8.4 Calculate the acquisition cost, $A(b_0, c_0)$, of the cheapest, acceptable bundle, and show that it increases with the price of cheese, p_c.

$A(b_0, c_0) = 1.6b_0 + p_c c_0 = \frac{8}{5}\left(\frac{5}{24}p_c c_0\right) + p_c c_0 = \frac{4}{3}p_c c_0$, and substituting for c_0, $A = \frac{4}{3}(3{,}072)^{0.25} p_c^{\,0.75}$, which is clearly increasing in p_c

X8.5 Confirm that for Leena, the price elasticity of demand $\varepsilon_{p_c} = \frac{dc^*}{dp_c} \cdot \frac{p_c}{c^*(p_c)} = -1$, but that for Michael, $\varepsilon_{p_c} = \frac{dc^H}{dp_c} \cdot \frac{p_c}{c^H(p_c)} = -0.25$.

For Leena, demand $c = \frac{3m}{4p_c}$, so $\frac{\partial c}{\partial p_c} = -\frac{3m}{4p_c^{\,2}}$, and the result follows directly. Similarly, with Michael's demands, $c = \left(\frac{3{,}072}{p_c}\right)^{0.25}$, $\frac{\partial c}{\partial p_c} = -768^{0.25} p_c^{\,-0.75}$, and again, the result follows directly.

X8.6 Suppose that the price of cheese increases from p_{c0} = 12 to p_{c1} = 16. Confirm that initially Leena and Michael would have chosen the same consumption bundles, but that after the price increase Leena reduces her consumption of cheese while maintaining her consumption of bread, whereas Michael increases his consumption of bread while reducing consumption of cheese, though by less than Leena.

When $p_c = 12$, we have the problem in Section 6.4, and know that for Leena the utility-maximizing, affordable bundle, $(b^*, c^*) = (10, 4)$. Setting $p_c = 12$ in this case, we find that for Michael, the least-cost acceptable bundle, $(b_0, c_0) = (10, 4)$.

When price $p_c = 16$, with income and prices unchanged, we know that Leena's demand for bread $\left(b = \frac{16}{p_b}\right)$ will not change, but her demand for cheese, given by the expression $c = \frac{48}{p_c} = 3$, will be less than before.

For Michael, after the price change, demand for bread, $b_0 = \frac{5}{24}p_c\left(\frac{3{,}072}{p_c}\right)^{0.25} = \frac{10}{3}(192)^{0.25} \approx 12.4$. Demand for cheese, $c_0 = \left(\frac{3{,}072}{p_c}\right)^{0.25} \approx 3.72$.

X8.7 Suppose that the least-cost, acceptable bundle changes from Z_0 to Z_1 after the price of cheese doubles. What other price change would lead to this change in compensated demands? What could you say about the acquisition cost of the new bundle in this case? Sketch a diagram showing the income and substitution effects for this alternative price change.

As well as the price of cheese doubling, the same effect on demand would occur if the price of bread were to be halved, since it is the relative price that matters in this case. We would show the income effect as a movement of the affordability constraint out so that it touches a higher indifference curve than was previously affordable, and the substitution effect as a shift along an indifference curve, substituting bread for cheese in the consumption bundle.

X8.8 Using diagrams, confirm that bundle Z_1 is not affordable at the original prices, and that bundle Z_0 is not affordable at the new prices.

As in Figure 8.3, we note that Z_0 lies above the affordability constraint through Z_1 and Z_1 lies above the affordability constraint through Z_0. So neither bundle would be affordable when the other one is chosen.

X8.9 Sketch a diagram that shows a budget constraint reflecting the new prices through the original, most-preferred affordable bundle, Z_0. Show that after the price increase, the most-preferred, affordable bundle on this new budget constraint will be preferred to Z_1. Illustrate the substitution and income effects in this case.

Since the original bundle is unaffordable at the new prices, the budget constraint through Z_0 reflecting the new prices lies further away from the origin than the budget constraint through Z_1. We might expect both the income and substitution effects to be larger with this price change.

X8.10 Consider the following problem. Uma has utility function $U(b, c) = (bc)^{0.5}$, where (just for a change), we define *b* as the number of books she purchases and *c* as the number of concerts she attends. Uma is willing to spend 200 on books and concerts. The prices of both books and concerts are initially 10.

a) Find Uma's ordinary demands, given these prices.

This problem should by now be reasonably familiar. Uma's MRS may be written as: $MRS(b, c) = -\frac{c}{b}$, so that the equation of her income expansion path will be $p_b b = p_c c$. In this case, $p_b = p_c = 10$, so the income expansion path has equation, $b = c$. This also means that she spends half of her money on books, and half on concerts, so her most-preferred, affordable consumption bundle, $(b^*, c^*) = (10, 10)$.

b) Show that Uma obtains a utility of 100; and that her compensated demands, given the constraint that she must achieve a utility of 100, are the same as her ordinary demands.

$U(10, 10) = (10*10)^{0.5} = 10$.

Compensated demands: Income expansion path, $b = c$. Utility constraint, $U(b, c) = (bc)^{0.5} = 10$. So $b^2 = c^2 = 100$; and $b = c = 10$.

The price of concert tickets now increases to 12.5.

c) Calculate Uma's ordinary demands after the price change. [*Note:* The demand for books should not change.]

From previous examples, we know that Uma's demand for books remains unchanged, while her demand for concert tickets is inversely related to their price. So the new most-preferred, affordable consumption bundle is $(b, c) = (10, 8)$.

d) Calculate Uma's compensated demands after the price change. (Remember that we assume that utility from consumption will be 100.)

Given the acceptability constraint, $U^2 = bc = 100$, and that on the income expansion path, $10b = 12.5c$ (or $b = 1.25c$), we can rewrite the constraint as $1.25c^2 = 100$, so that $c = 80^{0.5} \approx 8.94$; and then it follows that $b = 125^{0.5} \approx 11.2$. The least-cost, acceptable consumption bundle, $(b, c) \approx (11.2, 8.94)$.

e) Calculate the substitution and income effects of the price change for both books and concert tickets.

The substitution effects are the changes from the original demands to the compensated demands: for books, $\Delta b^S \approx 1.2$, and for concert tickets, $\Delta c^S \approx -1.06$. The income effects are the changes from the compensated demands to the final demands. For books, $\Delta b^M \approx -1.2$, and for concert tickets $\Delta c^M \approx -0.94$.

f) Sketch a diagram showing these effects.

The diagram should have the books axis on the horizontal and the concert ticket axis on the vertical. The initial affordability constraint, $b + c = 20$, passes through (10, 10) and has a slope of -1. At (10, 10), it is tangent to the indifference curve, $bc = 100$. The compensated constraint is flatter, and has a slope of -0.8. It is tangent to the indifference curve $bc = 100$ at approximately (11.2, 8.94). The final budget constraint has the equation $b + 1.25c = 20$. It is parallel to the compensated constraint, but is tangent to the indifference curve $bc = 80$ at (10, 8).

X8.11 Repeat X8.10 for Vishal, whose utility function $V = x^{0.5} + y^{0.5}$. [*Note:* In part (b), you will treat Vishal as a cost minimizer with a utility target of $2\sqrt{10}$.]

This problem should also be reasonably familiar. Vishal's MRS may be written as: $MRS(x, y) = -\left(\frac{y}{x}\right)^{0.5}$, so that the equation of his income expansion path will be $p_x^2x = p_y^2y$. In this case, $p_x = p_y = 10$, so the income expansion path has equation, $x = y$. This also means that he spends half of his money on good X, and half on good Y, so his most-preferred, affordable consumption bundle, $(x^*, y^*) = (10, 10)$.

$U(10, 10) = 10^{0.5} + 10^{0.5} = 2\sqrt{10}$.

Compensated demands: Income expansion path, $x = y$. Utility constraint, $U(x, y) = x^{0.5} + y^{0.5} = 2\sqrt{10}$. So $2\sqrt{x} = 2\sqrt{y} = 2\sqrt{10}$; and $x = y = 10$.

After the price change, Vishal's income expansion path becomes $100x = 156.25y$, so that $x = 1.5625y$.

To find the ordinary demands, we substitute into the new budget constraint, $10x + 12.5y = 200$, or $x + 1.25y = 20$. Then from the income expansion path, $2.8125y = 20$ and $y \approx 7.11$; and to lie on the income expansion path, $x = 1.5625y \approx 11.1$. The most-preferred, affordable consumption bundle $(x, y) \approx (11.1, 7.11)$

For the compensated demands, substituting into the utility constraint, $2.25\sqrt{y} = 2\sqrt{10}$, and $y \approx 7.90$. Then, given that $x = 1.5625y$, $x \approx 12.3$, so that the least-cost affordable consumption bundle $(x, y) \approx (12.3, 7.90)$.

The substitution effects are the changes from the original demands to the compensated demands: for good X, $\Delta x^S \approx 2.3$, and for good Y, $\Delta y^S \approx -2.1$. The income effects are the changes from the compensated demands to the final demands. For good X, $\Delta x^M \approx -1.2$, and for concert tickets $\Delta c^M \approx -0.79$.

In a diagram, show the consumption of good X on the horizontal axis and consumption of good Y on the vertical axis. The initial affordability constraint, $x + y = 20$, passes through (10, 10) and has a slope of -1. At (10, 10), it is tangent to the indifference curve, $x^{0.5} + y^{0.5} =$

$2\sqrt{10}$. The compensated constraint is flatter, and has a slope of -0.8. It is tangent to the indifference curve $x^{0.5} + y^{0.5} = 2\sqrt{10}$ at approximately (12.3, 7.90). The final budget constraint has equation x + 1.25y = 20. It is parallel to the compensated constraint, but is tangent to the indifference curve $x^{0.5} + y^{0.5} = 6$ at approximately (11.1, 7.11).

X8.12 Suppose that Wang's preferences are represented by utility function *U*(*b*, *c*) = 2*b* + 3*c*.

a) Initially prices are $p_b = 10$ and $p_c = 15$. Sketch indifference curves for *U* = 20, *U* = 30 and *U* = 40. Indicate on the diagram Wang's most preferred affordable consumption bundles, given that *m* = 150.

In a diagram with consumption of good B measured on the horizontal, and consumption of good C measured on the vertical axis, Wang's preferences are illustrated by line segments. The indifference curve U = 20, is a downward sloping line, stretching from (b, c) = $\left(0, \frac{20}{3}\right)$ to (10, 0); U = 30, is parallel to U = 20, stretching from (b, c) = (0, 10) to (15, 0); and U = 40, is also parallel to U = 20, stretching from (b, c) = $\left(0, \frac{40}{3}\right)$ to (20, 0).

When m = 150, we can write Wang's budget constraint as 10b + 15c = 150; or 2b + 3c = 30. This is the equation of one of the indifference curves, and so every consumption bundle on that indifference curve is in the most-preferred, affordable set.

b) Show the change in demand that occurs if p_c increases to 16.

The budget constraint tilts anticlockwise around its intersection on the b-axis. This means that it now lies below the indifference curve 2b + 3c = 30 whenever c > 0, so that there is a unique most-preferred, affordable consumption bundle, (15, 0).

c) Explain why there is no change in utility. What do you conclude about the income effect in this case?

One of the most-preferred affordable consumption bundles from before the price increase becomes the only member of the most-preferred, affordable set after the price change. So the utility generated from consumption remains the same. The compensated demands are the ordinary demands, so there can be no income effect.

X8.13 Suppose that Xavier's preference can be represented by the utility function *U*(*b*, *c*) = min(2*b*, 3*c*).

a) Initially prices are $p_b = 10$ and $p_c = 15$. Sketch indifference curves for *U* = 10, *U* = 15 and *U* = 20. Indicate Xavier's most preferred, affordable consumption bundle given that *m* = 150.

Indifference curves are L shaped, with their vertices on the line 2b = 3c. So, for U = 15, the vertex is at (b, c) = (7.5, 5), with a horizontal segment to the right of this point (with equation c = 5), and a vertical segment to the right of this point (with equation b = 7.5).

When m = 150, since by spending 30 on good B, and 30 on good C, Xavier generates utility U = 6, he will spend 75 on both goods, purchasing the consumption bundle (b, c) = (7.5, 5), and generating utility U = 15. The budget constraint, 10b + 15c = 150 is tangent to the indifference curve U = 15 at the most-preferred, affordable consumption bundle.

b) Show the change in demand that occurs if p_c increases to 20.

After the price increase, to generate utility U = 6, Xavier spends 30 on good B and 40 on good C. So, allocating 64.29 to the purchase of good B and 85.71 to the purchase of good A, Xavier purchases bundle (b, c) $\approx$ (6.43, 4.29), and generates utility U $\approx$ 12.9.

c) Illustrate the income expansion path on this diagram and the total effect of the price change. What do you conclude about the income effect in this case?

Irrespective of the price ratio, the income expansion path has the equation 2b = 3c. The effect of the price change is shown as a shift along the income expansion path, and so the income effect in this case is also the total effect of the price change (so that there is no substitution effect here).

X8.14 The only goods available to Zeki are biscuits and coffee, whose prices are p_b and p_c. Zeki has an amount of money *m* to finance purchases, and he chooses his most-preferred, affordable consumption bundle Z*(*b, *c**). His preferences are well behaved.**

a) Draw a diagram to show the shape of the indifference curve on which Zeki's most preferred, affordable consumption bundle lies. Demonstrate that:

Drawing the diagram with consumption of good B on the horizontal, and consumption of good C on the vertical axis, we simply require the indifference curve on which the most-preferred, affordable consumption bundle lies, to be downward sloping and convex to the origin, ensuring that there is a unique, most-preferred, affordable consumption bundle. We show this bundle Z as the point at which the budget constraint is tangent to an indifference curve.*

i. if the price of one good changes, then Zeki's demands will change; and

Since when the price of one good changes, if there is a price increase, then bundle Z will no longer be affordable; but if there is a price cut, Z* will no longer lie on the boundary of the affordable set, and so cannot be the most-preferred bundle in the affordable set.*

ii. the substitution effect will increase his demand for the good that has become relatively cheaper, and decrease the demand for the good that has become relatively more expensive.

The change in relative prices changes the slope of the budget constraint. The substitution effect is the change in demands, while holding utility constant (the change in the least-cost, acceptable bundle following a change in relative prices). With convex preferences, this will necessarily mean that there will be an increase in demand for the good that has become relatively cheaper, and a decrease in demand for the good that has become relatively more expensive.

b) Draw a separate diagram, with a single budget constraint. Choose a bundle on it, which will represent Zeki's initial most-preferred, affordable consumption bundle, Z_0. Add an indifference curve passing through that point. Zeki's favourite café now announces that it will reduce the price of coffee by 50% for the next month.

i. On your diagram, indicate the substitution effect.

Drawing a diagram with consumption of biscuits on the horizontal and consumption of coffee on the vertical axis, then the substitution effect of this price change should be shown as a shift up the indifference curve to a point where its gradient is twice what it was before the price change.

ii. Show Zeki's income effect, for the three cases: (1) coffee and biscuits are normal goods; (2) coffee is a normal good, but biscuits are inferior; and (3) biscuits are normal, but coffee is inferior.

(1) If both goods are normal, then the income effect of the price change will be an increase in their consumption, so that the income effect is a further increase in demand for coffee, and tempers the reduction in demand for biscuits associated with the relative price change.

(2) *If coffee is normal and biscuits are inferior, then the income effect for both goods amplifies the substitution effect. There is a further increase in the demand for coffee, and a further reduction in the demand for biscuits.*

(3) *If coffee is inferior and biscuits are normal, then the income effect for both goods operates in the opposite direction to the substitution effect. There is a reduction in the demand for coffee, and an increase in the demand for biscuits.*

iii. Explain why it must be that if coffee is a normal good, Zeki will drink more of it when the price falls.

If coffee is a normal good, then the income effect following a change in the money available to finance consumption will be in the same direction as the change in income: an increase in income leading to an increase in demand; and a reduction in income leading to a reduction in demand.

The substitution effect associated with a price change will be in the opposite direction: a price increase leads to a reduction in demand, and a price cut leads to an increase in demand.

Now, we consider that a change in price leads to a change in real income which has the opposite sign to the price change. Following a price cut, consumption bundles that were previously unaffordable now become affordable. A cut in price causes an increase in real income, which leads to an increase in demand. The substitution and income effects work in the same direction.

X8.15 We find that Zeki's preferences can be represented by the utility function, $U: U(b,c) = b^{\frac{1}{3}}c^{\frac{2}{3}}$. He is willing to spend *m* = 30, with the initial prices, p_{c0} = 2.5 and p_b = 1. During the coffee promotion, its price falls to p_{c1} = 1.25.

a) Obtain Zeki's

i. most preferred, affordable consumption bundle at the original prices;

It should be easy to check that Zeki's marginal utilities are: $MU_B : MU_B(b,c) = \frac{1}{3}b^{-\frac{2}{3}}c^{\frac{2}{3}}$ *and* $MU_C : MU_C(b,c) = \frac{2}{3}b^{\frac{1}{3}}c^{-\frac{1}{3}}$*; so that his marginal rate of substitution* $MRS = -\dfrac{c}{2b}$. *Then his income expansion path will be* $2p_bb = p_cc$*, so with* $p_{c0} = 2.5$ *and* $p_b = 1$*, the IEP is written as 2b = 2.5c or b = 1.25c. Given the budget constraint, b + 2.5c = 30, we obtain 3.75c = 30, so that c = 8, and b = 10. The most-preferred, affordable consumption bundle (b*, c*) = (10, 8). (Note that these are the demands that we would obtain by asserting that the expenditure share of coffee will be twice the expenditure share of biscuits.)*

ii. most preferred, affordable consumption bundle after the reduction in the price of coffee;

Using the expenditure share argument, Zeki will double purchases of coffee to 16, but that there will be no change in demand for biscuits.

iii. Hicksian demands based on the original utility achieved, and new prices.

Zeki's Hicksian demands will be the same as his Marshallian demand at the original prices. He derives utility U(10, 8) ≈ 8.62, and this become his utility target after the price change.

We can solve his expenditure minimization problem by finding the intersection of the indifference curve, $b^{\frac{1}{3}}c^{\frac{2}{3}} \approx 8.62$ *and the income expansion path* $2p_bb = p_cc$*, with* $p_b = 1$ *and* $p_c = 1.25$*, or b = 0.625c. Substituting into the constraint, c ≈ 10.08, and b ≈ 6.30, so that the least-cost, acceptable consumption bundle is (6.30, 10.08)*

b) Calculate the substitution effects, the income effects and the total effects of this price change.

The substitution effect is the change in the Hicksian demands. $\Delta b^S \approx$ --3.70; $\Delta c^S \approx 2.08$; the income effect is the difference between the final demands. For biscuits, $\Delta b^N \approx 3.70$; and $\Delta c^H \approx 4.92$; the total effect is the sum of the income and substitution effects, so here $\Delta b = 0$; $\Delta c^S \approx 7.01$.

c) Explain whether or not Zeki considers coffee and biscuits to be normal goods.

Since the income effects for both goods are positive, Zeki treats them as normal goods.

X8.16 Suppose that we have a situation where the price of biscuits $p_b = 1$, the money available for consumption $m = 30$, but the price of coffee p_c is not defined. Obtain Zeki's ordinary and compensated demand functions. Sketch their graphs, and explain why Zeki's ordinary demand curve is more elastic than his compensated demand curve.

We know that Zeki's ordinary demands satisfy the requirement that the expenditure share of biscuits is always 1/3, and the expenditure share of coffee is always 2/3. This gives rise to the ordinary demand functions: $b^M = \frac{10}{p_b}$ and $c^M = \frac{20}{p_c}$.

For the compensated demands, it will again be the case that the expenditure share of coffee will be twice the expenditure share of biscuits. In this case, though, the utility constraint does not depend on the price, so we simply call it U_0.

Then $2p_bb = p_cc$, so that $b = \frac{p_c}{2p_b}c$, and $b^{\frac{1}{3}}c^{\frac{2}{3}} = U_0$. Substituting, $c^H = \left(\frac{2p_b}{p_c}\right)^{\frac{1}{3}}U_0$. Both demand curves will be downward sloping, convex to the origin and never intersect the axes. However, the ordinary demand is more responsive to changes in price because it only takes account of the substitution effect and not the income effect.

X8.17 Suppose that Adele considers coffee to be an inferior good. Given the law of demand, why would you expect her compensated demand for coffee to be more elastic than her ordinary demand?

With an inferior good, the substitution effect of a price change operates in the opposite direction to the income effect. So the substitution effect, which is the change in compensated demands of a price change, is larger than the total effect of the price change, which is the change in ordinary demands. This means that the compensated demands are more responsive to price changes than the ordinary demands.

X8.18 Bruno has a utility function, U: $U(b, c) = b^{½}c^{½}$, where b is the quantity of beef and c the quantity of chicken that he eats.

a) Show that if Bruno has an amount $m = 64$ to spend, while facing prices $p_b = 2$ and $p_c = 8$, he will choose consumption bundle $(b^*, c^*) = (16, 4)$, and will generate utility $U(16, 4) = 8$.

Recognizing the utility function from previous examples, Bruno's demands will satisfy the requirement that half of his expenditure is on beef, and half on chicken. The results follow immediately.

b) Repeat part (a), but assume that the price p_b is allowed to vary, so that we obtain the ordinary demands, $b^M(p_b)$ and $c^M(p_b)$, as functions of the price p_b.

Bruno's ordinary demands are $b^M = \frac{32}{p_b}$ and $c^M = \frac{32}{p_c}$.

c) Obtain the compensated demands, $b^H(p_b)$ and $c^H(p_b)$.

Facing constraint $U^2 = bc = 64$, and income expansion path $p_b b = p_c c$, $c^2 = \frac{64p_b}{p_c}$, so that $c^H = 8\sqrt{\frac{p_b}{p_c}}$ and $b^H = 8\sqrt{\frac{p_c}{p_b}}$.

d) Confirm that the Hicks decomposition is valid for Bruno's demands for beef and chicken. Discuss how the income and substitution effects lead to changes in the ordinary demands as the price p_b increases.

Applying expression [8.16], $\frac{\partial c^H}{\partial p_c} = -4p_b^{0.5}p_c^{-1.5}$; $\frac{\partial c^M}{\partial p_c} = -32p_c^{-2}$, and $\frac{\partial c^M}{\partial m} = 0.5p_c^{-1}$ so $\frac{\partial c^H}{\partial p_c} - c^M\frac{\partial c^M}{\partial m} = -4p_b^{0.5}p_c^{-1.5} - 0.25m.p_c^{-2}$. Given $p_b = 2$, $p_c = 8$ and $m = 64$, we obtain $\frac{\partial c^H}{\partial p_c} - c^M\frac{\partial c^M}{\partial m} = -0.5 = -\frac{32}{p_c^2}$, confirming that the decomposition is valid in this particular case.

X8.19 We represent Chloe's preferences by the utility function, U: $U(b, c) = b^{0.5} + c^{0.5}$. She faces prices $p_b = p_c = 10$.

a) Obtain her Hicksian demand functions, and the amount, $A^*(U)$, that Chloe has to spend to obtain utility, U.

For income expansion path, $\frac{c}{b} = \left(\frac{p_b}{p_c}\right)^2 = 1$, to obtain utility U, Chloe chooses a consumption bundle (b^H, c^H): $b^H = c^H$ and $2b^{0.5} = U$; then $b^H = c^H = 0.25U^2$. Acquisition cost $A(b^H, c^H) = 5U^2$

b) Show that $\frac{dA^*}{dU} > 1$. What does this suggest about the marginal value of money for Chloe?

Differentiating, $\frac{dA}{dU} = 10U$, and for $U > 0.1$, this derivative is greater than one. As Chloe's utility objective increases, the amount that she needs to spend increases more rapidly. This suggests that the marginal value of money is decreasing.

Chapter 9

X9.1 Confirm that the utility function, U: $U(x, y) = x^a y^{1-a}$, is HOD 1.

A function, U: U = U(x, y), is homogeneous of degree 1 if $U(\lambda x_0, \lambda y_0) = \lambda U(x_0, y_0)$,
$U(x_0, y_0) = x_0^a y_0^{1-a}$; so $U(\lambda x_0, \lambda y_0) = (\lambda x_0)^a(\lambda y_0)^{1-a} = (\lambda^a x_0^a)(\lambda^{1-a} y_0^{1-a}) = \lambda^{a+1-a}(x_0^a y_0^{1-a})$
$= \lambda^1 U(x_0, y_0)$, so that the function is homogeneous of degree 1.

X9.2 Confirm that the members of the family of utility functions defined in Expression 9.1 are HOD ρ.

A function, U: U = U(x, y), is homogeneous of degree ρ if $U(\lambda x_0, \lambda y_0) = \lambda^\rho U(x_0, y_0)$,
For $U(x_0, y_0) = \left[\theta x_0^a + (1-\theta) y_0^a\right]^{\frac{\rho}{a}}$, $U(\lambda x_0, \lambda y_0) = \left[\theta(\lambda x_0)^a + (1-\theta)(\lambda y_0)^a\right]^{\frac{\rho}{a}} =$
$\left(\lambda^a\right)^{\frac{\rho}{a}}\left[\theta x_0^a + (1-\theta) y_0^a\right]^{\frac{\rho}{a}} = \lambda^\rho U(x_0, y_0)$. So, U is HOD$\rho$.
For $U(x_0, y_0) = x_0^{\rho\theta} y_0^{\rho(1-\theta)}$, $U(\lambda x_0, \lambda y_0) = (\lambda x_0)^{\rho\theta}(\lambda y_0)^{\rho(1-\theta)} = \lambda^{\rho(\theta+1-\theta)}(x_0^a y_0^{1-a}) = \lambda^\rho U(x_0, y_0)$. So, U is HOD$\rho$.

X9.3 Confirm that the utility function, U: $U(x, y) = \min[ax^\rho, by^\rho]$, is HOD ρ.

A function, U: U = U(x, y), is homogeneous of degree ρ if $U(x_0, y_0) = \lambda^\rho U(x_0, y_0)$.
We make the argument in two stages. Firstly, show that ax^ρ is HODρ in x, and by^ρ is HODρ in y. Secondly, confirm that then $\min[a(\lambda x_0)^\rho, b(\lambda y_0)^\rho] = \lambda^\rho (a x_0^\rho) = \lambda^\rho \min[a x_0^\rho, b y_0^\rho]$.
Since $a(\lambda x_0)^\rho = \lambda^\rho(a x_0^\rho)$, and $b(\lambda y_0)^\rho = \lambda^\rho(b y_0^\rho)$, the first step is correct.
For the second part, $\min[a(\lambda x_0)^\rho, b(\lambda y_0)^\rho] = \min[\lambda^\rho(a x_0^\rho), \lambda^\rho(b y_0^\rho)] = \lambda^\rho \min[(a x_0^\rho), (b y_0^\rho)]$. This concludes the proof.

X9.4 Show that the utility function, $U: U(x,y) = x^{\frac{1}{3}} + y^{\frac{1}{3}}$, is a member of the family of functions in Expression 9.1, and that it is HOD $\frac{1}{3}$.

Suppose that $\theta = 0.5$ and $\rho = \frac{1}{3}$, then $\left[0.5x^{\frac{1}{3}} + 0.5y^{\frac{1}{3}}\right]^{3.\left(\frac{1}{3}\right)} = 0.5\left[x^{\frac{1}{3}} + y^{\frac{1}{3}}\right] = 0.5U(x, y)$, and since any utility function is unique only to a monotonically increasing transformation, U is a member of the family of CES utility functions, which, from X9.2 is HOD $\frac{1}{3}$.

X9.5 Show that the utility functions, U: $U(x, y) = (x^{-1} + y^{-1})^{-1}$, and V: $V(x, y) = (x^{-2} + y^{-2})^{-0.5}$, can also be written as $U: U(x,y) = \frac{xy}{x+y}$, and $V: V(x,y) = \frac{xy}{(x^2+y^2)^{\frac{1}{2}}}$. Confirm that both U and V are HOD 1.

$(x^{-1} + y^{-1})^{-1} = \left(\frac{1}{x} + \frac{1}{y}\right)^{-1} = \left(\frac{x+y}{xy}\right)^{-1} = \frac{xy}{x+y}$;
$(x^{-2} + y^{-2})^{-0.5} = \left(\frac{1}{x^2} + \frac{1}{y^2}\right)^{-0.5} = \left(\frac{x^2+y^2}{(xy)^2}\right)^{-0.5} = \frac{xy}{(x^2+y^2)^{0.5}}$;
For a function, U: U = U(x, y), to be HOD1, we require $U(\lambda x_0, \lambda y_0) = \lambda U(x_0, y_0)$,
For $U(x_0, y_0) = (x_0^{-1} + y_0^{-1})^{-1}$, $U(\lambda x_0, \lambda y_0) = [(\lambda x_0)^{-1} + (\lambda y_0)^{-1})]^{-1} = (\lambda^{-1})^{-1}[x_0^{-1} + y_0^{-1}]^{-1} = \lambda U(x_0, y_0)$
For $V(x_0, y_0) = \frac{x_0 y_0}{(x_0^2 + y_0^2)^{\frac{1}{2}}}$, $V(\lambda x_0, \lambda y_0) = \frac{(\lambda x_0)(\lambda y_0)}{((\lambda x_0)^2 + (\lambda y_0)^2)^{\frac{1}{2}}} = \frac{\lambda^2 (x_0 y_0)}{(\lambda^2)^{\frac{1}{2}}(x_0^2 + y_0^2)^{\frac{1}{2}}} = \lambda U(x_0, y_0)$.
So both functions are HOD1.

X9.6 Without using Expressions 9.4 or 9.5 obtain the marginal utilities of goods X and Y for the utility functions, U:

a) $U(x, y) = x^{½}y^{½}$ b) $U: U(x,y) = x^{\frac{1}{3}} + y^{\frac{1}{3}}$ c) $U: U(x,y) = \left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^3$

d) $U: U(x,y) = \frac{xy}{x+y}$ **e)** $V: V(x,y) = \frac{xy}{(x^2+y^2)^{\frac{1}{2}}}$.

a) $\frac{\partial U}{\partial x} = \frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}}$; *and* $\frac{\partial U}{\partial y} = \frac{1}{2}x^{\frac{1}{2}}y^{-\frac{1}{2}}$

b) $\frac{\partial U}{\partial x} = \frac{1}{3}x^{-\frac{1}{3}}$; *and* $\frac{\partial U}{\partial y} = \frac{1}{3}y^{-\frac{1}{3}}$

c) $\frac{\partial U}{\partial x} = x^{-\frac{2}{3}}\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^2$; *and* $\frac{\partial U}{\partial y} = y^{-\frac{2}{3}}\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^2$

d) $\frac{\partial U}{\partial x} = \frac{y}{x+y} - \frac{xy}{(x+y)^2} = \left(\frac{y}{x+y}\right)^2$; *and* $\frac{\partial U}{\partial y} = \frac{x}{x+y} - \frac{xy}{(x+y)^2} = \left(\frac{x}{x+y}\right)^2$

e) *Begin by writing* $V(x, y) = (x^{-2} + y^{-2})^{-0.5}$. *Then*

$\frac{\partial V}{\partial x} = x^{-3}\left(x^{-2} + y^{-2}\right)^{-1.5} = x^{-3}\left(\frac{1}{x^2} + \frac{1}{y^2}\right)^{-1.5} = \frac{1}{x^3}\left(\frac{x^2y^2}{x^2+y^2}\right)^{1.5} = \left(\frac{y^2}{x^2+y^2}\right)^{1.5}$; *and by the same argument,*

$\frac{\partial V}{\partial y} = \left(\frac{x^2}{x^2+y^2}\right)^{1.5}$

X9.7 Confirm that for the functions in X9.6c–e, we can rely on Expressions 9.4 and 9.5 to calculate the marginal utilities.

X9.6c) $MU_X(x, y) = \frac{\partial U}{\partial x} = \left(\frac{U(x,y)}{x}\right)^{1-a} = \left(\frac{\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^3}{x}\right)^{\frac{2}{3}} = x^{-\frac{2}{3}}\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^2$; *and similarly,*

$MU_Y(x, y) = y^{-\frac{2}{3}}\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^2$

X9.6d) $MU_X(x, y) = \frac{\partial U}{\partial x} = \left(\frac{U(x,y)}{x}\right)^{1-a} = \left(\frac{\left(x^{-1} + y^{-1}\right)^{-1}}{x}\right)^2 = x^{-2}\left(x^{-1} + y^{-1}\right)^{-2}$; *and similarly,*

$MU_Y(x, y) = y^{-2}\left(x^{-1} + y^{-1}\right)^{-2}$

X9.6e) $MU_X(x, y) = \frac{\partial U}{\partial x} = \left(\frac{U(x,y)}{x}\right)^{1-a} = \left(\frac{\left(x^{-2} + y^{-2}\right)^{-0.5}}{x}\right)^3 = x^{-3}\left(x^{-2} + y^{-2}\right)^{-1.5}$; *and similarly,*

$MU_Y(x, y) = y^{-3}\left(x^{-2} + y^{-2}\right)^{-1.5}$

X9.8 Confirm that for the general form of CES utility functions in Expression 9.1:

$$MU_X(x,y) = \rho\theta\frac{U(x,y)^{\frac{\rho-a}{\rho}}}{x^{1-a}}, \text{ if } a \le 1; \text{ and } MU_Y(x,y) = \rho(1-\theta)\frac{U(x,y)^{\frac{\rho-a}{\rho}}}{y^{1-a}}, \text{ if } a \le 1$$

$MU_X(x,y) = \rho\theta x^{a-1}\left\{\theta x^a + (1-\theta)y^a\right\}^{\frac{\rho}{a}-1}$; *now we rewrite* $\frac{\rho}{a} - 1$ *as* $\frac{\rho}{a}\left(1 - \frac{a}{\rho}\right)$, *and then:*

$MU_X(x,y) = \rho\theta\frac{\left\{\theta x^a + (1-\theta)y^a\right\}^{\frac{\rho}{a}\left(1-\frac{a}{\rho}\right)}}{x^{1-a}} = \rho\theta\frac{U(x,y)^{\left(\frac{\rho-a}{\rho}\right)}}{x^{1-a}}$; *and likewise, we obtain*

$MU_Y(x,y) = \rho(1-\theta)\frac{\left\{\theta x^a + (1-\theta)y^a\right\}^{\frac{\rho}{a}\left(1-\frac{a}{\rho}\right)}}{y^{1-a}} = \rho(1-\theta)\frac{U(x,y)^{\left(\frac{\rho-a}{\rho}\right)}}{y^{1-a}}$

X9.9 Use Expression 9.6 to confirm that along any line passing through the origin that has the equation $y = kx$, the marginal rate of substitution, $MRS(x, y) = -k^{1-a}$.

MRS = $-\left(\frac{y}{x}\right)^{1-a} = -\left(\frac{kx}{x}\right)^{1-a} = -k^{1-a}$.

X9.10 For the following utility functions, obtain the marginal utilities of *x* and *y* and calculate the marginal rate of substitution:

a) $U: U(x, y) = 3x + 2y$; **b)** $U: U(x, y) = x^a y^{(1-a)}$; **c)** $U: U(x,y) = x^{\frac{1}{3}} + y^{\frac{1}{3}}$

d) $U: U(x, y) = [x^{½} + y^{½}]^2$ **e)** $U: U(x,y) = 5x^{\frac{1}{2}} + 3y^{\frac{1}{2}}$

a) $MU_X = 3$; $MU_Y = 2$; $MRS = -\frac{3}{2}$

b) $MU_X = ax^{a-1}y^{1-a}$; $MU_Y = (1-a)x^a y^{-a}$; $MRS = -\dfrac{ay}{(1-a)x}$

c) $MU_X = \frac{1}{3}x^{-\frac{2}{3}}$; $MU_Y = \frac{1}{3}y^{-\frac{2}{3}}$; $MRS = -\left(\dfrac{y}{x}\right)^{\frac{2}{3}}$

d) $MU_X = (x^{½} + y^{½})x^{-½}$; $MU_Y = (x^{½} + y^{½})y^{-½}$; $MRS = -\left(\dfrac{y}{x}\right)^{\frac{1}{2}}$

e) $MU_X = \frac{5}{2}x^{-\frac{1}{2}}$; $MU_Y = \frac{3}{2}y^{-\frac{1}{2}}$; $MRS = -\dfrac{5}{3}\left(\dfrac{y}{x}\right)^{\frac{1}{2}}$

X9.11 For the utility function U: $U(x, y) = \min(3x, 2y)$, show that:

a) the only possible values of the marginal utility of good X are 0 and 3, while the only possible values of the marginal utility of good Y are 0 and 2;

Write the function in the form: $U(x,y) = \begin{cases} 3x, \text{if } 3x \le 2y \\ 2y, \text{if } 3x > 2y \end{cases}$. *Then applying the usual rules of partial differentiation,* $MU_X(x,y) = \begin{cases} 3, \text{if } 3x \le 2y \\ 0, \text{if } 3x > 2y \end{cases}$ *; and* $MU_Y(x,y) = \begin{cases} 0, \text{if } 3x \le 2y \\ 2, \text{if } 3x > 2y \end{cases}$.

b) the marginal rate of substitution cannot be defined when $3x \le 2y$; and

MRS = $-\frac{MU_X}{MU_y} = -\frac{3}{0} = 0$ *when* $3x \ge 2y$

c) $MRS(x, y) = 0$ if $3x > 2y$.

MRS = $-\frac{MU_X}{MU_y} = -\frac{0}{2} = 0$ *when* $3x < 2y$

X9.12 Show that it is possible to write:

a) the equation of the indifference curve in explicit form as:

$$y = \begin{cases} \left[U_0{}^a - x^a\right]^{\frac{1}{a}}, \text{ if } a \le 1, \text{and } a \ne 0 \\ \frac{U_0{}^2}{x}, \text{ if } a = 0 \end{cases} \qquad [9.8]$$

Since $x^a + y^a = U_0{}^a$, *if* $a \le 1$ *and* $a \ne 0$, *while* $xy = U_0{}^2$, *if* $x = 0$, *expression [9.8] follows directly by further rearrangement.*

b) the derivative, $\frac{dy}{dx}$:

$$\left.\frac{dy}{dx}\right|_{U=U_0} = MRS(x,y,U_0) = -\left(\frac{y}{x}\right)^{(1-a)} \qquad [9.9]$$

Differentiating, $\frac{dy}{dx} = \begin{cases} \left(-ax^{a-1}\right)\left(\frac{1}{a}\right)\left[U_0{}^a - x^a\right]^{\frac{1}{a}-1} = -x^{-(1-a)}\left[U_0{}^a - x^a\right]^{\frac{1}{a}(1-a)}, \text{if } a \le 1, \text{and } a \ne 0 \\ -\frac{1}{x}\left(\frac{U_0{}^2}{x}\right), \text{if } a = 0 \end{cases}$.

Expression [9.9] follows immediately on substituting for y.

X9.13 Use the argument above to confirm that for a = ½, indifference curves are convex, but that for a = 2, indifference curves are concave.

From expression [9.11], we have found that the sign of the second derivative of the indifference curve, $\frac{d^2y}{dx^2}$ depends upon the sign of the expression 1 – a. When a = ½, 1 – a > 0, and so $\frac{d^2y}{dx^2}$ > 0; the indifference curve is convex. When a = 2, 1 – a < 0, and so $\frac{d^2y}{dx^2}$ < 0; the indifference curve is concave.

X9.14 Confirm that the indifference curve, with equation, $x^{0.5}y^{0.5}$ = 1, is convex.

Writing $y = x^{-1}$, $\frac{dy}{dx} = -x^{-2}$; and $\frac{d^2y}{dx^2} = 2x^{-3}$ > 0 for all x > 0, so that the indifference curve is convex.

X9.15 By evaluating Expression 9.11 or otherwise, confirm that indifference curves of the utility functions in X9.10 are all (weakly) convex.

We can write the value of parameter 'a' for each of these functions:

a) a = 1; b) a = 0; c) a = $\frac{1}{3}$; d) a = 0.5; e) a = 0.5.

In each case, a $\leq$ 1, so by the argument above, we are certain that the indifference curves associated with these utility functions will be convex.

X9.16 For the utility function, *U*: *U*(*x*, *y*) = *x* + *y*:

a) Confirm that the indifference curve that passes through (*x*, *y*) = (4, 4) also passes through the consumption bundles (8, 0) and (0, 8); and that *MRS* = –1.

The indifference curve passing through (4, 4) has equation x + y = 8. When x = 0, y = 8; and when x = 8, y = 0. From expression [9.6], MRS = -1.

b) Show this indifference curve on a diagram. [*Note:* It will be useful to draw the diagram at least approximately to scale, and to extend the *x*- and *y*-axes to 16.]

Showing x on the horizontal and y on the vertical axis, the indifference curve is a straight line meeting the axes at (8, 0) and (0, 8).

X9.17 For the utility function, *U*: *U*(*x*, *y*) = $x^{½} + y^{½}$:

a) Confirm that the indifference curve that passes through (*x*, *y*) = (4, 4) also passes through the consumption bundles (16, 0) and (0, 16).

The indifference curve passing through (4, 4) has equation $x^{½} + y^{½}$ = 4. When x = 0, $y^{½}$ = 4, so y = 16; and when x = 16, $x^{½}$ = 4, so y = 0.

b) Confirm that $MRS = -\left(\frac{y}{x}\right)^{\frac{1}{2}}$, so that *MRS*(16, 0) = 0; *MRS*(4, 4) = –1; and *MRS*(0, 16) is not defined. What do you conclude about the slope of the indifference curve at each of these three consumption bundles?

From expression [9.6], $MRS = -\left(\frac{y}{x}\right)^{\frac{1}{2}}$. The values of the MRS follow immediately. These indicate that the indifference curve is flat where it meets the horizontal axis, vertical where it meets the vertical axis, and has gradient -1 where it intersects the line y = x.

c) Remembering that the indifference curve is convex and downward sloping, sketch it on the diagram used in X9.16.

The curve is downward sloping and convex; the y-axis is tangent at (0, 16); the line x + y = 8 is tangent at (4, 4); and the x-axis is tangent at (16, 0).

X9.18 Repeat X9.17 for the utility function, $U: U(x,y) = x^{\frac{1}{3}} + y^{\frac{1}{3}}$, but showing that:

a) **The indifference curve passing through (x, y) = (4, 4) also passes through the consumption bundles (32, 0) and (0, 32).**

The indifference curve passing through (4, 4) has equation $x^{\frac{1}{3}} + y^{\frac{1}{3}} = 2\left(2^2\right)^{\frac{1}{3}} = 2^{\frac{5}{3}}$. *When x = 0,* $y^{\frac{1}{3}} = 2^{\frac{5}{3}}$ *so y = 32; and when y = 0,* $x^{\frac{1}{3}} = 2^{\frac{5}{3}}$, *so x = 32.*

b) $MRS = -\left(\frac{y}{x}\right)^{\frac{2}{3}}$**, so that the indifference curve is flat where it meets the horizontal axis, vertical where it meets the vertical axis, and just touches the line $y = 8 - x$ at (x, y) = (4, 4).**

From expression [9.6], $MRS = -\left(\frac{y}{x}\right)^{\frac{2}{3}}$. *The values of the MRS follow immediately. These indicate that the indifference curve is flat where it meets the horizontal axis, vertical where it meets the vertical axis, and has gradient -1 where it intersects the line y = x.*

c) **The indifference curve passes close to the points (16, 0.28) and (0.28, 16). Hence, add this indifference curve to the diagram used in X9.16.**
The curve is downward sloping and convex; the y-axis is tangent at (0, 32); the line x + y = 8 is tangent at (4, 4); and the x-axis is tangent at (32, 0). At (16, 0.28), MRS $\approx$ *-0.067, and at (0.28, 16), MRS* $\approx$ *14.8.*

X9.19 For the utility function, U: $U(x, y) = (xy)^{½}$:

a) **Confirm that the indifference curve that passes through (x, y) = (4, 4) also passes through the consumption bundles (16, 1), (8, 2), (2, 8) and (1, 16). Drawing a new diagram, to the same scale as the diagram for X9.16, sketch this indifference curve.**
The indifference curve passing through (4, 4) has equation $x^{½}y^{½} = 4$. *When x = 1,* $y^{½} = 4$, *so y = 16; when x = 2,* $x^{½} = \sqrt{2}$, *so* $y^{½} = \sqrt{8}$, *and y = 8; the other results follow by the same argument.*

b) **Confirm that for this indifference curve, as $x \to \infty$, $y \to 0$; and that as $x \to 0$, $y \to \infty$.**
As $x \to \infty$, $y = \frac{16}{x} \to 0$; *as* $x \to 0$, $y = \frac{16}{x} \to \infty$;

c) **Confirm that** $MRS = -\left(\frac{y}{x}\right)$**, so that MRS(16, 1) = –0.0625; MRS(4, 4) = –1; and MRS(1, 16) = –16.**
From expression [9.6], $MRS = -\left(\frac{y}{x}\right)$. *The values of the MRS follow immediately. These confirm that the indifference curve is downward sloping and convex.*

X9.20 Repeat X9.19 for the utility function, $U: U(x,y) = \frac{xy}{x+y} = \left(\frac{1}{x} + \frac{1}{y}\right)^{-1}$**, showing that:**

a) **The indifference curve that passes through (x, y) = (4, 4) also passes through the consumption bundles (18, 2.25), (10, 2.5), (6, 3), (3, 6), (2.5, 10) and (2.25, 18). Hence sketch its graph on the diagram used for X9.19.**

The indifference curve passing through (4, 4) has equation $\left(x^{-1} + y^{-1}\right)^{-1} = 2$. *We can rewrite this indifference curve explicitly in terms of x as* $y = \frac{2x}{x-2}$

x	*18*	*10*	*6*	*3*	*2.5*	*2.25*
y	*2.25*	*2.5*	*3*	*6*	*10*	*18*

b) **Confirm that** $MRS = -\left(\frac{y}{x}\right)^2$**, so that MRS(18, 2.25) =** $-\frac{1}{64}$**; MRS(10, 2.5) =** $-\frac{1}{16}$**; MRS(6, 3) =** $-\frac{1}{4}$**; MRS(4, 4) = –1; MRS(6, 3) = –4; MRS(2.5, 10) = –16; and MRS(2.25, 18) = –64.**

From expression [9.6], $MRS = -\left(\frac{y}{x}\right)^2$. The values of the MRS follow immediately. These confirm that the indifference curve is downward sloping and convex.

c) Confirm that if $x \to \infty$, then $y \to 2$, but that if $x \to 2$, $y \to \infty$. On the diagram, draw in these two lines, which the indifference curve approaches but does not cross.

Given that $y = \frac{2x}{x-2}$, then as $x \to \infty$, $\frac{x}{x-2} \to 1$, so $y \to 2$. As $x \to 2$, $x - 2 \to 0$, so $y \to \infty$. The indifference curve approaches, but does not cross the lines $y = 2$ (which is horizontal) and $x = 2$ (which is vertical).

X9.21 Using the same diagram as in X9.19 and X9.20, and for the utility function U: $U(x, y) = \min(x, y)$, draw the indifference curve that passes through the bundle $(x, y) = (4, 4)$. Discuss the behaviour of the *MRS* function.

The indifference curve will be L shaped with vertex (4, 4). The horizontal arm is that part of the line, y = 4, to the right of (4, 4), while the vertical arm is that part of the line, x = 4, above (4, 4).

X9.22 Confirm that all indifference curves of the utility function, $U: U(x,y) = \left[x^a + y^a\right]^{\frac{1}{a}}$, for which $0 < a < 1$, have the properties stated above.

When $x = y = x_0$, $U(x, y) = \left(2x_0^{\ a}\right)^{\frac{1}{a}} = 2^{\frac{1}{a}} x_0$. On the indifference curve $U: U(x,y) = \left[x^a + y^a\right]^{\frac{1}{a}} = 2^{\frac{1}{a}} x_0$, if $x = 0$, then $y = 2^{\frac{1}{a}} x_0$. If $y = 0$, then $x = 2^{\frac{1}{a}} x_0$

We know that MRS = $-\left(\frac{y}{x}\right)^{1-a}$, so when $x = y$, MRS = -1. Since $x = y = x_0$, it follows that the equation of the tangent will be $x + y = 2x_0$. When $x = 0$, MRS is undefined; and when $y = 0$, MRS = 0.

X9.23 Confirm that all indifference curves of the utility function, $U: U(x,y) = \left[x^a + y^a\right]^{\frac{1}{a}}$, for which $a < 0$, have the properties stated above.

When $x = y = x_0$, $U(x, y) = \left(2x_0^{\ a}\right)^{\frac{1}{a}} = 2^{\frac{1}{a}} x_0$. We can write the equation of the indifference curve $U: U(x,y) = x^a + y^a = 2x_0^{\ a}$ as $y^a = 2x_0^{\ a} - x^a$. Since $a < 0$, it will be convenient to rewrite the right-hand side of this expression as $\frac{2}{x_0^{\ -a}} - \frac{1}{x^{-a}}$, where $-a > 0$. We can then rewrite this expression as a single fraction: $\frac{2x^{-a} - x_0^{\ -a}}{x_0^{\ -a} x^{-a}}$. To obtain $y > 0$, we require $2x^{-a} - x_0^{\ -a} > 0$ so that

$2^{-\frac{1}{a}} x > x_0$, and

$x > 2^{\frac{1}{a}} x_0$. Unless this threshold value of x is achieved, the level of utility associated with the indifference curve cannot be reached.

A very similar argument can then be made for y, replacing x by y throughout.

From the preceding argument, we see that as $x \to 2^{\frac{1}{a}} x_0$, $y \to \infty$; and similarly, that as $y \to 2^{\frac{1}{a}} x_0$, $x \to \infty$. We know that MRS = $-\left(\frac{y}{x}\right)^{1-a}$, so when $x = y$, MRS = -1; and as $x \to 2^{\frac{1}{a}} x_0$, MRS $\to -\infty$; and when $x \to \infty$, $y \to 2^{\frac{1}{a}} x_0$, and MRS $\to 0$.

X9.24 For the following utility functions – and except for functions (a) and (b), for which you may find graphical analysis more straightforward – use Expression 9.16 to obtain the Marshallian or ordinary demands for a consumer, for whom $m = 200$, $p_y = 5$ and p_x is allowed to vary:

a) $U: U(x, y) = 3x + 2y$; **b)** $U: U(x, y) = \min(2x, 3y)$; **c)** $U: U(x, y) = x^{0.5}y^{0.5}$;

d) $U:U(x,y)=x^{\frac{1}{3}}+y^{\frac{1}{3}}$; **e)** $U: U(x, y) = [x^{½} + y^{½}]^2$; **f)** $U: U(x, y) = (x^{-1} + y^{-1})^{-1}$;

g) $U:U(x,y)=\frac{xy}{(x^2+y^2)^{0.5}}$.

a) We know in this case that the marginal utilities are constant, and that goods X and Y are perfect substitutes. With $p_y = 5$, abstaining from consumption of good X, the consumer can afford the consumption bundle (0, 40), generating a utility U(0, 40) = 80. This consumer can generate the same utility from the consumption bundle $\left(\frac{80}{3},0\right)$, which is affordable if $p_x \le 7.5$. Marshallian demands: $x^M(p_x)=\begin{cases}\frac{200}{p_x}, \text{if } p_x<7.5\\ \left[0,\frac{80}{3}\right], \text{if } p_x=7.5\\ 0, \text{if } p_x>7.5\end{cases}$;

$$y^M(p_x)=\begin{cases}0, \text{if } p_x<7.5\\ [0,40], \text{if } p_x=7.5\\ 40, \text{if } p_x>7.5\end{cases}$$

b) Define a composite unit, which contains one unit of good Y and 1.5 units of good X. Such a unit generates utility U(1.5, 1) = 3; but increasing consumption of only one good will not increase utility further (while decreasing consumption of either good will reduce utility). The acquisition cost of each composite unit A(1.5, 1) = 5 + 1.5p_x. The consumer will purchase C = $\frac{200}{5+1.5p_x}$ *composite units, giving Marshallian demands* $x^M = \frac{300}{5+1.5p_x}$, $y^M = \frac{200}{5+1.5p_x}$.

c) We have a = 0, so that $x^M(p_x, 5, 200) = \frac{100}{p_x}$; *and* $y^M(p_x, 5, 200) = 20$.

d) With a = $\frac{1}{3}$, $x^M(p_x, 5, 200) = 200\left[\frac{p_x^{-\frac{3}{2}}}{p_x^{-\frac{1}{2}}+5^{-\frac{1}{2}}}\right]$; *and* $y^M(p_x, 5, 200) = 200\left[\frac{5^{-\frac{3}{2}}}{p_x^{-\frac{1}{2}}+5^{-\frac{1}{2}}}\right]$

e) With a = ½, $x^M(p_x, 5, 200) = 200\left[\frac{p_x^{-2}}{p_x^{-1}+5^{-1}}\right]$; *and* $y^M(p_x, 5, 200) = 200\left[\frac{5^{-2}}{p_x^{-1}+5^{-1}}\right]$

f) With a = -1, $x^M(p_x, 5, 200) = 200\left[\frac{p_x^{-0.5}}{p_x^{0.5}+5^{0.5}}\right]$; *and* $y^M(p_x, 5, 200) = 200\left[\frac{5^{-0.5}}{p_x^{0.5}+5^{0.5}}\right]$

g) With a = -2, $x^M(p_x, 5, 200) = 200\left[\frac{p_x^{-\frac{1}{3}}}{p_x^{\frac{2}{3}}+5^{\frac{2}{3}}}\right]$; *and* $y^M(p_x, 5, 200) = 200\left[\frac{5^{-\frac{1}{3}}}{p_x^{\frac{2}{3}}+5^{\frac{2}{3}}}\right]$

X9.25 By obtaining the partial derivative, $\frac{\partial x^M}{\partial p_x}$, or otherwise, confirm that for the utility functions in X9.24c, X9.24e and X9.24f, the demand, $x^M(p_x; 5, 200)$, decreases as the price, p_x, increases.

From X9.24c), $x^M(p_x, 5, 200) = \frac{100}{p_x}$, *so* $\frac{dx}{dp_x}=-\frac{100}{p_x^2}<0$

From X9.24e)., $x^M(p_x, 5, 200) = \frac{200}{p_x^2}\left(\frac{5p_x}{5+p_x}\right)=\frac{1,000}{(5p_x+p_x^2)}$. *Differentiating,* $\frac{\partial x^M}{\partial p_x}=-\frac{200(5+2p_x)}{(5p_x+p_x^2)^2}<0$.

From X9.24f), $x^M(p_x, 5, 200) = \frac{200}{p_x^{0.5}}\left(\frac{1}{5^{0.5}+p_x^{0.5}}\right)=\frac{200}{((5p_x)^{0.5}+p_x)}$. *Differentiating,*

$$\frac{\partial x^M}{\partial p_x}=-\frac{100(5^{0.5}+2p_x^{0.5})}{p_x^{0.5}((5p_x)^{0.5}+p_x)^2}<0.$$

X9.26 By obtaining the partial derivative, $\frac{\partial y^M}{\partial p_x}$, or otherwise, confirm that:

a) **for the utility function in X9.24e, the demand, $y^M(p_x; 5, 200)$, is an increasing function of the price, p_x;**

From X9.24e), $y^M(p_x, 5, 200) = \frac{200}{25}\left(\frac{5p_x}{5+p_x}\right) = \frac{40p_x}{(5+p_x)} = 40 - \frac{200}{(5+p_x)}$. *Differentiating,* $\frac{\partial y^M}{\partial p_x} = \frac{200}{(5+p_x)^2} > 0$.

b) **for the utility function in X9.24c, the demand, $y^M(p_x; 5, 200)$, remains constant as the price, p_x, changes; and**

From X9.24c), $y^M = 20$, *so* $\frac{dy^M}{dp_x} = 0$

c) **for the utility function in X9.24f, the demand, $y^M(p_x; 5, 200)$, is a decreasing function of the price, p_x.**

From X9.24f), $y^M(p_x, 5, 200) = \frac{200}{5^{0.5}}\left(\frac{1}{5^{0.5}+p_x^{0.5}}\right)$. *Differentiating,* $\frac{\partial y^M}{\partial p_x} = -\frac{100}{(5p_x)^{0.5}(5^{0.5}+p_x^{0.5})^2} < 0$.

X9.27 Repeat X9.24, but obtaining the Hicksian demands, x^H and y^H, for the following cases, with $p_x = p_y = 5$ and with the minimum acceptable utility, U_0, is allowed to vary:

a) U: $U(x, y) = 3x + 2y$; b) U: $U(x, y) = \min(2x, 3y)$; c) U: $U(x, y) = x^{0.5}y^{0.5}$;

d) $U: U(x,y) = x^{\frac{1}{3}} + y^{\frac{1}{3}}$; e) U: $U(x, y) = [x^{½} + y^{½}]^2$; f) U: $U(x, y) = (x^{-1} + y^{-1})^{-1}$;

g) $U: U(x,y) = \frac{xy}{(x^2+y^2)^{0.5}}$.

a) With $p_x = p_y = 5$, the consumer will find good X generates more utility per pound spent than good Y. So in order to generate utility U_0, the consumer will purchase the bundle $(x^H, y^H) = \left(\frac{U_0}{3}, 0\right)$.

b) The consumer will generate $U = 1$ by purchasing the bundle $\left(\frac{1}{2}, \frac{1}{3}\right)$. So to generate the utility, U_0, the consumer demands $(x^H, y^H) = \left(\frac{U_0}{2}, \frac{U_0}{3}\right)$.

c) $x^H = y^H = U_0$. ($a = 0$)

d) $x^H = y^H = 0.125U_0^3$. ($a = \frac{1}{3}$)

e) $x^H = y^H = 0.25U_0$. ($a = 0.5$)

f) $x^H = y^H = 2U_0$. ($a = -1$)

g) $x^H = y^H = 4U_0$. ($a = -2$)

X9.28 For the utility function in X9.27c, obtain the Hicksian demands, x^H and y^H, when the price $p_y = 5$, the minimum acceptable utility $U_0 = 20$, and the price, p_x, is allowed to vary.

The problem is to minimize expenditure $A = p_xx + 5y$, while achieving the target utility $U_0 = (xy)^{0.5} = 20$.

We know that the income expansion path has equation $p_xx = 5y$. So $\sqrt{\frac{p_x}{5}}x = 20$; and $x^H = 20\sqrt{\frac{5}{p_x}}$;

$y^H = 20\sqrt{\frac{p_x}{5}}$

X9.29 Repeat X9.28 for the utility function in X9.27e, with $U_0 = 80$.

Applying expression [9.19], with $a = 0.5$, $U_0 = 80$ and $p_y = 5$,

$$x^H = 80\left(\frac{p_x^{-1}}{p_x^{-1}+0.2}\right)^2; \; y^H = 80\left(\frac{0.2}{p_x^{-1}+0.2}\right)^2$$

X9.30 Repeat X9.28 for the utility function in X9.27f, with $U_0 = 10$.

Applying expression [9.19], with $a = -1$, $U_0 = 10$ and $p_y = 5$,

$$x^H = 10\left(\frac{p_x^{0.5}}{p_x^{0.5}+5^{0.5}}\right)^{-1};\ y^H = 10\left(\frac{5^{0.5}}{p_x^{0.5}+5^{0.5}}\right)^{-1}$$

X9.31 Confirm that the indirect utility function is:

a) homogeneous of degree –1 in prices (so that if both prices, p_x and p_y, increase by k%, but m, the money available to finance consumption, remains constant, the indirect utility, V, decreases by k%);

From expression [9.21], $V(p_x, p_y, m) = m\left[p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right]^{\frac{1-a}{a}}$. *Now, for this function V to be homogeneous of degree -1 in prices,* $V(kp_x, kp_y, m) = k^{-1}V(p_x, p_y, m)$.

Since $m\left[(kp_x)^{-\frac{a}{1-a}}+(kp_y)^{-\frac{a}{1-a}}\right]^{\frac{1-a}{a}} = \left(k^{-\frac{a}{1-a}}\right)^{\frac{1-a}{a}}\left\{m\left[p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right]^{\frac{1-a}{a}}\right\} = k^{-1}V(p_x, p_y, m)$*, the condition is met, and V is HOD1 in prices.*

b) homogeneous of degree 0 in prices and the sum available to finance consumption.

From expression [9.21], $V(p_x, p_y, m) = m\left[p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right]^{\frac{1-a}{a}}$. *Now, for a function to be homogeneous of degree 0 in prices and money to finance consumption,* $V(kp_x, kp_y, km) = V(p_x, p_y, m)$.

Since $km\left[(kp_x)^{-\frac{a}{1-a}}+(kp_y)^{-\frac{a}{1-a}}\right]^{\frac{1-a}{a}} = k^1\left(k^{-\frac{a}{1-a}}\right)^{\frac{1-a}{a}}\left\{m\left[p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right]^{\frac{1-a}{a}}\right\} = k^{(1-1)}V(p_x, p_y, m)$*, the condition is met, and V is HOD1 in prices.*

X9.32 Explain why it is sensible that the indirect utility should not change as m, p_x and p_y increase by the same proportion.

Utility is derived from consumption. If both the money available to finance consumption and prices increase by the same quantity, then the real resources available to the consumer – the combinations of goods and services in the affordable set – do not change. That is, the constraint on generating utility is not affected by an increase in the price level (which is matched by an equi-proportional increase in the money to finance consumption) and so the value of the objective, or the utility that can be generated, will not change.

X9.33 Suppose that $U_0 = m\left[p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right]^{\left(\frac{1-a}{a}\right)}$, the indirect utility achievable when there is an amount m to finance consumption. Show that $E(p_x, p_y, U_0) = m$.

Given that $E(p_x, p_y, U_0) = U_0\left[p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right]^{-\frac{1-a}{a}}$*, then if* $U_0 = m\left[p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right]^{\left(\frac{1-a}{a}\right)}$,

$E(p_x, p_y, U_0) = m\left[p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right]^{\frac{1-a}{a}-\frac{1-a}{a}} = m\left[p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right]^0 = m$.

9.34 For the utility function, U: $U(x, y) = [x^{½} + y^{½}]^2$, with prices initially set at $p_x = 2$ and $p_y = 3$, a utility-maximizing consumer has $m = 30$ to spend. Confirm each of the following statements.

a) The marginal utilities of goods X and Y are $MU_X = x^{-½}[x^{½} + y^{½}]$ and $MU_Y = y^{-½}[x^{½} + y^{½}]$.

This follows directly from evaluation of expression [9.4].

b) The marginal rate of substitution, $MRS = -\left(\frac{y}{x}\right)^{\frac{1}{2}}$.

This follows directly from expression [9.6].

c) **The income expansion path may be written $y=\frac{4}{9}x$.**

This follows from expression [9.14].

d) **The utility-maximizing consumption bundle (x^*, y^*) = (9, 4), and the consumer derives a utility of 25 from consuming it.**

This follows from expressions [9.16] and [9.21].

e) **A cost-minimizing consumer who purchases the bundle (x^*, y^*) = (9, 4) must have a minimum acceptable utility, V = 25.**

This follows from expressions [9.19] and [9.22].

X9.35 Confirm that:

a) **$x^H = f(p_x).g(p_x)$, where** $f(p_x)=U_0\left(p_x^{-\frac{1}{1-a}}\right) and\, g(p_x)=\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-\frac{1}{a}}$.

This follows directly by simplifying the factor $\left(p_x^{-\frac{a}{1-a}}\right)^{\frac{1}{a}}$ *in expression [9.19a].*

b) $\frac{df}{dp_x}=-\frac{U_0}{1-a}\left(p_x^{-\frac{2-a}{1-a}}\right) and\, \frac{dg}{dp_x}=\frac{1}{1-a}p_x^{-\frac{1}{1-a}}\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-\frac{1+a}{a}}$.

We obtain $\frac{df}{dp_x}$ by application of the power rule of differentiation. We obtain $\frac{dg}{dp_x}$ by application of the chain rule.

$$\frac{dg}{dp_x}=\left(-\frac{a}{1-a}p_x^{-\left(\frac{a}{1-a}+1\right)}\right)\left[-\frac{1}{a}\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-\left(\frac{1}{a}-1\right)}\right]=\frac{1}{1-a}p_x^{-\frac{1}{1-a}}\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-\frac{1+a}{a}}$$

c) $\frac{\partial x^H}{\partial p_x}=-\frac{U_0}{1-a}\left(p_x^{-\frac{2-a}{1-a}}\right)\left(p_y^{-\frac{a}{1-a}}\right)\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-\frac{1+a}{a}}$. **[*Note:* You will need to apply the product rule, and then to simplify the expression.]**

$$\frac{\partial x^H}{\partial p_x}=f(p_x).\frac{\partial g}{\partial p_x}+\frac{\partial f}{\partial p_x}g(p_x)=U_0\left(p_x^{-\frac{1}{1-a}}\right).\left[\frac{1}{1-a}p_x^{-\frac{1}{1-a}}\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-\frac{1+a}{a}}\right]-\left[\frac{U_0}{1-a}\left(p_x^{-\frac{2-a}{1-a}}\right)\right]\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-\frac{1}{a}}$$

$$=\frac{U_0}{1-a}\left(p_x^{-\frac{2-a}{1-a}}\right)\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-\frac{1+a}{a}}\left[p_x^{-\frac{a}{1-a}}-\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)\right]$$

$$=-\frac{U_0}{1-a}\left(p_x^{-\frac{2-a}{1-a}}\right).\left(p_y^{-\frac{a}{1-a}}\right).\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-\frac{1+a}{a}}$$

d) **If** $U_0=m\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{\frac{1-a}{a}}$**, then:**

$$\frac{\partial x^H}{\partial p_x}=-\frac{m}{1-a}\left(p_x^{-\frac{2-a}{1-a}}\right).\left(p_y^{-\frac{a}{1-a}}\right)\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-2}. \quad \textbf{[9.26]}$$

This result follows directly from substitution.

e) $\frac{\partial x^H}{\partial p_x}<0$.

Every factor in the partial derivative has a positive value; by assumption, utility and prices are all positive (so that powers of prices are always positive values); and a < 1. The sign of the derivative is therefore given by the initial negative sign.

f) **Subtracting Expression 9.26 from Expression 9.25:**

$$\frac{\partial x^M}{\partial p_x}-\frac{\partial x^H}{\partial p_x}=-m\left(p_x^{-\frac{2}{1-a}}\right)\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-2}. \quad \textbf{[9.27]}$$

$$\frac{\partial x^M}{\partial p_x}-\frac{\partial x^H}{\partial p_x}=-m\left(p_x^{-\frac{2-a}{1-a}}\right).\left[p_x^{-\frac{a}{1-a}}+\left(\frac{1}{1-a}\right)p_y^{-\frac{a}{1-a}}\right]\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-2}+\frac{m}{1-a}\left(p_x^{-\frac{2-a}{1-a}}\right).\left(p_y^{-\frac{a}{1-a}}\right)\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-2}$$

$$=-m\left(p_x^{-\frac{2-a}{1-a}}\right).p_x^{-\frac{a}{1-a}}.\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-2}=-m\left(p_x^{-\frac{2}{1-a}}\right)\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-2}$$

This is the difference between the total effect and the substitution effect, and so we should expect to find that it is equal to the income effect of the price change.

X9.36 Confirm that:

a) $\frac{\partial x^M}{\partial m}=\left(p_x^{-\frac{1}{1-a}}\right)\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-1}$;

Going back to expression [9.16], $x^M=m\left(p_x^{-\frac{1}{1-a}}\right)\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-1}$, *which is linear in m, and the partial derivative follows immediately.*

b) $-x^M.\frac{\partial x^M}{\partial m}=-m\left(p_x^{-\frac{2}{1-a}}\right)\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-2}$; **[9.28]**

This is simply the result of the multiplication of the expressions discussed in part a).

c) and $\frac{\partial x^H}{\partial p_x}-x^M.\frac{\partial x^M}{\partial m}=\frac{\partial x^M}{\partial p_x}$.

Note that expressions [9.27] and [9.28] are identical. The result follows immediately.

X9.37 For each of the following utility functions, obtain the income and substitution effects. [*Note:* You can treat all of these as CES utility functions, with the last three being special cases.] The value of parameter *a* in each is: (a) 0.5; (b) –1; (c) 0; (d) 1; and (e) undefined (–∞).

a) $U(x, y) = [x^{½} + y^{½}]^2$; **b)** $U(x, y) = [x^{-1} + y^{-1}]^{-1}$; **c)** $U(x, y) = x^{0.5}y^{0.5}$;

d) $U(x, y) = x + y$; **e)** $U(x, y) = \min[x, y]$.

We apply expressions [9.27] and [9.28]. The substitution effect,

$\frac{\partial x^H}{\partial p_x}=-\frac{U_0}{1-a}\left(p_x^{-\frac{2-a}{1-a}}\right)\left(p_y^{-\frac{a}{1-a}}\right)\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-\frac{1+a}{a}}$, *although we make the assumption that* $U_0=m\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{\frac{1-a}{a}}$, *so that the substitution effect can be rewritten in terms of the money available to finance consumption, m. The substitution effect is then*

$\frac{\partial x^H}{\partial p_x}=-\frac{m}{1-a}\left(p_x^{-\frac{2-a}{1-a}}\right)\left(p_y^{-\frac{a}{1-a}}\right)\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-2}$, *with income effect,*

$-x^M.\frac{\partial x^M}{\partial m}=-m\left(p_x^{-\frac{2}{1-a}}\right)\left(p_x^{-\frac{a}{1-a}}+p_y^{-\frac{a}{1-a}}\right)^{-2}$

a) a = 0.5: $\frac{\partial x^H}{\partial p_x}=-2U_0\left(p_x^{-3}\right)\left(p_y^{-1}\right)\left(p_x^{-1}+p_y^{-1}\right)^{-3}=\frac{\partial x^H}{\partial p_x}=-2m\left(p_x^{-3}\right)\left(p_y^{-1}\right)\left(p_x^{-1}+p_y^{-1}\right)^{-2}$;

$-x^M.\frac{\partial x^M}{\partial m}=-m\left(p_x^{-4}\right)\left(p_x^{-1}+p_y^{-1}\right)^{-2}$

b) a = -1: $\frac{\partial x^H}{\partial p_x}=-0.5U_0\left(p_x^{-1.5}\right)\left(p_y^{0.5}\right)=-0.5m\left(p_x^{-1.5}\right)\left(p_y^{0.5}\right)\left(p_x^{0.5}+p_y^{0.5}\right)^{-2}$;

$-x^M.\frac{\partial x^M}{\partial m}=-m\left(p_x^{-1}\right)\left(p_x^{0.5}+p_y^{0.5}\right)^{-2}$

c) With these special cases, it will be useful to refer to the results that we have already obtained. Here U(x, y) = $(xy)^{0.5}$. *From expression [9.16], we obtain the ordinary demands,* $x^M=\frac{m}{2p_x}$ *and* $y^M=\frac{m}{2p_y}$, *and indirect utility* $V(p_x, p_y, m)=\frac{m}{2\sqrt{p_xp_y}}$. *Since a = 0, we did not use* *expression [9.19] to calculate the compensated demands, instead calculating these directly in X9.27c), obtaining* $x^H = U_0\sqrt{\frac{p_y}{p_x}}$.

Differentiating the ordinary demand, x^M*, with respect to* p_x*, we obtain* $\frac{\partial x^M}{\partial p_x} = -\frac{m}{2p_x^2}$*; differentiating the compensated demand,* x^H*, with respect to price,* p_x*, we obtain* $\frac{\partial x^H}{\partial p_x} = -0.5U_0\frac{p_y^{0.5}}{p_x^{1.5}}$*; and substituting for* U_0*, we can write this expression as* $\frac{\partial x^H}{\partial p_x} = -\frac{m}{4p_x^2}$*; lastly, evaluating the expression,* $x^M\frac{\partial x^M}{\partial m}$*, we obtain* $-x^M\frac{\partial x^M}{\partial m} = -\frac{m}{4p_x^2}$*, allowing us to confirm that the Hicks' decomposition holds.*

d) *We recognize the utility function, U: U(x, y) = x + y, as a special case where goods X and Y are perfect substitutes. We know that if* $x^M > 0$*, then* $p_x \le p_y$*. Assuming that* $p_x < p_y$*, we obtain ordinary demands* $(x^M, y^M) = \left(\frac{m}{p_x}, 0\right)$*, with indirect utility* $V(p_x, p_y, m) = \frac{m}{p_x}$*. Similarly, seeking to obtain utility* U_0*, the compensated demands* $(x^H, y^H) = (U_0, 0)$*.*

Differentiating these demands, we obtain $\frac{\partial x^M}{\partial p_x} = -\frac{m}{p_x^2}$*;* $\frac{\partial x^H}{\partial p_x} = 0$*; and* $-x^M\frac{\partial x^M}{\partial m} = -\frac{m}{p_x^2}$*. The decomposition holds, but it is trivial.*

e) *We recognize the utility function, U: U(x, y) = min(x + y), as a special case where goods X and Y are perfect complements. We know that the income expansion path has equation, x = y, and that there is no substitution effect of a price change.*

To obtain the ordinary demands, we find the consumption bundle on the income expansion path where the budget constraint, $p_x x + p_y y = m$*, so that* $x^M = y^M = \frac{m}{p_x + p_y}$*. The indirect utility can then be written as* $V(p_x, p_y, m) = \frac{m}{p_x + p_y}$*.*

Seeking to obtain utility U_0*, the compensated demands* $(x^H, y^H) = (U_0, U_0)$*.*

Differentiating these demands, we obtain $\frac{\partial x^M}{\partial p_x} = -\frac{m}{(p_x + p_y)^2}$*;* $\frac{\partial x^H}{\partial p_x} = 0$*; and* $-x^M\frac{\partial x^M}{\partial m} = -\frac{m}{(p_x + p_y)^2}$*. As in part d), the decomposition is trivial.*

X9.38 We define the Hicks decomposition of the change in the ordinary demand, x^M, following a change in price p_y as:

$$\frac{\partial x^M(p_x, p_y, m)}{\partial p_y} = \frac{\partial x^H(p_x, p_y, U_0)}{\partial p_y} - y^M(p_x, p_y, m)\frac{\partial x^M(p_x, p_y, m)}{\partial m} \qquad [9.30]$$

For a consumer with a CES utility function, maximizing utility in the usual way, show that:

a) **The total effect,** $\frac{\partial x^M(p_x, p_y, m)}{\partial p_y} = \frac{a}{1-a}m(p_x p_y)^{-\frac{1}{1-a}}\left(p_x^{-\frac{a}{1-a}} + p_y^{-\frac{a}{1-a}}\right)^{-2}$**.** **[9.31]**

Recall from expression [9.16] that the ordinary demand, $x^M = mp_x^{-\frac{1}{1-a}}\left(p_x^{-\frac{a}{1-a}} + p_y^{-\frac{a}{1-a}}\right)^{-1}$*.*

Then differentiating x^M *with respect to price,* p_y*:*

$\frac{\partial x^M}{\partial p_y} = mp_x^{-\frac{1}{1-a}}\left(-\frac{a}{1-a}p_y^{-\frac{1}{1-a}}\right)\left[(-1)\left(p_x^{-\frac{a}{1-a}} + p_y^{-\frac{a}{1-a}}\right)^{-2}\right]$ *(this is quite a straightforward application of the power and chain rules of differentiation). Collecting terms, we obtain expression [9.31].*

b) **The substitution effect,** $\frac{\partial x^H(p_x, p_y, m)}{\partial p_y} = \frac{1}{1-a}m(p_x p_y)^{-\frac{1}{1-a}}\left(p_x^{-\frac{a}{1-a}} + p_y^{-\frac{a}{1-a}}\right)^{-2}$**.** **[9.32]**

Recall from expression [9.19] that the compensated demand, $x^H = U_0 p_x^{-\frac{1}{1-a}}\left(p_x^{-\frac{a}{1-a}} + p_y^{-\frac{a}{1-a}}\right)^{-\frac{1}{a}}$*.*

Then differentiating x^H *with respect to price,* p_y*, note that once again, price* p_y *appears only in the expression in brackets, so that we again used the power and chain rules of differentiation, obtaining:* $\frac{\partial x^H}{\partial p_y} = U_0 p_x^{-\frac{1}{1-a}}\left(-\frac{a}{1-a}p_y^{-\frac{1}{1-a}}\right)\left[\left(-\frac{1}{a}\right)\left(p_x^{-\frac{a}{1-a}} + p_y^{-\frac{a}{1-a}}\right)^{-\frac{1+a}{a}}\right]$*. Collecting terms, this simplifies* *to* $\frac{\partial x^H}{\partial p_y} = \frac{U_0}{1-a}\left(p_x p_y\right)^{-\frac{1}{1-a}}\left(p_x^{-\frac{a}{1-a}} + p_y^{-\frac{a}{1-a}}\right)^{\frac{1+a}{a}}$ *. We evaluate this expression for the achievable indirect utility, given prices and the amount available to finance consumption:* $U_0 = m\left(p_x^{-\frac{a}{1-a}} + p_y^{-\frac{a}{1-a}}\right)^{\frac{1-a}{a}}$*, obtaining expression [9.32].*

c) **The income effect,** $-y^M\left(p_x, p_y, m\right)\frac{\partial x^M\left(p_x, p_y, m\right)}{\partial m} = -m\left(p_x p_y\right)^{-\frac{1}{1-a}}\left(p_x^{-\frac{a}{1-a}} + p_y^{-\frac{a}{1-a}}\right)^{-2}$**.** **[9.33]**

This expression follows easily, given that the ordinary demand is linear in the money available to finance consumption.

d) **The substitution effect is always positive, and the income effect is always negative.**

Inspecting expression [9.32], we see that for it to take a negative value, 1 – a < 0, so that a > 1. By assumption, we exclude such a value. Similarly since every part of expression [9.33] must take a positive value, its sign is determined by the negative sign.

e) **The total effect is always less than the substitution effect.**

Expressions [9.31] and [9.32] are identical, except for the factors $\frac{a}{1-a}$ *in [9.31] and* $\frac{1}{1-a}$ *in [9.32]. Subtracting one from the other,* $\frac{1}{1-a} - \frac{a}{1-a}$ *= 1, so that the substitution effect is always greater.*

f) **The total effect is positive if *a* > 0, and negative if *a* < 0.**

The total effect is the sum of a positive substitution effect and a negative income effect. Its sign (in expression [9.31]) is determined by the factor $\frac{a}{1-a}$*. Given that a < 1, the denominator is always greater than zero, so the sign of this factor is the same as the sign of a; that is, there is a positive total effect when a > 0, and a negative total effect when a < 0.*

X9.39 **In Figure 9.2, we illustrate the effect of a price change on the Marshallian demands,** $x^M(p_x, p_y, m)$**. In both panels,** $m = 2$**, and initially** $p_x = p_y = 1$**, with** p_x **falling to** $\frac{2}{3}$**. In Figure 9.2a, the utility function is** U**:** $U(x, y) = [x^{0.5} + y^{0.5}]^2$**; while in Figure 9.2b, the utility function is** V**:** $V(x, y) = [x^{-1} + y^{-1}]^{-1}$**.**

a) **Confirm that the budget constraint is initially** $x + y = 2$**, but that after the price change it may be written** $2x + 3y = 6$**.**

Writing the constraint as $p_x x + p_y y = m$*, initially,* $x + y = 2$*; and after the price change,* $\frac{2}{3}x + y = 2$*, so* $2x + 3y = 6$*.*

b) **Use Expression 9.16 to calculate the ordinary demands before and after the price change.**

Given the ordinary demands, $\left(x^M, y^M\right) = \left(m\left(\frac{p_x^{-\frac{1}{1-a}}}{p_x^{-\frac{a}{1-a}} + p_y^{-\frac{a}{1-a}}}\right), m\left(\frac{p_y^{-\frac{1}{1-a}}}{p_x^{-\frac{a}{1-a}} + p_y^{-\frac{a}{1-a}}}\right)\right)$*, when a = 0.5, we evaluate these expressions as* $\left(x^M, y^M\right) = \left(m\left(\frac{p_x^{-2}}{p_x^{-1} + p_y^{-1}}\right), m\left(\frac{p_y^{-2}}{p_x^{-1} + p_y^{-1}}\right)\right)$*. We rewrite these in the slightly simpler form,* $\left(x^M, y^M\right) = \left(\frac{m}{p_x}\left(\frac{p_y}{p_x + p_y}\right), \frac{m}{p_y}\left(\frac{p_x}{p_x + p_y}\right)\right)$ *Initially, with budget constraint x + y = 2, we obtain demands* $\left(x^M, y^M\right) = \left(2\left(\frac{1}{1+1}\right), 2\left(\frac{1}{1+1}\right)\right)$ *= (1, 1).*

After the price change, with new budget constraint 2x + 3y = 6, we obtain: $\left(x^M, y^M\right) = \left(\frac{6}{2}\left(\frac{3}{2+3}\right), \frac{6}{3}\left(\frac{2}{2+3}\right)\right) = (1.8, 0.8)$.

When a = -1, we evaluate these expressions as $\left(x^M, y^M\right) = \left(m\left(\frac{p_x^{-0.5}}{p_x^{0.5} + p_y^{0.5}}\right), m\left(\frac{p_y^{-0.5}}{p_x^{0.5} + p_y^{0.5}}\right)\right)$.

We rewrite these in the slightly simpler form, $\left(x^M, y^M\right) = \left(\frac{m}{p_x^{0.5}\left(p_x^{0.5} + p_y^{0.5}\right)}, \frac{m}{p_y^{0.5}\left(p_x^{0.5} + p_y^{0.5}\right)}\right)$. *Initially, with budget constraint x + y = 2, we obtain demands* $\left(x^M, y^M\right) = \left(\frac{2}{1+1}, \frac{2}{1+1}\right)$ *= (1, 1). After the price change, the demands are:* $\left(x^M, y^M\right) = \left(\frac{6}{2^{0.5}\left(2^{0.5} + 3^{0.5}\right)}, \frac{6}{3^{0.5}\left(2^{0.5} + 3^{0.5}\right)}\right) \approx (1.35, 1.10)$.

c) Confirm that when *a* = 0.5, the consumer's indirect utility increases from 4 to 5; while when *a* = –1, indirect utility increases from 0.5 to 0.606 (approximately).

When a = 0.5, the indirect utility, $V(p_x, p_y, m) = U(x^H, y^M) = m\left(\frac{p_x + p_y}{p_x p_y}\right)$ *(taken from expression [9.21]). Initially, V(1, 1, 2) = 4; but after the price change, V(2, 3, 6) =* $6\left(\frac{5}{6}\right) = 5$. *It is straightforward to confirm these calculations by using the ordinary demands as the arguments of the utility function as well.*

When a = -1, the indirect utility, $V(p_x, p_y, m) = U(x^H, y^M) = m\left(p_x^{0.5} + p_y^{0.5}\right)^{-2}$. *Initially, V(1, 1, 2) =* $2(1 + 1)^{-2}$ *= 0.5; but after the price change, V(2, 3, 6) =* $6\left(2^{0.5} + 3^{0.5}\right)^{-2} \approx 0.606$.

X9.40 For Cobb-Douglas preferences, we write the utility function, *U*: $U = x^{0.5}y^{0.5}$. Confirm that $\frac{\partial y^M}{\partial p_x} = 0 = \frac{\partial x^M}{\partial p_y}$, so that the goods lie on the boundary between gross complements and gross substitutes.

This follows directly from the argument of X9.38f), where we showed that if a > 0, the goods are gross substitutes, but that if a < 0, the goods are gross complements. This utility function represents the case where a = 0, and we have already found the Marshallian demands to be $\left(x^M, y^M\right) = \left(\frac{m}{2p_x}, \frac{m}{2p_y}\right)$. *For both demand functions, the price of the other good is not an argument, and so the partial derivatives with respect to the other good are both zero.*

X9.41 For a pair of *net complements*, $\frac{\partial x^H}{\partial p_x}, \frac{\partial y^H}{\partial p_x} < 0$; and $\frac{\partial x^H}{\partial p_y}, \frac{\partial y^H}{\partial p_y} < 0$, so that as the price of one good increases, the Hicksian demands for both goods decrease. Assuming that there are only two goods in the consumption bundle, explain why it is impossible for them to form a pair of net complements.

Two goods in a pair cannot both be net complements. The substitution effect is shown as a movement along an indifference curve, and so, assuming that preferences are well behaved, this effect must lead to an increase in consumption of one good and a decrease in consumption of the other one.

X9.42 Consider the Cobb Douglas utility function, $U(x, y) = x^{½}y^{½}$. Calculate the *MRS* in terms of the ratio, $\frac{y}{x}$, and hence obtain an expression for $g\left(\frac{y}{x}\right)$. Show that the elasticity of substitution $\sigma = 1$.

We have already shown that MRS = $-\left(\frac{y}{x}\right)$ = -g; or that g = -MRS. Writing the elasticity of substitution as $\sigma = \frac{dg}{dMRS} \cdot \frac{MRS}{g}$, here $\frac{dg}{dMRS} = -1$, so $\sigma = -\left(\frac{MRS}{-MRS}\right) = 1$.

X9.43 Suppose that the elasticity of substitution $\sigma = 0$. What do you infer about the effect of a change in *MRS* on the composition, *g*, of the consumption bundle? What can you say about goods for which $\sigma = 0$?

The composition of the bundle is entirely unresponsive to the change in MRS. The MRS changes without the composition of the bundle changing, and this is consistent with the MRS not being properly defined for the chosen consumption bundle. We expect to observe this when goods are perfect complements. At the vertex of the indifference curve, the marginal rate of substitution switches from being zero to being undefined, and so consumption of that bundle is consistent with any price ratio being set.

X9.44 Suppose that the elasticity of substitution $\sigma \to \infty$. What do you infer about the effect of a change in *MRS* on the composition, *g*, of the consumption bundle? What can you say about goods for which $\sigma \to \infty$?

The composition of the bundle responds discontinuously to the change in MRS. The MRS changes infinitesimally but there is a finite change in the composition of the bundle. This is consistent with the substitution effect of a price change being very large relative to the price change. We expect to observe this characteristic of demands when goods are close to being perfect substitutes. The indifference curves are then almost linear and so the marginal rate of substitution scarcely varies. A small change in MRS leads to a very large change in the composition of the consumption bundle.

X9.45 Consider the general CES utility function, *U*: $U(x, y) = [x^a + y^a]^{1/a}$. By calculating *MRS* in terms of the ratio, $\frac{y}{x}$, show that the elasticity of substitution, $\sigma = \frac{1}{1-a}$. Obtain the value of σ when *a* = 1 and as $a \to -\infty$. Explain why these calculations are consistent with the arguments that you developed in Exercises X9.43 and X9.44.

We have demonstrated that for this function, MRS = $-\left(\frac{y}{x}\right)^{1-a}$ = $-g^{1-a}$; so that g = $-MRS^{\frac{1}{1-a}}$. Defining the elasticity of substitution,

$$\sigma = \frac{dg}{dMRS} \cdot \frac{MRS}{g} = -\frac{1}{1-a} MRS^{\frac{a}{1-a}} \cdot \frac{MRS}{\left(-MRS^{\frac{1}{1-a}}\right)} = \frac{1}{1-a} \frac{MRS^{\frac{1}{1-a}}}{MRS^{\frac{1}{1-a}}} = \frac{1}{1-a}.$$

When a = 1, then elasticity of substitution is undefined. But then the goods are perfect substitutes, and we expect MRS to take a unique value, as discussed in X9.44. Similarly, as a $\to \infty$, $\frac{1}{1-a} \to 0$, and the goods behave as perfect complements. We have already seen that the elasticity of substitution is then zero.

X9.46 Using the definition of the Hicksian demands in Expression 9.19, show that $g^H = \frac{y^H}{x^H} = \left[\frac{p_x}{p_y}\right]^{\frac{1}{1-a}} = -MRS^{\frac{1}{1-a}}$. Confirm that for the Hicksian demands, the elasticity of substitution, $\sigma = \frac{1}{1-a}$.

From equation [9.19], $p_y^{-\frac{1}{1-a}} x^H = p_x^{-\frac{1}{1-a}} y^H$; and from the optimization condition defining the income expansion path, we see that $g^H = \frac{y^H}{x^H} = \left[\frac{p_x}{p_y}\right]^{\frac{1}{1-a}} = -MRS^{\frac{1}{1-a}}$. The elasticity of substitution, $\sigma = \frac{dg}{dMRS} \cdot \frac{MRS}{dg} = -\frac{1}{1-a} MRS^{\frac{a}{1-a}} \cdot \frac{MRS}{-MRS^{\frac{1}{1-a}}}$.

X9.47 Given that in X9.46 the income expansion path is linear, explain why the effect of a price change on both the Hicksian and the Marshallian demands across a group of consumers will depend on the total sum of money used to finance consumption, and not on its distribution.

We have seen that the income expansion path is linear, that ordinary demand functions are linear in income, and also that (given our usual formulation of utility so that the functions are homogeneous of degree 1), indirect utility is linear in income, while compensated demands are linear in utility. Assuming that all consumers have the same preferences, then all will respond to a change in income with proportional increases in their demands; and all will demand the goods in the same proportion. This means that a group of consumers who between them have an amount of money to spend, M, will collectively make the same choices as a single consumer who has this amount M available to finance consumption.