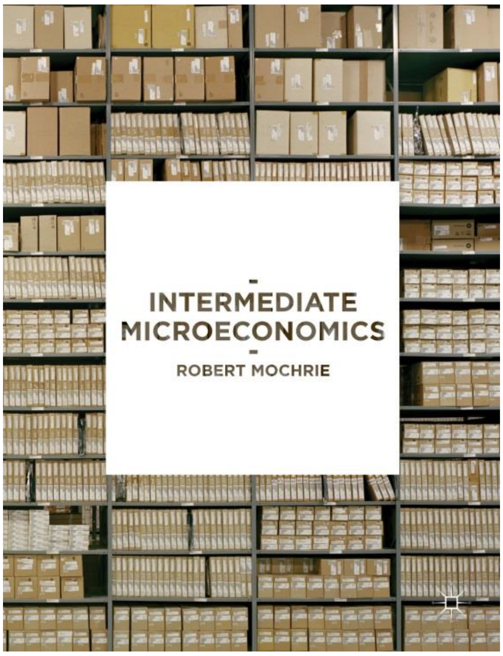
Solutions Manual: Part IV

Market Power

Summary answers to the 'By yourself' questions



Chapter 15

X15.1 **Suppose that we are considering the monopoly enjoyed by the only bakery on an island. Discuss its decision making process, explaining why we consider that it will be a price maker, rather than a price taker.**

We define the market for bread on this island. As the only bakery, the firm's supply is also the market supply. If we imagine the bakery as fixing the amount of bread that it will sell, the market price will emerge as the price at which that quantity can be sold. More generally, we expect the bakery to choose the output at which it maximizes profit given market clearing.

X15.2 **Suppose that a monopoly faces inverse demand, $p = p(q_f) = p_0 - p_1q_f$ and total costs, $C_f = c_0q_f$.**

a) **Obtain the firm's marginal and average costs, MC_f and AC_f, and its total, marginal and average revenues, TR_f, MR_f, and AR_f.**

Marginal cost, $MC_f : MC_f = \frac{dC_f}{dq_f} = c_0$. *Average cost,* $AC_f : AC_f = \frac{C_f}{q_f} = c_0$.

Total revenue, $TR_f = p.q_f = (p_0 - p_1q_f)q_f$*; so marginal revenue,* $MR_f : MR_f = \frac{dTR_f}{dq_f} = p_0 - 2p_1q_f$*;*

and average revenue, $AR_f : AR_f = \frac{R_f}{q_f} = p_0 - p_1q_f$.

b) **Calculate the profit-maximizing level of output, the price that the firm will then charge, and the profits that it will make.**

First-order condition for profit maximization is that marginal revenue equals marginal cost. (We do not check the second-order condition, that the derivative of marginal cost with respect to output is greater than the derivative of marginal revenue with respect to output at the profit maximizing output.)

Here, $p_0 - 2p_1q_f = c_0$*, so rearranging,* $2p_1q_f = p_0 - c_0$*, and we obtain profit maximizing output* $q_f{}^* = \frac{p_0 - c_0}{2p_1}$. *Substituting into the expression for inverse demand,*

$p_f{}^* = p_0 - p_1\left(\frac{p_0 - c_0}{2p_1}\right) = \frac{1}{2}(p_0 + c_0)$*; and so we obtain profit* $\Pi(q_f{}^*) = (p_f{}^* - c_0)q_f{}^* =$

$\left[\frac{1}{2}(p_0 + c_0) - c_0\right]\left(\frac{p_0 - c_0}{2p_1}\right) = \frac{1}{4p_1}(p_0 - c_0)^2$.

c) **Sketch a diagram showing MC_f, AC_f, MR_f, and AR_f. Indicate clearly the profit-maximizing output, and the resulting price and profits.**

In a diagram with output q_f shown on the horizontal axis and measures of cost and revenue shown on the vertical axis, the average cost and marginal cost are coincident, a horizontal line with equation $MC_f = AC_f = c$. The average revenue (or inverse demand) curve is a downward sloping line, passing through the vertical axis at $(0, p_0)$, and with gradient $-p_1$. The marginal revenue curve is also a downward sloping line, passing through $(0, p_0)$, but with gradient $-2p_1$, it is twice as steep as the average revenue curve. The profit maximizing output is where the marginal cost curve cuts through the marginal revenue curve. We see that the average cost is then greater than the average revenue, ensuring that the firm is able to make profits, shown by a rectangle formed by the vertical axis, the average cost curve, a vertical line passing through the point where MC = MR, and closed by a horizontal line at height $AR(p_f{}^*) = ½(p_1 + c_0)$.

X15.3 **Repeat X15.2 for the inverse demand $p = p_0 - p_1q_f$ and the total costs $C_f = c_0 + c_1q_f + c_2q_f{}^2$.**

Marginal cost, $MC_f : MC_f = \frac{dC_f}{dq_f} = c_1 + 2c_2q_f$. *Average cost,* $AC_f : AC_f = \frac{C_f}{q_f} = \frac{c_0}{q_f} + c_1 + c_2q_f$.

Total revenue, $TR_f = p.q_f = (p_0 - p_1q_f)q_f$; so marginal revenue, $MR_f : MR_f = \frac{dTR_f}{dq_f} = p_0 - 2p_1q_f$;

and average revenue, $AR_f : AR_f = \frac{R_f}{q_f} = p_0 - p_1q_f$.

First-order condition for profit maximization is that marginal revenue equals marginal cost. (We do not check the second-order condition, that the derivative of marginal cost with respect to output is greater than the derivative of marginal revenue with respect to output at the profit maximizing output.)

Here, $p_0 - 2p_1q_f = c_1 + 2c_2q_f$, so rearranging, $2(p_1 + c_2)q_f = p_0 - c_1$, and we obtain profit maximizing output $q_f{}^* = \frac{p_0 - c_1}{2(p_1 + c_2)}$. *Substituting into the expression for inverse demand,* $p_f{}^* = p_0 - p_1\left(\frac{p_0 - c_1}{2(p_1 + c_2)}\right) = \frac{(p_1 + 2c_2)p_0 + p_1c_1}{2(p_1 + c_2)}$; *and we obtain profit $\Pi(q_f^*) = -(p_1 + c_2)q_f^2 + (p_0 - c_1)q_f - c_0 = -\frac{(p_0 - c_1)^2}{4(p_1 + c_2)} + \frac{(p_0 - c_1)^2}{2(p_1 + c_2)} - c_0 = \frac{(p_0 - c_1)^2}{4(p_1 + c_2)} - c_0$.*

In a diagram with output q_f shown on the horizontal axis and measures of cost and revenue shown on the vertical axis, the average cost curve never reaches the vertical axis. With fixed costs, the vertical axis forms an asymptote, and the curve is downward sloping. In the same way, as output increases without limit, average fixed cost falls to zero, and the curve approaches the upward-sloping line $AVC = c_1 + c_2q_f$, which meets the vertical axis at $(0, c_1)$ and has gradient c_2. The marginal cost curve is a line, also passing through $(0, c_1)$, but with gradient $2c_2$; it is therefore twice as steep as the average variable cost curve. We can show that the average cost is minimized at output $q = \sqrt{\frac{c_0}{c_2}}$, where average cost equals marginal cost.

The average revenue (or inverse demand) curve is a downward sloping line, passing through the vertical axis at $(0, p_0)$, and with gradient $-p_1$. The marginal revenue curve is also a downward sloping line, passing through $(0, p_0)$, but with gradient $-2p_1$, it is twice as steep as the average revenue curve.

The profit maximizing output is where the marginal cost curve cuts through the marginal revenue curve. The presence of fixed costs means that it is possible that the firm will make losses; we define the firm's total profit as the area of the rectangle bounded by the vertical axis, horizontal lines the average revenue at the profit maximizing output, the average cost at the profit maximizing output and a vertical line at the profit maximizing output

X15.4 **Repeat Exercise X15.2 for the inverse demand $p = Aq_f^{-1}$, with these total costs: (a) $C_f = c_0q_f$; (b) $C_f = c_0q_f^{2/3}$.**

a) *Marginal cost,* $MC_f : MC_f = \frac{dC_f}{dq_f} = c_0$. *Average cost,* $AC_f : AC_f = \frac{C_f}{q_f} = c_0$.

Total revenue, $TR_f = p.q_f = (Aq_f^{-1})q_f = A$; so marginal revenue, $MR_f : MR_f = \frac{dTR_f}{dq_f} = 0$; *and*

average revenue, $AR_f : AR_f = \frac{R_f}{q_f} = \frac{A}{q_f}$

Since marginal revenue is zero, and marginal cost is greater than zero, the first-order conditions for profit maximization cannot be met; but any increase in output reduces profits. The firm cannot do better than produce zero output, incurring no costs, and making profit $\pi(0) = A$.

As in previous cases, in a diagram with output on the horizontal axis and measures of cost and revenue on the vertical axis, the marginal and average cost curves are horizontal lines. The marginal revenue curve is the horizontal axis; and the average revenue curve is a rectangular hyperbola, always downward sloping, and with the axes forming asymptotes.

b) *Marginal cost,* $MC_f : MC_f = \frac{dC_f}{dq_f} = \frac{2}{3}c_0q_f^{-\frac{1}{3}}$. *Average cost,* $AC_f : AC_f = \frac{C_f}{q_f} = c_0q_f^{-\frac{1}{3}}$. *Measures of revenue are as noted in part a).*

Once again, we have a situation where marginal revenue is less than marginal cost at all levels of output, so that the firm produces zero output to maximize profits. The marginal and average cost curves are downward sloping, curved, and never touch the axes. At every level of output, average cost is 50% greater than marginal cost.

X15.5 **Repeat Exercise X15.2 for the inverse demand $p = p(q_f) = p_0 - p_1q_f$, with these total costs: (a) $C_f = c_0q_f^2$; (b) $C_f = c_0q_f^{3/2}$.**

We have the same inverse demand as in X15.2. So total revenue, $TR_f = p.q_f = (p_0 - p_1q_f)q_f$; and marginal revenue, $MR_f : MR_f = \frac{dTR_f}{dq_f} = p_0 - 2p_1q_f$; and average revenue, $AR_f : AR_f = \frac{R_f}{q_f} = p_0 - p_1q_f$.

a) *The cost function is a special case of the cost function in X15.3, and we obtain marginal cost, $MC_f : MC_f = \frac{dC_f}{dq_f} = 2c_0q_f$. Average cost, $AC_f : AC_f = \frac{C_f}{q_f} = c_0q_f$.*

   The condition MR = MC is satisfied when $p_0 - 2p_1q_f = 2c_0q_f$. Rearranging this expression, we obtain $q_f{}^* = \frac{p_0}{2(p_1+c_0)}$. The profit maximizing price, $p^* = p_0 - p_1q_f^* = \frac{p_1+2c_0}{2(p_1+c_0)}p_0$, so that the total profit, $\pi = (p - c_0q_f)q_f = \frac{p_0{}^2}{4(p_1+c_0)}$.

   In a diagram, we have already described all of these cost and revenue curves. All are straight lines. The average revenue curve passes through $(0, p_0)$, and is downward sloping with gradient $-p_1$. The marginal revenue curve also passes through $(0, p_0)$, but is twice as steep, so has gradient $-2p_1$. The average cost curve and marginal cost curve both start at the origin; and the marginal cost curve is twice as steep as the average cost curve, with gradients $2c_0$ and c_0, respectively.

b) *We obtain marginal cost, $MC_f : MC_f = \frac{dC_f}{dq_f} = \frac{3}{2}c_0q_f{}^{\frac{1}{2}}$. Average cost, $AC_f : AC_f = \frac{C_f}{q_f} = c_0q_f{}^{\frac{1}{2}}$.*

   The condition MR = MC is satisfied when $p_0 - 2p_1q_f = \frac{3}{2}c_0q_f{}^{\frac{1}{2}}$. We can solve this by using the quadratic formula, but omit this here, instead concentrating on the graphical analysis. The marginal and average revenue curves have the same shape as described above. The average cost and marginal cost curves both start from the origin, and both are upward sloping and concave –they are arms of two distinct parabolas. Marginal cost is everywhere 50% greater than average cost, and marginal revenue is everywhere less than average revenue. We can therefore be sure that where marginal revenue equals marginal cost, the firm makes profits.

X15.6 **Confirm that in Figure 15.2: (a) there is no producer surplus; and (b) the maximum consumer surplus is equivalent to the area enclosed by the marginal cost and average revenue curves.**

For every level of output, q, $MC(q) = AC(q) = c_0$. Defining the rate of change of producer surplus as the difference between marginal and average cost, since $MC(q) - AC(q) = 0$, the total surplus when producing output q_0 may be written PS = $\int_0^{q_0} 0 dq = 0$.

In the same way defining the rate of change of consumer surplus as the difference between average revenue and marginal revenue at any level of output, we define total consumer surplus as $CS = \int_0^{q_0} [AR(q) - MR(q)]dq$. Now define q_0: $AR(q_0) = c_0$, which is to say that we set the firm's output so that the market price is equal to the firm's marginal cost. Then $AR(q_0)$ is the firm's WTA for every level of output, and q_0 is the maximum output. The consumer surplus is then increasing as q increases towards q_0.

X15.7 **Suppose that the market demand of a monopoly is linear (and decreasing in price), while the average cost is linear (and increasing in output). Sketch a diagram to show the average and marginal revenues and the average and marginal costs. Indicate the profit-maximizing and the welfare-maximizing levels of output; and label the areas on your diagram that represent the firm's profits, its producer surplus, the consumers' surplus, and the welfare loss from the monopoly.**

In a diagram with firm output, q, measured on the horizontal axis, and measures of cost and revenue on the vertical axis, the average and marginal revenue curves are downward sloping straight lines, intersecting the vertical axis at the same point, but with the marginal revenue curve twice as steep as the average revenue curve. In the same way, the average and marginal cost curves are straight lines, intersecting the vertical axis at the same point, which has to be lower than the intersection of the average and marginal revenue curves with the axis. The average and marginal cost curves are upward sloping, with the marginal cost curve twice as steep as the average cost curve.

The firm maximizes profit at output q, chosen so that MR(q*) = MC(q*). The firm charges price AR(q*) > AC(q*), the cost per unit of output, and so there are positive profits. We define the consumer surplus as the triangle bounded by the average and marginal revenue curves to the left of output q*; the producer surplus as the triangle bounded by the average and marginal cost curves to the left of output q*; the firm's profits as the triangle formed by the vertical axis and the marginal revenue and marginal cost curves; and the welfare loss as the triangle to the right of output q*, bounded by the average revenue and marginal cost curves.*

X15.8 **Why might it be reasonable to conclude that in Exercises X15.1–X15.6 we are considering the long run? How reasonable is it to expect there to be a monopoly in these circumstances?**

In all of these examples, we have excluded fixed costs of production, indicating that there are no fixed factors, which defines the long-run. Monopoly occurs when a firm faces no competitors. We have noted that in perfect competition, in the short-run, there is no possibility of firm entry, but that long-run entry eliminates profits. In monopoly, we exclude entry. Given that the monopoly makes profits, we need to find some explanation, such as increasing returns to scale, or statutory protection of the monopoly.

X15.9 **Adapt Figure 15.3 to represent the monopoly making short-run decisions. Illustrate two distinct cases: (1) the monopoly makes profits in the short run; and (2) the monopoly makes losses, but wishes to continue production.**

Moving from the long-run to the short-run, we show both the average variable cost curve and the average cost curve. We expect the average variable cost curve to have similar properties to the average cost curve in Figure 15.3; possibly U-shaped, but certainly eventually increasing, if not everywhere increasing, while reflecting the element of average fixed costs, the average cost curve will be never reach the vertical axis, which acts as its asymptote. The average cost curve will certainly have a minimum, and output increases without limit, it approaches the average variable cost curve.

In order for the monopoly to make profits in the short run, the average cost curve lies below the average revenue curve at the profit-maximizing output (for which marginal cost equals marginal revenue). For the firm to make a loss, the average cost curve lies above the average revenue curve at the profit-maximizing output.

X15.10 **Applying the same argument to marginal revenue as to marginal costs (in Figure 15.3), interpret the area between the output axis and the marginal revenue curve.**

We define marginal revenue as the rate of change of revenue as output increases. The area underneath the marginal revenue curve (between the vertical axis and the line $q = q_0$*)*

therefore represents the sum of revenue increments across a continuum of infinitesimally small increases in output. It is therefore the total revenue, $R(q_0)$. We conclude that $R(q_0) = \int_0^{q_0} MR(q)dq$.

X15.11 **Explain why the area between the marginal revenue and marginal cost curves yields the firm's profit.**
Essentially, we build on the argument already developed. The expression, $\{MR(q) – MC(q)\}dq$ represents the increase in profit when output increases by some infinitesimally small amount, from q to $q + dq$. The integral $\Pi(q_0)=\int_0^{q_0}(MR(q)-MC(q))dq$ is then the sum of these increases in profits as output increases from $q = 0$ to $q = q_0$. The firm's profit is shown in a diagram as the area bounded by the vertical axis and its output and the marginal revenue and marginal cost curves. We should note here that it is possible that for large enough values of q_0, $MR(q_0) < MC(q_0)$, so that increasing output leads to a reduction in profit.

X15.12 **Interpret the area in Figure 15.3 between the inverse demand (average revenue) curve and the marginal cost curve in terms of consumers' *WTP* and the monopoly's *WTA*. Hence justify the argument that the area labelled 'Welfare loss' represents the cost to society of production being undertaken by a monopoly.**
The inverse demand curve shows the price that the marginal consumer is willing to pay, given the firm's current output; the marginal cost curve shows the firm's WTA, given its output. We know that so long as WTP > WTA, there is a surplus from completing at transaction, so that we can write the rate of change of surplus, Σ, with respect to output, q, as the derivative, $\frac{d\Sigma}{dq}=AR(q)-MC(q)$; so that the total surplus from output q_0 may be written $\Sigma(q_0)=\int_0^{q_0}[AR(q)-MC(q)]dq$. Assume that $AR(q) > MC(q)$ for all values of q: $q < q^M$, but that $AR(q) < MC(q)$ for higher values of output. It then follows that surplus increases until $q < q_0$, and decreases at higher output levels, so that surplus is maximized when $q = q^M$. If the firm chooses to produce output q_f: $q_f* < q^M$, total surplus is less than its maximum value; and we define the welfare loss $\Omega(q_0)=\int_{q_0}^{q^M}[AR(q)-MC(q)]dq$*

X15.13 **Suppose that it is proposed that the monopolist should be required to produce output Q^M, the welfare-maximizing output. Write down an expression for the loss incurred by the monopolist in terms of the marginal revenue and cost functions, illustrating this area on a diagram. How might it be possible to compensate the monopolist for this loss?**
The monopolist maximizes profits at output q_f; required to produce output q^M, the monopolist loses the difference in profits $\Pi(q^M)-\Pi(q_f*)=\int_{q_f*}^{q^M}[MR(q)-MC(q)]dq$. Note that this reduction in profits is less than the increase in consumer surplus; if consumers agree to compensate the monopolist for increasing output, agreeing collectively to transfer this amount (collectively and voluntarily) to the produce, on condition that the producer increases output, then the producer would be no worse off than before being required to increase output and consumers (as a group) are better off.*

X15.14 **Given the marginal revenue and marginal cost curves in each of these three cases:**

a) **Find the output for which marginal revenue equals marginal cost.**
b) **Obtain the following functions of output: (i) total revenue; (ii) average revenue; (iii) total cost; (iv) profit; and (v) consumer surplus.**

c) Sketch diagrams showing the marginal and average revenue curves and the marginal cost curve, indicating the total cost of the profit-maximizing output, the firm's profit, and the consumer surplus.

d) Estimate the welfare loss from monopoly:

i. *MC*(*q*) = 2 + 0.2*q*; *MR*(*q*) = 4;

ii. *MC*(*q*) = 3; *MR*(*q*) = 6 – 0.15*q*;

iii. *MC*(*q*) = 2 + 0.3*q*; *MR*(*q*) = 8 – 0.9*q*.

i. If MR = MC, then 2 + 0.2q = 4, so q = 10.

We assume no fixed costs. Total revenue R = $\int_0^{q_0} 4.dq = [4q]_0^{q_0} = 4q_0$ *; so we write R: R(q) = 4q. Average revenue AR: AR(q) =* $\frac{R(q)}{q}$ *= 4.*

Total cost C = $\int_0^{q_0} (2+0.2q).dq = [2q+0.1q^2]_0^{q_0} = 2q_0 + 0.1q_0^{\,2}$ *. We write C: C(q) = 2q + 0.1q*^{2}*.*

Profit $\Pi = \int_0^{10} [MR(q) - MC(q).]dq = \int_0^{10} [2-0.2q]dq = [2q-0.1q^2]_0^{10} = 20-10 = 10$ *.*

Consumer surplus, CS(10) = $\int_0^{10} [AR(q) - MR(q)]dq = \int_0^{10} [4-4]dq = 0$ *.*

We illustrate all of this information in a diagram with the level of output on the horizontal axis and measures of cost and revenue on the vertical axis. Average and marginal revenues are horizontal lines, AR(q) = MR(q) = 4; marginal cost is shown by an upward sloping line, passing through (0, 2) and with gradient 0.2, so that it intersects MR at q_f** = 10. This is the profit maximizing output, and we can see the firm's profit (and the producer surplus) as the triangle formed by the vertical axis and the marginal revenue and marginal cost curves. The total cost of the profit maximizing output is the area below that triangle, bounded by the line q = 10. Consumer surplus is here zero.*

ii. If MR = MC, then 6 – 0.15q = 3, so 0.15q* = 3; and q = 20.*
We assume no fixed costs.

Total revenue R = $\int_0^{q_0} (6-0.15q).dq = [6q-0.075q^2]_0^{q_0} = 6q_0 - 0.075q_0^{\,2}$ *; so we write R: R(q) = 6q – 0.075q*^{2}*. Average revenue AR: AR(q) =* $\frac{R(q)}{q}$ *= 6 – 0.075q.*

Total cost C = $\int_0^{q_0} 3.dq = [3q]_0^{q_0} = 3q_0$ *. We write C: C(q) = 3q.*

Profit $\Pi =$

$$\int_0^{20} [MR(q) - MC(q).]dq = \int_0^{20} [3-0.15q]dq = [3q-0.075q^2]_0^{20} = 60 - 0.075*400 = 30.$$

Consumer surplus, CS(20) =

$$\int_0^{20} [AR(q) - MR(q)]dq = \int_0^{20} [6-0.075q-(6-0.15q)]dq = \int_0^{20} 0.075q.dq = [0.0375q^2]_0^{20} = 15..$$

We illustrate all of this information in a diagram with the level of output on the horizontal axis and measures of cost and revenue on the vertical axis. Average and marginal revenues are downward sloping lines, with equations AR(q) = 6 – 0.075q, and MR(q) = 6 – 0.15q. Both lines pass through the point (0, 6) on the vertical axis, with the *marginal revenue curve twice as steep as the average revenue; marginal cost is shown by horizontal line MC(q) = 3, which intersects the marginal revenue curve when output, q_f^* = 20. This is the profit maximizing output, and we can see the consumer surplus as the triangle formed by the vertical axis, and the marginal revenue and marginal cost curves, which meet at q_f^* = 20. This triangle has area CS(20) =15. For the firm, the total cost of production is equal to its revenues; so that there is no producer surplus. However, it continues to make a profit because it is able to sell output at price p= 4.5. We show the total profit, Π = 30, as the triangle formed by the marginal cost and marginal revenue curves, and the vertical axis.*

iii. *MC(q) = 2 + 0.3q; MR(q) = 8 – 0.9q.*
If MR = MC, then 2 + 0.3q = 8 – 0.9q*, so 1.2q* = 6; and q* = 5.*
We assume no fixed costs.

Total revenue R = $\int_0^{q_0} (8-0.9q).dq = \left[8q - 0.45q^2\right]_0^{q_0} = 8q_0 - 0.45q^2$ *; so we write R: R(q) = 6q – 0.45q*^{2}*. Average revenue AR: AR(q) =* $\frac{R(q)}{q}$ *= 8 – 0.45q.*

Total cost C = $\int_0^{q_0} [2+0.3q].dq = \left[2q+0.15q^2\right]_0^{q_0} = 2q_0 + 0.15{q_0}^2$. *We write C: C(q) = 2q + 0.15q*^{2}*.*

Profit Π = $\int_0^5 [MR(q) - MC(q).]dq = \int_0^5 [6-1.2q]dq = \left[6q - 0.6q^2\right]_0^5 = 30 - 0.6 * 25 = 15$.

Consumer surplus, CS(5) =

$$\int_0^5 [AR(q) - MR(q)]dq = \int_0^5 [8 - 0.45q - (8-0.9q)]dq = \int_0^5 0.45q.dq = \left[0.225q^2\right]_0^5 = 5.625..$$

We illustrate all of this information in a diagram with the level of output on the horizontal axis and measures of cost and revenue on the vertical axis. Average and marginal revenues are downward sloping lines, with equations AR(q) = 8 – 0.45q, and MR(q) = 8 – 0.9q. Both lines pass through the point (0, 8) on the vertical axis, with the marginal revenue curve twice as steep as the average revenue; marginal cost is shown by the upward sloping line, with equation MC(q) = 2 + 0.3q. This passes through the point (0, 2) on the vertical axis, and has gradient 0.3. It intersects the marginal revenue curve when output, q_f^* = 5. This is the profit maximizing output, and we can see the consumer surplus as the triangle formed by the vertical axis and the marginal revenue and marginal cost curves, which meet at q_f^* = 5, and which has area CS(20) = 5.625. The firm makes a profit because it is able to sell output at price p= 5.75. We show the total profit, Π = 15, as the triangle formed by the marginal cost and marginal revenue curves, and the vertical axis.

X15.15 **For a monopoly facing the inverse demand function, $p = 100 - q_f$.**

a) **Obtain:**

i. the total revenue and marginal revenue functions;

Total revenue R: $R(q_f) = p(q_f).q_f = (100 - q_f).q_f$

Marginal revenue MR: $MR(q_f)$ = $\frac{dR}{dq_f} = 100 - 2q_f$

ii. the demand function and the price elasticity of demand (as a function of price).

Demand function: q_f: q(p) = 100 – p;

elasticity of demand, $\varepsilon_p = \frac{dq_f}{dp}.\frac{p}{q_f} = -1.\frac{p}{100-p} = \frac{p}{p-100}$

b) Sketch the inverse demand curve and the marginal revenue curve.
On a diagram with output measured on the horizontal axis and measures of revenue on the vertical axis, inverse demand will be shown by a downward sloping line, meeting the vertical axis at (0, 100), and with slope -1. Marginal revenue will also be downward sloping line, passing through (0, 100), but with slope – 2, so that it is twice as steep as inverse demand. Note that inverse demand meets the horizontal axis at (100, 0) while marginal revenue meets the axis at (50, 0).

c) Indicate on your diagram the range of outputs for which demand is elastic. Confirm that this is also the range of outputs for which *MR* > 0.
Demand is elastic when 2p – 100 > 0, or when p > 50. We note from the diagram that when this condition is met, q_f = 100 – p >50, so that MR(q_f) > 0.

X15.16 **Suppose that a firm faces inelastic demand at output q_f. By considering the effect on its revenue and costs of reducing its output, show that this firm will increase profits by reducing output.**
If a firm faces inelastic demand, then the responsiveness of demand to price increases is low; so for a monopolist reducing output, price increases more rapidly than output falls. Reducing output, revenues increase, and costs necessarily fall, so that profits increase.

X15.17 **Using Expression 15.9, and the first-order condition for profit maximization, show that the price-cost margin**

$$\frac{p_f - MC(q_f*)}{MC(q_f*)} = -\frac{1}{1+\varepsilon} \qquad [15.10]$$

Confirm that in Figure 15.4 the price cost margin is also the ratio of profits to total costs.
Defining elasticity $\varepsilon = \frac{dq}{dp}.\frac{p}{q} = \left(\frac{dp}{dq}\right)^{-1}.\left(\frac{q}{p}\right)^{-1}$. *Differentiating the (total) revenue function, we obtain* $MR = \frac{dR}{dq} = q\frac{dp}{dq} + p = p\left(1 + \frac{q}{p}.\frac{dp}{dq}\right) = p\left(1+\varepsilon^{-1}\right)$. *With profit maximization, MR(q*) = MC(q*), so that* $\frac{p_f - MC(q_f*)}{MC(q_f*)} = \frac{p_f - p(1+\varepsilon^{-1})}{p(1+\varepsilon^{-1})} = -\frac{\varepsilon^{-1}}{1+\varepsilon^{-1}} = -\frac{1}{1+\varepsilon}$.

X15.18 **Using Expression 15.10, confirm that at the profit maximizing output, q_f*, the monopoly's demand function must be elastic, or that the price elasticity of demand, $\varepsilon_p < -1$. Show that as $\varepsilon_p \to -1$, the price cost margin increases without limit; and that as $\varepsilon_p \to -\infty$, the price-cost margin approaches zero.**
We know that for profit maximization, the profit margin must take a positive value, so that $\frac{1}{1+\varepsilon_p} < 0$*; and so 1 +* ε_p *< 0; and* ε_p *< -1, meaning that demand is elastic. As* $\varepsilon_p \to -1$, *1 +* $\varepsilon_p \to 0$, *and the price cost margin,* $\frac{p_f - MC(q_f*)}{MC(q_f*)} = -\frac{1}{1+\varepsilon} \to \infty$. *We explain this result by noting that as* $\varepsilon \to -1$, *MR* $\to 0$ *and so at any price p > 0, the margin increases without limit.*

X15.19 **Given the demand function $q_f = Ap_f^{-b}$:**

a) Confirm that the price elasticity of demand, $\varepsilon_p = -b$.
Price elasticity of demand: $\varepsilon_p = \frac{dq_f}{dp_f}.\frac{p_f}{q_f} = -bAp_f^{-b-1}.\frac{p_f}{Ap_f^{-b}} = -b$.

b) Obtain the inverse demand function: and show that when *b* = 1, total revenue *TR*(q_f) = *A*, so that marginal revenue *MR*(q_f) = 0. Confirm that the monopoly then maximizes profits by setting output to zero.

Inverse demand function: since $p_f^{-b} = \frac{q_f}{A}$, *inverse demand,* $p_f = \left(\frac{q_f}{A}\right)^{-\frac{1}{b}} = \left(\frac{A}{q_f}\right)^{\frac{1}{b}}$. *Then we write total revenue R: R =* $p_f.q_f$ *=* $q_f\left(\frac{q_f}{A}\right)^{-\frac{1}{b}} = A^{\frac{1}{b}}.q_f^{\,1-\frac{1}{b}}$. *So when b = 1, R =* $A^{\frac{1}{b}}.q_f^{\,1-\frac{1}{1}}$ *= A. Since revenue is constant,* $MR(q_f) = \frac{dR}{dq_f} = 0$. *Since the firm generates no additional revenue from sales, it minimizes costs (and maximizes profits) by setting* $q_f = 0$.

c) **Assume that *b* > 1 and that the firm has total cost function *C* = cq_f. Calculate the profit maximizing output and price, and illustrate these in a diagram showing both the average and marginal revenue curves and the average and marginal cost curves.**

When b > 1, marginal revenue, $MR(q_f) = \frac{dR}{dq_f} = \left(1-\frac{1}{b}\right)A^{\frac{1}{b}}.q_f^{-\frac{1}{b}}$. *With total cost, C:* $C(q_f) = cq_f$, *marginal cost,* $MC(q_f) = c$. *So for the profit maximizing condition,* $MC(q_f) = MR(q_f)$ *to be satisfied, it must be that* $\left(1-\frac{1}{b}\right)A^{\frac{1}{b}}.q_f^{-\frac{1}{b}} = c$, *and that* $\left(1-\frac{1}{b}\right)A^{\frac{1}{b}}c^{-1}. = q_f^{\frac{1}{b}}$; *and* $q_f = \frac{\left(1-\frac{1}{b}\right)^b A}{c^b}$.

Since price $p_f = \left(\frac{A}{q_f}\right)^{\frac{1}{b}}$, *we obtain* $p_f = \left(\frac{A}{q_f}\right)^{\frac{1}{b}} = \frac{c}{1-\frac{1}{b}} = \frac{bc}{b-1}$.

In a diagram, with the level of output on the horizontal axis, and measures of value on the vertical axis, the average and marginal revenues are rectangular hyperbolae, and so are downward sloping, convex, and bounded by the axes. The marginal revenue curve lies closer to the origin than the average revenue curve (indeed at any level of output, q_f, $MR(q_f) = \frac{b-1}{b}q_f$*). The average cost and marginal cost curves are horizontal lines, a distance c above the axis.*

X15.20 **Given the demand function $q_f = a - bp$:**

a) **Confirm that the price elasticity of demand, $\varepsilon_p = -bp/(a - bp)$.**

We write the price elasticity of demand, $\varepsilon_p = \frac{dq_f}{dp}.\frac{p}{q_f} = -b.\frac{p}{a-bp}$.

b) **Obtain the inverse demand function, the average revenue function, and the marginal revenue function. Sketch a diagram showing all of these.**

Inverse demand, p: $p(q_f) = \frac{a-q_f}{b}$. *This will also be the average revenue. With (total) revenue,* R_f: $R(q_f) = p(q_f).q_f$, *we write* $R(q_f) = \left(\frac{a-q_f}{b}\right)q_f$, *and on differentiating with respect to output* q_f, *we obtain marginal revenue,* MR_f: $MR(q_f) = \frac{a-2q_f}{b}$.

In a diagram, with output on the horizontal axis and value measures on the vertical axis, inverse demand (average revenue) and marginal revenue are downward sloping lines, both starting from $\left(0,\frac{a}{b}\right)$ *on the value axis, and the marginal revenue is twice as steep as the average revenue.*

c) **Confirm that if the marginal revenue, $MR(q_f) > 0$, then the price elasticity of demand, $\varepsilon_p < -1$.**

For $MR(q_f) > 0$, $\frac{a-2q_f}{b} > 0$, *so that* $q_f < \frac{a}{2}$. *But then* $p(q_f) > \frac{a}{2b}$, *and so*

$\varepsilon_p = \frac{-bp}{a-bp} = 1 - \frac{a}{a-bp} < 1 - \frac{a}{a-b\left(\frac{a}{2b}\right)} = 1 - \frac{a}{\frac{a}{2}} = 1-2$; *and* $\varepsilon_p < -1$.

d) **Using the result from part (c), confirm that if the firm maximizes profits, price $p_f > a/2b$.**

For the firm to maximize profits, the condition $MC(q_f*) = MR(q_f*) > 0$ must be satisfied, and this can only happen if $p_f > \frac{a}{2b}$.

X15.21 **Accepting the argument that Standard Oil was able to extract better terms from its suppliers than could its potential competitors, explain how this might have affected its production function, giving the appearance of increasing returns to scale in production. How might Standard Oil also have increased prices to its customers above the level that competitors might be able to charge?**
[*Hint*: think about the impact of lower input prices on the costs of production.]
Negotiating better terms with suppliers than its competitors, we would have expected Standard Oil to have lower per unit (average) costs of production, and lower marginal costs. If these terms resulted simply from the large volume of orders that Standard Oil was placing, then this would be evidence of increasing returns. However, in effect Standard Oil required its suppliers to enter into contracts in which suppliers had to grant those terms or else lose the contracts; and so the terms resulted from Standard Oil's economic power, rather than the scale of its activities.

X15.22 **Suppose that a second firm plans to enter the market. It is able to produce a good of exactly the same quality, and faces exactly the same production (and cost) functions. The monopolist has already announced that if any other firm enters the market, it will increase its output so that it just breaks even. Why would it be difficult for entry to occur under such circumstances? Do you think that such a policy should be considered as 'restraint of trade'?**
Facing the threat of entry, the monopolist proposes to expand its output, so that the market clearing price given its own and its competitor's output will be equal to the monopolist's average cost. We have already noted that the monopolist faces increasing returns to scale in production, and therefore, assuming that is the larger firm, were the monopolist to expand output, the entrant will make losses. No entrant, who considers the monopolist's threat to be credible will choose to enter the market. It may seem that the monopolist is preventing entry in this case, but we might also argue that the monopoly identified a profit opportunity before anyone else, so that the monopoly's profits reflect its choices.

X15.23 **Suppose that a firm does enter the market, producing an output q_1. Sketch a diagram showing the output of the monopolist, which now sets output so as just to break even.**
In a diagram with the monopolist's output measured on the horizontal axis, and measures of revenue and cost on the vertical axis, we show the monopolist's average revenue and marginal cost curves as being downward sloping, with average revenue above marginal revenue, and with both intersecting the vertical axis at the same point. We also show the average and marginal cost curves as being downward sloping, with the average cost curve above the marginal cost curve, and both intersecting the vertical axis at the same point. The cost curves should be flatter than the revenue curves, with the average cost curve lying above the average revenue curve at output q_f*, where the marginal revenue and marginal cost curves intersect. We can now be certain that the firm makes profits at output q_f*. Allowing for entry, with the entrant's output q_1, the monopolist increases its output to q_0, chosen so that the inverse market demand, $p(q_0 + q_1) = AC(q_0)$.

X15.24 **We show the firm as being able to sell an output Q^M at price p^M. Applying the argument used in X15.12, show that this is the welfare-maximizing output. Confirm that when producing output, Q^M, the monopoly makes a loss. How might government induce the monopolist to produce output, Q^M?**

We see that the marginal surplus from transactions at output Q^M is zero; since $AR(Q^M) = MC(Q^M)$. This is the first-order condition for a welfare maximum. However, we also see that since $AC(Q^M) > MC(Q^M)$, the firm makes a loss. The government could encourage production at this level by providing the firm with some form of subsidy.

X15.25 **Suppose that the government does decide to set up a system of regulation of a market such as energy distribution, sometimes argued to be a natural monopoly. Explain why it might be difficult for the regulator to set the price p^M, which maximizes welfare. [*Hint:* Think about how easy it would be in reality to define that price, and also about the impact that this form of regulation would have upon the behaviour of the monopoly.]**
To set the welfare maximizing price, p^M, the regulator would need to know both the revenue and cost functions. We might suppose that it will be possible to estimate the demand function, but the firm might well have information about costs that it does not want to share with the regulator. In particular, remember that we assume that the firm always uses the most efficient combination of factor inputs. In this situation, the firm might not report cost savings that it has made through improved efficiency; or, knowing that it would not benefit from doing so, the firm may not attempt to improve the efficiency of production.

X15.26 **In our example of the fruit market in Chapter 1, how likely it is that any firm would be able to sustain a competitive advantage over periods of: (a) a day; (b) a month; or (c) a year? How do you think this affects the behaviour of the firms in the market?**
We can be quite certain that a day is a short enough period of time that no competitor will be able to match this – this is the very short run in our analysis, when the firms simply sell the stock that they have to hand. Similarly, a year seems far too long to sustain a competitive advantage: other firms would have time to observe and copy the innovation. It may well be that a period of some weeks is about as long as a competitive advantage could exist before being noticed by other businesses.

X15.27 **Why might we argue that it is essential to tolerate some degree of market power if there is to be innovation and invention in the economy?**
This builds on the discussion of monopoly that we set out here. We have seen that market power is a transitory phenomenon in perfect competition, have argued that there is natural monopoly where production is characterized by increasing returns to scale. There are also some services to which access might be considered essential (such as clean water and waste disposal); for such businesses, a statutory monopoly is one way ensuring provision. We can also think of market power as a generalization of monopoly power, and that it will accrue to innovators. We have argued that it is difficult for innovators to make profits in highly competitive industries because they lack opportunities to generate returns. This suggests that some toleration of market power may have long-run benefits in the sense of granting innovators a greater opportunity to exploit the advances that they have made.

X15.28 **What justification, if any, would you give for each of the following measures?**

a) ***Patents* – these prevent any business other than the one holding the patent from producing a specified output or using a specified technique of production.**
A patent restricts the ability of competitors to exploit techniques developed by a firm. It might reduce competition in supply very substantially, and so be a source of substantial market power. It is argued that this provides inventors with the protection necessary for them to exploit the intellectual property inherent in their inventions, and so to earn adequate returns to their efforts.

b) ***Copyright* – this prohibits the copying of a particular expression of an idea in text or music.**

This book is protected by copyright, the purpose of which is to prevent the copying of material without permission of the author (or the publisher as my agent). Again, this is intended to give me and my publisher the opportunity to exploit the novel expression of ideas by having a monopoly on the sale of the book. Note that it is a very restricted right: there are many competing textbooks, all of which have idiosyncratic characteristics, ensuring product differentiation. Copyright is also how computer programmers tend to protect the expression of ideas.

c) ***Trademarks and registered designs*** **– these restrict the freedom of other producers to copy the appearance of goods.**
Trademarks are perhaps the most general expression of an idea. The immediacy with which some trademarks can be recognized (the design of the Coca-Cola logo, for example) means that producers have an interest in ensuring that they cannot simply be replicated. The nature of design means that this is a difficult right to assert in many cases, and the extent of the protection offered is therefore very limited.

X15.29 **What might be the effects on the music industry of the abolition of copyright? Consider in particular the impact on the distribution of recordings and the decision to create and perform new material.**
We have started to observe cases where artists are starting to use novel distribution platforms to disseminate music. Abolishing copyright would effectively make it impossible for recordings to be made, or for music to be sold in manuscript. It would lead to music being marketed along with other goods, possibly connected with concert tours, festivals and other major events. We might expect there to be further substantial changes in control of distribution – we have seen a shift from production companies controlling revenues to web-based distribution.

Chapter 16

X16.1 **For price discrimination to take place, there must be no possibility of resale among consumers. Explain why this condition is necessary.**

Suppose that resale is possible. Then a consumer, able to purchase the good at a low price, can sell it on to a consumer willing to pay a higher price. While we still end up with the efficient outcome in which the people who value the good most consume it, we do reach it with a sole (final) supplier.

X16.2 **Explain why a firm supplying personal services, such as a hairdressing salon or a beauty therapist, might find it relatively straightforward to engage in price discrimination compared with a firm selling an easily divisible but durable physical product, such as screws and nails.**

This turns on the question of resale. Personal services cannot be easily transferred from the purchaser to someone else – they are created at the point of consumption. So it is in principle possible for a salon to charge prices that differ across consumers. A supplier of screws and nails offers a good that can be stored and which is also fungible – screws and nails can be sold in bulk or in very small quantities. A purchaser of a bulk load can therefore resell them in much smaller quantities.

X16.3 **Suppose that the willingness to pay for a product $WTP(q) = 200 - 3q_f$, while the marginal cost of production $MC(q) = 20$. Draw a diagram showing the average and marginal revenue curves when there is no price discrimination, and the marginal cost curve. From the diagram calculate: (1) the firm's profit maximizing output in the absence of price discrimination, and the price that the firm then charges; (2) the firm's profit, and the consumer surplus without price discrimination; and (3) the welfare loss when there is price discrimination. Explain how the outcome changes when price discrimination is permitted.**

On a diagram with firm output measured on the horizontal axis and revenue and cost measures on the vertical axis, the marginal cost curve (and the average cost curve) are both the horizontal line, 20 units above the output axis. With WTP(q) representing both the inverse demand curve and the average revenue curve (given uniform pricing), then the average revenue curve is a downward-sloping line, which meets the vertical axis at (0, 200), and which has slope -3. The marginal revenue curve is also a downward sloping straight line passing through (0, 200), and which is twice as steep as the average revenue curve, so that it has gradient -6.

The firm maximizes profit at output q_f^*: $MC(q_f^*) = MR(q_f^*)$; and this means that $200 - 6q_f^* = 20$; so that $q_f^* = 30$. The firm then charges price $WTP(30) = 200 - 90 = 110$. The firm's profit is calculated as the area of the right-angled triangle formed by the vertical axis, the marginal cost curve and the marginal revenue curve. This has height 180 and width 30 and so has area $\Pi = ½*180*30 = 2{,}700$. The consumer surplus is the triangle formed by the line $p = 110$, the vertical axis and the average revenue curve. This has height 90 and width 30 and so has area $\Pi = ½*90*30 = 1{,}350$.

The welfare loss is the area formed by the triangle formed by the line $q_f^* = 30$, the average revenue curve and the marginal cost curve. As a right-angled triangle with base 30, and height 90, this has area $W = 1{,}350$.

Assume that we allow the firm to engage in perfect price discrimination. We redraw the diagram, simply showing the WTP curve. The monopolist continues sales until at output, q_f^D, $WTP(q_f^D) = MC(q_f^D)$. We see that then $q_f^D = 180$, and the monopolist's profits are equal to the sum of the three areas stated above: $\Pi^D = \Pi + CS + W = 5{,}400$.

X16.4 **Consider the situation of the firm in Exercise X16.3. The government has calculated that allowing perfect price discrimination and then collecting the tax from the firm will cost W = 400. Should the government permit price discrimination in these circumstances?**
Yes. The potential welfare gain from trading is 1,350. The cost of managing price discrimination is 400; so that there is a net gain of 950. Perfect price discrimination would be welfare increasing.

X16.5 **Why might it be particularly difficult to engage in price discrimination in sales of music downloads? A few musicians have experimented with allowing people to download music in return for a donation, rather than by setting a fixed price. Explain why these experiments do not involve the practice of price discrimination.**
We have argued that resale should not be possible if there is to be price discrimination. In principle, it is possible to generate an infinite number of copies of digital files at close to zero cost. Asking for donations does not constitute price discrimination since there is no price set; while it is reasonable to allow distribution at zero price, there is no way of confirming the WTP of donors.

X16.6 **Explain why it is rational for bidder B to respond.**
Suppose B bids and wins the auction. So long as the winning bid is less than WTP_B*, B obtains surplus from taking part in the auction.*

X16.7 **Suppose that WTP_A = 100 and WTP_B = 120. Suppose that bidder A responds to price p_n = 97.5. How should bidder B respond to price p_{n+1} = 100? How does the auction end?**
The conditions for B to bid and win are satisfied here; should B know A's valuation, then B will know that any offer $p_{n+2} > 100$ *will win; and any offer* $p_{n+2} < 120$*, will generate surplus.*

X16.8 **In a situation where there are 100 bidders, suppose that the highest valuation $v_{(100)}$ = 150 and the second highest valuation $v_{(99)}$ = 148. How would you expect this auction to end? Suggest a general rule for ending the auction if it is run as a *clock auction*, in which some mechanism raises the price continuously, with bidders choosing the time when they withdraw from the auction.**
We expect bidder 99 to continue bidding until it is impossible to enter a bid of no more than 148, while bidder 100 will continue until it is impossible to enter a bid of greater than 150. Allowing bids of only whole units, we expect the auction to end with bidder 100 offering either 148 or 149. In a clock auction, we expect the bidder 100 to obtain the good at a price of 148 (or again, allowing only unit bids, perhaps 149).

X16.9 **Consider a slightly different bidding process:**
(1) Each bidder writes down a bid, places it in a sealed envelope, and delivers it to the auctioneer.
(2) When the auction closes to further bids, the auctioneer opens the envelopes.
(3) The auctioneer gives the good to the highest bidder.
(4) The highest bidder pays the auctioneer the second-highest bid.
Suppose that one bidder still has to decide what bid to make. Why might the bidder regret making a bid greater than *WTP*? [*Hint:* Think what happens if the bidder wins.] Why might the bidder regret making a bid less than *WTP*? [*Hint:* Think what happens if another bidder with a lower *WTP* wins.] What conclusions do you draw about this form of auction compared with the traditional one with verbal bids?
Bid more than WTP, and it is possible to win the auction, but, if there are two bids greater than WTP, still to pay a price so that there the winner is worse off than by losing the auction.

Stop bidding before bids reach WTP, and is possible that the second-highest bid will be less than WTP, so that the loser is worse off than by winning the auction.
We have said that in a traditional auction bidders should never bid more than their WTP. We see the same rule here, and so we expect the same outcomes. The two forms of auction are effectively equally good for the seller.

X16.10 **Suppose that a customer buys only one unit of the good when the second unit is offered for free. What might we reasonably assume about the value of the second unit to this customer?**
The second unit has no value; WTP(2) < 0.

X16.11 **What justifications might we give for a retailer choosing to use this form of promotional pricing? Think in particular about the possible effects on its revenues, but also consider the implications for its costs.**
With these types of arrangement, the retailer will place a large order, sufficient to obtain a substantial discount on the supply price. This means that it does not have to sell twice as many units as it would at full price to maintain profits. We should also note that such offers allow the retailer to advertise these offers, attracting more consumers, and that, given that it sells many products, it may well expect an increase in sales across the range of products.

X16.12 **Consider the situation facing a consumer who chooses among consumption bundles consisting of coffee and doughnuts. The price per cup of coffee falls from £2.50 to £2.00 on the fifth cup of coffee. (Note that this means that the fifth cup of coffee is effectively free, since the lower price is offered on all five cups purchased.) The price of a doughnut remains constant at £1.20, with no quantity discount being offered. Sketch the affordability (budget) constraint for a consumer willing to spend £24.00. Discuss the difficulties that such a constraint might cause in trying to solve the standard optimization problems.**
Show the quantity of coffee consumed on the horizontal axis and the quantity of doughnuts on the vertical axis. The consumer can afford 12 coffees (and no doughnuts); or else 20 doughnuts (and no coffees). The affordability constraint passes through (12, 0) and (0, 20). Defining the slope of the constraint as the ratio of prices, $-\frac{p_c}{p_d}$, we see that there are two segments to the constraint. For s < 5, $-\frac{p_c}{p_d}=-\frac{25}{12}$, while for $c \geq 5$, $-\frac{p_c}{p_d}=-\frac{5}{3}$. We see that the left-hand segment of the constraint runs from (0, 20) on the vertical axis to approximately (5, 9.58); but that there is then a discontinuity in the constraint, so that the right hand segment runs from approximately (5, 11.67) to (12, 0).
The discontinuity means that the affordable set is no longer convex. We cannot simply rely on the first-order condition for maximization implied by the equal gradient condition, but must instead be aware of the possibility of a corner solution at the discontinuity in the constraint; or of finding internal solutions in both segments of the constraint.

X16.13 **For the marginal consumer, suppose that the highest attainable indifference curve touches both segments of the affordability constraint. Sketch a diagram showing this situation for a consumer choosing among bundles of coffee and doughnuts. Indicate on your diagram the effect of a small increase in the relative price of coffee.**
This is possible because the left-hand segment is steeper than the right-hand segment. It is possible to draw an indifference curve that touches both segments. A small increase in the price of coffee will change the relative prices in both segments. We might expect that this will have the effect of making one of the local optima strictly preferred to the other, but

without knowing more about income and substitution effects, we cannot say which this will be.

X16.14 **Figure 16.2 is incomplete. It shows that price discrimination is possible, but does not show that it is desirable.**

a) **Sketch a diagram with two linear individual (inverse) demand curves, D_1 and D_2, which have different slopes and different intercepts on the vertical (value) axis, and which intersect. Label them so that the price elasticity of demand of D_1 is less than for D_2 where they intersect. [Hint: To make your diagram consistent with Figure 16.2, make sure that you have chosen a level of marginal cost for which consumer 2 would demand none of the good at price p_1, so that $q_2(p_1) = 0$; and also so that at the minimum quantity at which a quantity discount will be offered, Q^*, $WTP_1(Q^*) < p_2$.]**

b) **Add the marginal revenue curves, MR_1 and MR_2, to this diagram.**

c) **Assume that the firm faces constant returns to scale in production. Show its marginal cost curve; and hence identify the profit maximizing outputs, q_1^* and q_2^*, and the prices, p_1 and p_2, that the firm should charge each customer.**

d) **Identify areas on the diagram that represent the profit per customer. How might the use of quantity discounts increase the firm's profits?**

It may be useful to note at this point that demand for good 1 is expected to be less than demand for good 2. Consumer 1 has a higher reservation price than consumer 2, and we have set up the market so that for output q_1^*, chosen so that the inverse demand, $p_1 = p(q_1^*) = MC = c$, $q_2(p_1) = 0$. The firm offers two contracts, one in which the access fee is equal to the consumer surplus for consumers of type 1: this can be shown by the triangle formed by the marginal cost curve, the vertical axis and the inverse demand curve for consumer 1. The second contract requires consumers to consume a minimum amount, Q^*, again paying a price p_1. This is chosen so that consumers of type 1 will prefer contract 1, and consumers of type two, paying an access fee equal to the area formed by the marginal cost curve, their inverse demand curve and the vertical axis, prefer contract 2. These two triangles represent the profit per consumer.

X16.15 **On a diagram showing consumption bundles consisting of quantities of goods A and B, indicate the effect of the imposition of an access fee for good B. Sketch in an indifference curve demonstrating that the consumer would prefer to pay the user fee and buy a consumption bundle consisting of a mixture of goods, rather than simply spending the whole amount of money on good A.**

On purchasing any quantity of good B, the consumer has to pay the access fee. The consumer's budget constraint is therefore determined by the amount of money available to finance consumption after payment of the usage fee. In addition, though, if the consumer does not consume good B at all, then the amount available to finance consumption will include what would otherwise be paid as a usage fee.

In a diagram with consumption of good A on the horizontal and consumption of good B on the vertical, axes, amount of money to finance consumption, m, prices p_A and p_B, and usage fee f, the constraint can be written $\begin{cases} p_A a + p_B b = m - f, \text{if } p_B > 0 \\ p_A a = m, \text{if } p_B = 0 \end{cases}$. *In the diagram, the constraint is a straight line running from* $\left(0, \frac{m-f}{p_B}\right)$ *on the vertical axis towards, but not quite reaching* $\left(\frac{m-f}{p_A}, 0\right)$ *on the horizontal axis. There is a discontinuity in the constraint at b = 0, with the constraint including the point* $\left(\frac{m}{p_A}, 0\right)$.

The consumer chooses a mixture of the two goods. This means that the utility that can be derived from some mixture is greater than the utility that can be derived from consumption *of good A alone. Denote the best, affordable consumption bundle as point Z on the constraint. Sketching a smooth, downward sloping indifference curve, which is tangent to the affordability constraint at point Z, we require the indifference curve to meet the horizontal axis to the right of* $\left(\frac{m}{p_A}, 0\right)$.

X16.16 **Explain why it is more likely that consumers will choose to pay the access fee for good *B* when the goods are complements, rather than substitutes.**
We can be certain that the consumer will pay the access fee where the goods are complements, and utility possesses the constant elasticity of substitution property. The relatively low value of elasticity of substitution when goods are complements means that indifference curves do not reach the axes; and this is sufficient to ensure that the interior solution, at point Z in X16.15, will be preferred to the alternative of consuming only good A. Thinking of the example of perfect substitutes, it is of course possible that the consumer will simply choose good A; a sufficient condition being that the marginal rate of substitution is greater than the price ratio, so that indifference curves are steeper than the affordability constraint.

X16.17 **Write down the precise conditions that must be satisfied in order for the two consumers to choose the different supply contracts.**
There are two conditions which must be satisfied for both Omar and Paul. Omar must prefer contract 1 to both contract 2 and staying out of the market. Paul must prefer contract 2 both to contract 1 and staying out of the market.
For Omar, $CS_O(A_1, p_1) \geq \max\{CS_O(A_2, p_2), 0\}$*, while for Paul,* $CS_P(A_2, p_2) \geq \max\{CS_P(A_1, p_1), 0\}$

X16.18 **Consider the situation facing a car manufacturer. It believes that its customers consider cars to be a bundle of characteristics, which we summarize as speed and comfort. Assume that the firm has to trade off these characteristics, so that each model of car is located at a particular point on a production possibility frontier, which shows feasible combinations of speed and comfort.**

a) **Sketch a diagram with a concave production possibility frontier.**
In a diagram with maximum speed measured on the horizontal axis and comfort measured on the vertical axis, we show the production possibility frontier as a downward sloping curve, which, moving from left to right, becomes increasingly steep.

b) **Suppose that the firm produces four models. Explain how doing so would allow the firm to meet the desires of consumers with different preferences.**
In the diagram, we can show different preferences through differences in the gradient of indifferences curves through any characteristics bundle, and effectively differences in the marginal rate of substitution of comfort for speed. The steeper the indifference curve, and so the larger the marginal rate of substitution, the closer the characteristics bundle at which the first-order condition that the production possibility frontier and the indifference curve share a common tangent will be to the horizontal (speed) axis. Such consumers prefer fast cars. Similarly, the flatter the indifference curve, and so the smaller the marginal rate of substitution, the closer the characteristics bundle at which the first-order condition that the production possibility frontier and the indifference curve share a common tangent will be to the vertical (comfort) axis. Such consumers prefer comfortable cars.
Note that with the indifference curves reflecting well-behaved preferences, so that they are concave, the intersection of each consumer's affordable and preferred sets is again the single characteristics bundle formed by the most-preferred feasible bundle.
We might also note here that with well-behaved preferences, we do not need the tangency condition to be satisfied. Suppose that Rita has a stronger preference for speed to Sandy, and

that they are offered the choice between characteristics bundles A, B, C and D, as in the diagram. Say that Rita expresses a preference for bundle C over all of the others. This means that Rita's indifference curve through C passes above the indifference curves for the other three combinations, and it is irrelevant to her choice whether C is her most preferred combination because only A, B, C and D are available.
In the same way, if Sandy expresses a preference for bundle A over all of the others. This means that her indifference curve through A passes above the indifference curves for the other three combinations, and it is irrelevant to her choice whether A is her most preferred combination because only A, B, C and D are available.

X16.19 **In previous examples, we have emphasized that a firm can exploit market power where customers respond differently to changes in prices. Apply these arguments to the case of bundling. Explain how the provision of a small set of bundles of different quality might enable the airline to extract additional surplus from consumers whose demand for services is inelastic.**
The argument is essentially the same as the argument in X16.18. The airline offers a small number of bundles, and it is these bundles, rather than the whole of the characteristics sets from which they chosen bundle must emerge.
A customer with inelastic demand for travel generates a much higher consumer surplus from any individual journey than passengers with a relatively elastic demand. The airline takes advantage of this by offering a bundle of services that has a relatively high fixed price; and another one that has a relatively low fixed price (but for which additional services might be purchased separately). Designed properly, the additional services offered in the high price bundle will be preferred (at the price charged) only by the consumer with low price elasticity of demand. The simpler bundle will be preferred only by the consumer with high price elasticity of demand, and both will prefer to purchase those bundles than to choose alternative forms of travel, or not to make the journey.

X16.20 **Criticize the argument that price discrimination is possible because travellers differ in their elasticity of demand for travel.**
There is some merit in the argument. People travelling on business tend to need to be at particular destinations at particular times. People travelling for leisure purposes may have much more flexibility. Note though, that payment for business travel is typically made by an employer or a client, or at the very least will be an expense that can be set against business costs for tax purposes. This means that the personal costs incurred in business travel tend to be related to the time spent and the discomfort associated with it. Here we see another justification for the bundling of services: they provide an indirect way of compensating people for taking part in an activity in which they would not otherwise choose to engage.

X16.21 **Confirm that if we restrict the value of the parameters, a_S, b_S, a_F and b_F so that $\frac{a_F}{b_F} > \frac{a_S}{b_S}$ and the firm adopts uniform pricing, so that $p_S = p_F = p$, then $\varepsilon_F > \varepsilon_S$.**
Given $q_S = a_S - b_S p_S$ and $q_F = a_F - b_F p_F$, then the elasticities of demand in each sector may be written $\varepsilon_p = \frac{\partial q_K}{\partial p} \cdot \frac{p}{q_K} = -b_K \cdot \frac{p}{a_K - b_K p} = -\frac{b_K p}{a_K - b_K p} = \frac{b_K p}{b_K p - a_K}$. The difference in the elasticities can then be written as $\varepsilon_F - \varepsilon_S = \frac{b_F p}{b_F p - a_F} - \frac{b_S p}{b_S p - a_S} = \frac{b_F (b_S p - a_S) - b_S (b_F p - a_F)}{(b_S p - a_S)(b_F p - a_F)} p = \frac{b_S a_F - b_F a_S}{(b_S p - a_S)(b_F p - a_F)} p$. By inspection, we see that the term determining the sign of this expression is $b_S a_F - b_F a_S$; and that if this term is greater than zero, the result follows. For $b_S a_F - b_F a_S > 0$, we require $b_S a_F > b_F a_S$, and the result follows on dividing through by $b_S b_F$. Where this conditions is satisfied, demand in the student sector will be more elastic than in the full price sector of the market, since all

price elasticities of demand in this situation are negative numbers taking a value of less than minus one.

X16.22 **Using the demand function in X16.21:**

a) **Sketch a diagram showing the demand curves in the two sectors. [*Hint:* Remember that we usually show the inverse demand or average revenue curve, so measure price on the vertical axis and quantity on the horizontal axis.]**
It is perhaps also easiest to draw the diagram with two panels. The conditions that we have set out in X16.22 are certainly satisfied if the demand curve in market segment F meets the vertical (value) axis at a higher point than the demand curve for market segment S.

b) **Use the diagram to explain the effect of a price increase on demand in each sector. In which sector is demand more elastic?**
In each market segment, we expect the firm to set output so that marginal cost and marginal revenue are equal. With marginal revenue curves being lines that start from the same point on the vertical axis as the demand curves, but which are twice as cheap, this implies that there will be a higher price set in sector F, where demand is less elastic. It also implies that following an increase in price of a fixed amount in each sector, the effect on demand in sector F will be less than in sector S.

X16.23 **Given the assumption that the firm faces constant marginal costs, *c*, obtain the condition for profit maximization in each sector, *MR* = *MC*, and hence the profit maximizing outputs and prices. Interpret the prices that the firm sets in each sector in terms of the marginal cost and the price that chokes off demand. Show that the total profit in each sector, Π = $\frac{1}{4b}(a-bc)^2 = \frac{1}{b}.(q^*)^2$, where *q** is the profit maximizing output for the sector.**
Given that we can write the sector demand q = a – bp, we can also write the inverse demand as $p=\frac{a-q}{b}$*, and the sector revenue as* $R=pq=\left(\frac{a-q}{b}\right)q$*, so that the marginal revenue, MR =* $\frac{dR}{dq}=\frac{a-2q}{b}$*. Imposing the first-order condition, MR = MC, we obtain a – 2q* = bc, so that q* = ½(a – bc). We see that the firm then makes profits $\Pi(q^*)$ = R(q*) – C(q*) =*
$$\left(\frac{a-\frac{1}{2}(a-bc)}{b}-c\right)\left(\frac{1}{2}(a-bc)\right)=\frac{1}{4b}(a-bc)^2$$

X16.24 **Confirm that the firm produces output *Q** = $\frac{1}{2}\left(a_S+a_F-(b_S+b_F)c\right)$, and that it makes profits** $\Pi^*=\frac{1}{4}\left[\frac{a_S{}^2}{b_S}+\frac{a_F{}^2}{b_F}-2(a_S+a_F)c+(b_F+b_S)c^2\right]$.
The firm produces outputs q_S^* = ½(a_S – b_Sc) and q_F^* = ½(a_F – b_Fc), so that total output Q = q_S^* + q_F^* = ½(a_S – b_Sc) + ½(a_F – b_Fc) = ½[a_S + a_F – (b_S + b_F)c]. The firm then makes profits*
$$\Pi^*=\frac{1}{4b_F}(a_F-b_Fc)^2+\frac{1}{4b_S}(a_S-b_Sc)^2=\frac{a_F{}^2-2a_Fb_Fc+b_F{}^2c_F{}^2}{4b_F}+\frac{a_S{}^2-2a_Sb_Sc+b_S{}^2c_S{}^2}{4b_S}=\frac{1}{4}\left[\frac{a_S{}^2}{b_S}+\frac{a_F{}^2}{b_F}-2(a_S+a_F)c+(b_F+b_S)c^2\right]$$

X16.25 **Confirm that Expressions 16.9 and 16.10 define the profit maximizing output and the profit maximum.**
Facing inverse demands $p=\frac{a_s+a_F-q}{b_S+b_F}$*, we can write the marginal revenue as* $MR(q)=\frac{a_s+a_F-2q}{b_S+b_F}$*, so that when MR(q*) = MC(q*), q* = ½[a_S + a_F – (b_s + b_F)c]. (This is expression 16.9)*
Then the firm's profits can be written as Π^* = Π(q) = q*[p(q*) – c] =* $\frac{[a_s+a_F-(b_S+b_F)c]^2}{4(b_S+b_F)}$*, and expression 16.10 follows immediately.*

X16.26 **Show that the firm sells the same output when engaging in price discrimination as when it sets a uniform price for all customers.**

This follows directly from the solutions of X16.24 – 25.

X16.27 **Write an expression for the difference in the profits made with price discrimination, given in X16.24, and the profits made with uniform pricing, given in Expression 16.10. Show that the assumption $\frac{a_F}{b_F} > \frac{a_S}{b_S}$, introduced in X16.21, is sufficient to ensure that the firm makes more profit when using price discrimination.**

We have the expressions for profits under price discrimination, $\Pi_D = \frac{1}{4}\left[\frac{a_S{}^2}{b_S} + \frac{a_F{}^2}{b_F} - 2(a_S + a_F)c + (b_F + b_S)c^2\right] = \frac{1}{b_S}\left(\frac{a_S - b_S c}{2}\right)^2 + \frac{1}{b_F}\left(\frac{a_F - b_F c}{2}\right)^2$ *; and profits under uniform pricing* $\Pi_U = \frac{1}{b_F + b_S}\left(\frac{a_S + a_F - (b_F + b_S)c}{2}\right)^2 = \frac{1}{4}\left[\frac{(a_S + a_F)^2}{b_S + b_F} - 2(a_S + a_F)c + (b_S + b_F)c^2\right]$*. So for* $\Pi_D > \Pi_U$*,* $\frac{a_S{}^2}{b_S} + \frac{a_F{}^2}{b_F} - \frac{(a_S + a_F)^2}{b_S + b_F} > 0$*. Multiplying through this condition by* $b_S b_F(b_S + b_F)$*, it may be written* $a_S{}^2 b_F{}^2 - 2a_S a_F b_F b_S + a_F{}^2 b_S{}^2 > 0$*, or else as* $(a_S b_F - a_F b_S)^2 > 0$*, which certainly takes a positive value if* $\frac{a_F}{b_F} > \frac{a_S}{b_S}$*. We are interested only in this situation, rather than where* $\frac{a_F}{b_F} < \frac{a_S}{b_S}$*, because this ensures that the price elasticity of demand in the full-price segment of the market is greater than the elasticity of demand in the student segment.*

X16.28 **Explain why one effect of price discrimination is that the firm increases revenue from students, while reducing it in the full-price sector.**

Third degree price discrimination involves reducing the price and increasing sales to the student sector and increasing price and reducing sales to the full-price sector. We have seen that demand must be elastic where profits are maximized, so that demand is sufficiently responsive to changes in price that revenue is a decreasing function of price in each sector. Introducing discrimination, the firm increases sales by the same amount in the student sector as it reduces them in the full-price sector. To increase profits, it follows that it must increase revenue by more than enough in the student sector to compensate for the loss of revenue from the full price sector.

X16.29 **Relate the operation of third-degree price discrimination to the differences in the price elasticity of demand in the two sectors.**

In general, we expect to see higher prices charged in the sector in which demand is less elastic, and lower prices where the demand is more elastic.

Chapter 17

X17.1 **How does an increase in *Aulds'* output affect *Blacks'* total revenue and profit, assuming that *Blacks* does not change its output? [Hint: Use partial differentiation.]**
If Blacks keeps its output constant, the effect of a change in Aulds' output on revenue and profit will be the same, since we will assume that Blacks' costs depend only on its own output. Then $\frac{\partial R_B}{\partial q_A} = \frac{\partial \Pi_B}{\partial q_A} = -bq_B$. *Blacks' revenue and profits both fall as Aulds increases its profits.*

X17.2 **Sketch a diagram with Aulds' output on the horizontal axis, and the conjecture q_B^c on the vertical axis. In your diagram, sketch the reaction function for Aulds, as given in Expression 17.6.**
In a diagram with Aulds' output on the horizontal axis and Blacks' output on the vertical axis, we see that Expression 17.6 is the equation of a straight line. Setting q_A to zero, we require $q_B^c = \frac{a-c}{b}$, *while setting q_B^c to zero, Aulds would produce* $q_A = \frac{a-c}{2b}$. *The reaction function is then represented by a downward sloping line connecting* $\left(0, \frac{a-c}{b}\right)$ *and* $\left(\frac{a-c}{2b}, 0\right)$, *and with gradient -2. Anticipating a one loaf increase in Blacks' output, Aulds reduces its output by half a loaf.*

X17.3 **Obtain the firm revenue and profit functions, and hence the reaction function for Aulds, given:**

a) **inverse market demand $p = 120 - Q^D$ with firm costs $C_A = 30q_A$;**
We write revenue, $R_A = (120 - q_A - q_B^c)q_A$ *and profit* $\Pi_A = (120 - q_A - q_B^c)q_A - 30q_A = (90 - q_A - q_B^c)q_A$. *Differentiating the profit function, we obtain* $\frac{\partial \Pi_A}{\partial q_A} = 90 - 2q_A - q_B^c$. *To find the reaction function, we set the partial derivative to zero, so that* $90 - 2q_A - q_B^c = 0$, *and rearranging,* $q_A(q_B^c) = \frac{90 - q_B}{2}$.

b) **inverse market demand $p = 500 - 2Q^D$, with firm costs $C_A = 20q_A$;**
We write revenue, $R_A = (500 - 2q_A - 2q_B^c)q_A$ *and profit* $\Pi_A = (500 - 2q_A - 2q_B^c)q_A - 20q_A = 2(240 - q_A - q_B^c)q_A$. *Differentiating the profit function, we obtain* $\frac{\partial \Pi_A}{\partial q_A} = 2(240 - 2q_A - q_B^c)$. *To find the reaction function, we set the partial derivative to zero, so that* $240 - 2q_A - q_B^c = 0$, *and rearranging,* $q_A(q_B^c) = \frac{240 - q_B}{2}$.

c) **inverse market demand $p = 200 - 0.5Q^D$, with firm costs $C_A = 8q_A$.**
We write revenue, $R_A = (200 - 0.5q_A - 0.5q_B^c)q_A$ *and profit* $\Pi_A = (200 - 0.5q_A - 0.5q_B^c)q_A - 8q_A = 0.5(384 - q_A - q_B^c)q_A$. *Differentiating the profit function, we obtain* $\frac{\partial \Pi_A}{\partial q_A} = 0.5(384 - 2q_A - q_B^c)$. *To find the reaction function, we set the partial derivative to zero, so that* $384 - 2q_A - q_B^c = 0$, *and rearranging,* $q_A(q_B^c) = \frac{384 - q_B}{2}$.

In each case, sketch the reaction function.
In all three cases, showing Aulds' output on the horizontal and Blacks' output on the vertical we note that the graph of the reaction function is a downward sloping line with gradient -2; so that all three graphs can be shown on the same axis.

a) The line connects (0, 90) (on the vertical axis) with (45, 0) (on the horizontal axis).
b) The line connects (0, 240) with (120, 0).
c) The line connects (0, 384) with (192, 0).

X17.4 **Confirm that if Blacks exits the market, so that Aulds has a monopoly, then the profit maximizing output $q_A = \frac{a-c}{2b}$.**

We obtain this result by evaluating Expression 17.6 for output $q_B^c = 0$.

X17.5 **Substitute the reaction function given in Expression 17.6 into the profit function given in Expression 17.4, writing Aulds' profits, Π_A, as a function of the conjecture, q_B^c. Show that Π_A is a decreasing function of q_B^c.**

Since $q_A(q_B^c) = \frac{a-c-bq_B^c}{2b}$, we can write $\Pi_A(q_B^c) = \left(a-c-b\left[\frac{a-c-bq_B^c}{2b} + q_B^c\right]\right)\frac{a-c-bq_B^c}{2b}$. We confirm that the expression in brackets simplifies to

$\Pi_A(q_B^c) = \frac{1}{2}(a-c-bq_B^c)\frac{a-c-bq_B^c}{2b} = \frac{1}{4b}(a-c-bq_B^c)^2$. *Differentiating this expression with respect to Blacks' conjectured output, we obtain*

$\frac{\partial \Pi_A}{\partial q_B^c} = -b.\frac{1}{2b}(a-c-bq_B^c) = -\frac{1}{2}(a-c-bq_B^c) = -bq_A(q_B^c)$. *We can be certain that $\frac{\partial \Pi_A}{\partial q_B^c} < 0$ because the expression is a negative multiple of Aulds' profit maximizing output.*

X17.6 **Replicate the argument by which we obtained the reaction function for Aulds and confirm that Blacks' reaction function is given by Expression 17.7. Without further calculations, sketch the reaction function for Blacks' in each of the cases in Exercise X17.3.**

We write revenue, $R_B = (a - bq_A - bq_B^c)q_B$ and profit $\Pi_B = (a - bq_A^c - bq_B)q_B - cq_B = (a - c - bq_A^c - bq_B)q_B$. Differentiating the profit function, we obtain $\frac{\partial \Pi_B}{\partial q_B} = a - c - bq_A^c - 2bq_B$. To find the reaction function, we set the partial derivative to zero, so that $a - c - bq_A^c - 2bq_B^c = 0$, and rearranging, $q_B(q_A^c) = \frac{a-c-bq_A}{2b}$.

Applying the formula that we have just obtained, we obtain reaction functions:

a) $q_B(q_A^c) = \frac{90-q_A}{2}$; *b)* $q_B(q_A^c) = \frac{240-q_A}{2}$; *c)* $q_B(q_A^c) = \frac{384-q_A}{2}$

X17.7 **Without using Expression 17.10, confirm that in the three examples used in Exercise X17.3, there are consistent conjectures for outputs $q_A = q_B =$ (a) 30; (b) 80; (c) 128. In each case find the market price and the profit that each firm makes.**

a) Assume that conjectures are consistent. Then each firm's output is the other firm's conjecture, and we can rewrite both reaction functions in terms of actual outputs, obtaining $2q_A + q_B = 90$ and $q_A + 2q_B = 90$. We see that both expressions are true when $q_A = q_B = 30$. The firms then set price $p(30, 30) = 60$ and make profits $\Pi_A(30, 30) = \Pi_B(30, 30) = 900$.

b) We write both reaction functions in terms of actual outputs, obtaining $2q_A + q_B = 240$ and $q_A + 2q_B = 240$. We see that both expressions are true when $q_A = q_B = 80$. The firms then set price $p(80, 80) = 500 - 320 = 180$ and make profits $\Pi_A(80, 80) = \Pi_B(80, 80) = (180 - 20)*80 = 12{,}800$.

c) We write both reaction functions in terms of actual outputs, obtaining $2q_A + q_B = 384$ and $q_A + 2q_B = 384$. We see that both expressions are true when $q_A = q_B = 128$. The firms then set price $p(128, 128) = 200 - 128 = 72$ and make profits $\Pi_A(128, 128) = \Pi_B(128, 128) = (72 - 8)*128 = 8{,}192$.

X17.8 **For each case in X17.7, find the firms' maximum profits.**

These are the same as in X17.7: a) 900; b) 12,800; c) 8,192

X17.9 **Suppose that Aulds' conjecture $q_B^c < q_B^*$. Show on a sketch that Aulds will then produce output $q_A^*(q_B^c) > q_A^*$. Illustrate this situation in a diagram and show:**

a) This situation will not be an equilibrium. [*Hint:* Assume that Blacks' conjecture $q_A^c = q_A^*(q_B^c)$.]

In a diagram with the output of Aulds measured on the horizontal axis and the output of Blacks measured on the vertical axis, we draw in the reaction functions for the two firms as downward sloping lines. The reaction function for Aulds has gradient -2, whereas the reaction function for Blacks has gradient -0.5. The reaction function for Aulds meets the vertical axis twice as far away, and the horizontal axis half as far away from the origin as the reaction function for Blacks. The equilibrium outputs (q_A, q_B*) occur at the intersection of the best replies.*
If conjecture $q_B^c < q_B$, then since q_A*(q_B^c) is decreasing in q_B^c, it follows that q_A*(q_B^c) > q_A*(q_B*). From our diagram, we note that if $q_A > q_A$*(q_B*), then the graph of the reaction function for Blacks lies above the graph of the reaction function for Aulds. Blacks, anticipating Aulds' output, produces more than Aulds conjectured. But this will mean that Aulds should anticipate this response and produce some amount less than q_A*(q_B^c).*

b) If the firms are allowed to take turns in changing their conjectures according to the output that the other one last proposed, the equilibrium will be reached.
We begin with the assumption that $q_A > q_A$, the optimal output. We have already argued that in this case Blacks' best reply involves a higher level of production than Aulds has conjectured. So the proposed output pair is (q_A, q_B*(q_A)). We show Blacks' deviation from Aulds' conjecture by a vertical line segment from Aulds' reaction function to Blacks' reaction function.*
We now note that at the level of output q_{B1}= q_B(q_A), Aulds will wish to reduce its planned output to q_{A1} = q_A*(q_{B1}). This reduction in output means that Blacks will wish to increase output further; and we can show the successive changes as a series of steps, with smaller and smaller adjustments as the production plan converges on the equilibrium.*

X17.10 **Adapt Figure 17.1 to demonstrate that *A*ulds' conjecture is consistent, then $q_B^{c_1} = q_B^c$. In addition, show that if $q_B^c > q_B$*, $q_B^c > q_B^{c_1} > q_B$ ***
The argument here is that if Aulds' conjecture is consistent, it expects Blacks to produce the output that it chooses to produce. It follows that Blacks' conjecture will then also be consistent; for if not, it would have chosen a different level of output, and there would be no equilibrium.
The second statement follows from the relationship between the reaction functions. Aulds expects Blacks to produce more than the equilibrium output. It then plans to produce less than the equilibrium output; then calculates that it is optimal for Blacks to produce less than the original conjecture, but more than the equilibrium output.

X17.11 **Given the examples in Exercise X17.3, sketch the original reaction functions, indicating the equilibrium outputs for the firms.**
As in X17.9, in a diagram with the output of Aulds measured on the horizontal axis and the output of Blacks measured on the vertical axis, we draw in the reaction functions for the two firms as downward sloping lines. The reaction function for Aulds has gradient -2, whereas the reaction function for Blacks has gradient -0.5. The reaction function for Aulds meets the vertical axis twice as far away, and the horizontal axis half as far away from the origin as the reaction function for Blacks. The equilibrium outputs (q_A, q_B*) occur at the intersection of the best replies.*

Then sketch the new reaction functions after the following changes to the market environment:

a) Inverse market demand is initially $p = 120 – Q^D$, but becomes $p = 240 – Q^D$, with firm costs $C_A = 30q_A$.

The initial reaction functions are, for Aulds, $2q_A + q_B = 90$ and, for Blacks, $q_A + 2q_B = 90$. For Aulds, the reaction function runs from (0, 90) to (45, 0), while for Blacks, the reaction function runs from (0, 45) to (90, 0). They intersect at (30, 30).
After the change in demand, Aulds' profit function can be rewritten $\Pi_A = (240 - q_A - q_B)q_A - 30q_A$, and so differentiating with respect to q_A, $\frac{\partial \Pi_A}{\partial p_A} = 210 - 2q_A - q_B$; and we obtain a similar expression for Blacks: $\frac{\partial \Pi_B}{\partial p_B} = 210 - q_A - 2q_B$. Setting both derivatives to zero, it follows that $q_A = 210 - 2q_B$, and that $3q_B = 210$; so that we have a new equilibrium, $(q_A^*, q_B^*) = (70, 70)$.

b) Inverse market demand is initially $p = 500 - 2Q^D$, but becomes $p = 500 - 4Q^D$, with firm costs $C_A = 20q_A$.
The initial reaction functions are, $2q_A + q_B = 240$ and $q_A + 2q_B = 240$. For Aulds, the reaction function runs from (0, 240) to (120, 0), while for Blacks, the reaction function runs from (0, 120) to (240, 0). They intersect at (80, 80).
After the change in demand, Aulds' profit function can be rewritten $\Pi_A = (500 - 4q_A - 4q_B)q_A - 20q_A$, and so differentiating with respect to q_A, $\frac{\partial \Pi_A}{\partial p_A} = 480 - 8q_A - 4q_B$; and we obtain a similar expression for Blacks: $\frac{\partial \Pi_B}{\partial p_B} = 480 - 4q_A - 8q_B$. Setting both derivatives to zero, it follows that $q_A = 120 - 2q_B$, and that $3q_B = 120$; so that we have a new equilibrium, $(q_A^*, q_B^*) = (40, 40)$.

c) Inverse market demand $p = 200 - 0.5Q^D$, with firm costs initially $C_A = 8q_A$, becoming $C_A = 16q_A$.
The initial reaction functions are, $2q_A + q_B = 384$ and $q_A + 2q_B = 384$. For Aulds, the reaction function runs from (0, 384) to (192, 0), while for Blacks, the reaction function runs from (0, 192) to (384, 0). They intersect at (128, 128).
After the change in demand, Aulds' profit function can be rewritten $\Pi_A = (200 - 0.5q_A - 0.5q_B)q_A - 16q_A$, and so differentiating with respect to q_A, $\frac{\partial \Pi_A}{\partial p_A} = 184 - q_A - 0.5q_B$; and we obtain a similar expression for Blacks: $\frac{\partial \Pi_B}{\partial p_B} = 184 - 0.5q_A - q_B$. Setting both derivatives to zero, it follows that $q_A = 184 - 0.5q_B$, and that ¾$q_B = 92$; so that we have a new equilibrium, $(q_A^*, q_B^*) = \left(\frac{368}{3}, \frac{368}{3}\right)$.

X17.12 **Confirm the following:**

a) *Aulds*' profit function is concave in its own output, q_A, and has a maximum when the value of q_B is held constant.
Writing profit $\Pi_A(q_A, q_B) = [a - c - b(q_A + q_B)]q_A$, then differentiating twice with respect to output q_A, we obtain $\frac{\partial^2 \Pi_A}{\partial {q_A}^2} = -2b < 0$. Since $\frac{\partial^2 \Pi_A}{\partial {q_A}^2} = < 0$, the profit function is concave. The first partial derivative, $\frac{\partial \Pi_A}{\partial q_A} = a - c - 2bq_A - bq_B = 0$ if $q_A = \frac{a - c - bq_B}{2b}$; and so where the first-order condition for a stationary value is satisfied, the function is concave. This is enough to confirm that the profit function has a maximum.

b) On the reaction function, $\frac{dq_B}{dq_A} = 0$. Explain what this means.

Where the partial derivative $\frac{\partial q_B}{\partial q_A} = 0$, the rate of change of Blacks' output required for Aulds' profit to remain constant, as Aulds increases its output, is zero. On the reaction function, we find the largest value of output, q_B, consistent with Aulds reaching any target profit level.

c) **In the area enclosed by the isoprofit curve, $\Pi_A = \Pi_A^0$, and the q_A-axis, the firm's profits are at least as great as Π_A^0, so that $\Pi_A \geq \Pi_A^0$.**
Writing profit $\Pi_A(q_A, q_B) = [a - c - b(q_A + q_B)]q_A$, then, holding output q_A constant, and reducing q_B, so that we move into the interior of the area bounded by any given isoprofit curve, since $\frac{\partial \Pi_A}{\partial q_B} < 0$, the firm's profits increases. It follows that within the area bounded by the isoprofit curve and the horizontal axis, Aulds makes profits that are at least as great as on the isoprofit curve.

X17.13 **Confirm that *Aulds* maximizes monopoly profits on the segment of the market left by *Blacks* by choosing output combinations that lie on the best-reply line $q_A^*(q_B) = \frac{a-c-bq_B}{2b}$. Discuss briefly how we have dealt with the constraint in this case.**
With Blacks producing output q_B, Aulds faces inverse demand $p(q_A; q_B) = a - bq_B - bq_A$. The average revenue function for Aulds takes the same form, and we obtain $MR(q_A; q_B) = a - bq_B - 2bq_A$. Applying the standard first-order condition that marginal revenue equals marginal cost, we obtain the output on the best reply line, $q_A^*(q_B) = \frac{a-c-bq_B}{2b}$. We treat the constraint as a fixed level of output here; we might consider it more generally as the minimum level of output acceptable to Blacks.

X17.14 **Write down the residual demand that faces *Blacks* when *Aulds* is committed to producing an amount q_A. Hence confirm that *Blacks* will maximize profits in the residual market segment by setting output $q_B^*(q_A) = \frac{a-c-bq_A}{2b}$.**
This is a very similar problem to X17.13. With Aulds producing output q_A, Blacks faces inverse demand $p(q_B; q_A) = a - bq_A - bq_B$. The average revenue function for Blacks takes the same form, and we obtain $MR(q_B; q_A) = a - bq_A - 2bq_B$. Applying the standard first-order condition that marginal revenue equals marginal cost, we obtain the output on the best reply line, $q_B^*(q_A) = \frac{a-c-bq_A}{2b}$.

X17.15 **What can we say about the residual profit maximizing outputs and the planned outputs if the market is in equilibrium? Calculate the output that the two firms will produce when maximizing profits.**
Both firms choose their outputs so that they maximize profits subject to the constraint that their competitor will set output that is no less than the equilibrium output. Both firms then choose planned outputs equal to the profit maximizing output when the other firm produces its planned output. The profit maximizing output for each firm is as found in previous, similar problems, since $2bq_A + bq_B = a - c$; and $bq_A + 2bq_B = a - c$. Then $q_A = \frac{a-c}{b} - 2q_B$; and $2b\left(\frac{a-c}{b} - 2q_B\right) + bq_B = a - c$. It follows at $q_A = q_B = \frac{a-c}{3b}$.

X17.16 **We consider a situation in which there are *n* firms in the market. We identify these firms as *f* = 1, 2, …, n. (Until now, we have assumed that *n* = 2.) We write the inverse demand function for the market as *p* = 150 – *Q*, where *p* is the market-clearing price, and the *n* firms' total output $Q = \sum_{f=1}^{n} q_f$. Each firm can produce any amount of output, q_f, at constant marginal cost, 30. We write each firm's total cost function as $C_f = 30q_f$.**

a) **Suppose that firms 2, 3, … , *n* have chosen their outputs. Write down an expression for firm 1's profits in terms of the output of all *n* firms.**

$\Pi_1 = \left(150 - \sum_{f=1}^{n} q_f\right) q_1 - 30q_1 = \left(120 - q_1 - \sum_{f=2}^{n} q_f\right) q_1$. *Note how we rewrite the total industry output in terms of firm 1's and all other firms' outputs. From the perspective of firm 1 all other firms' outputs are held constant.*

b) Partially differentiate firm 1's profit function with respect to output q_1.

Partially differentiating, $\frac{\partial \Pi_1}{\partial q_1} = 120 - 2q_1 - \sum_{f=2}^{n} q_f$

c) Setting this derivative to zero, obtain firm 1's best-reply function.

If $\frac{\partial \Pi_1}{\partial q_1} = 120 - 2q_1 - \sum_{f=2}^{n} q_f = 0$, *then rearranging,* $q_1 = \frac{1}{2}\left(120 - \sum_{f=2}^{n} q_f\right)$. *Note that for n = 2, this reduces to the form that we have already seen in previous problems.*

d) Discuss how firm 1's output changes as the total output of every other firm increases.
For every unit increase in the total output of other firms, firm 1 reduces its output by half a unit.

X17.17 **Write firm 1's best-reply function as $q_1^*(q_2, q_3, \ldots, q_n)$. Suppose that all firms except firm 1 have chosen to produce output, q_1^*. Rewrite the expression for firm 1's best reply in X17.13. Find firm 1's profit maximizing output when there are 2, 3, 4, 5 and 6 firms in the market. Discuss how the total output Q and the market price p change as firms enter the market. Show that consumer surplus therefore increases with more firms in the market.**
With $q_2 = q_3 = \ldots = q_n = q_1^*$, *then* $q_1^* = \frac{1}{2}\left[120 - (n-1)q_1^*\right]$, *so that for equilibrium,* $\frac{n+1}{2} q_1^* = 60$, *and* $q_1^* = \frac{120}{n+1}$. *Then with all firms setting output* q_1^*, *market price p =* $150 - 120\frac{n}{n+1} = \frac{30n+150}{n+1} = 30\left(\frac{n+5}{n+1}\right)$.

We see that industry output $Q^* = \frac{120n}{n+1} = 120\left(1 - \frac{1}{n+1}\right)$, *and so industry output increases, and market price falls as entry occurs. Representing the consumer surplus as the area of the triangle formed by the demand curve, the market price and the vertical axis in a diagram, we conclude that as more firms enter the market, the area of the triangle increases.*

X17.18 **What do you think the long-run equilibrium will be in this market? [*Hint:* Remember that to explain the rise of monopoly, we argued that entry must either be prevented or be unprofitable.]**
Firms continue to make profits in this market, so entry will continue without limit, but with firms taking an infinitesimally small market share.

X17.19 **The follower in this situation, *Blacks*, is able to choose its profit maximizing output after the leader, *Aulds*. Criticize the argument that this is preferable to being the leader, because it is possible to verify the leader's output and so choose the profit maximizing output.**
The leader anticipates how the follower will react to its own output choice. So, while the follower chooses the profit-maximizing choice given the leader's output choice, the leader chooses its profit maximizing choice, anticipating the response of the follower. We can think of the leader as being able to maximize its profits across these two separate steps.

X17.20 **Confirm that where the two firms set their outputs, q_A and q_B, simultaneously, $q_A = q_B = 40$. Calculate the market price and firm profits.**

We recognize this as the model that we have already analysed. Writing the profits for Aulds as $\Pi_A = (120 - q_A - q_B)q_A$, and $\Pi_B = (120 - q_A - q_B)q_B$, we obtain best reply functions (in implicit form $120 - 2q_A - q_B = 0$ and $120 - q_A - 2q_B = 0$. Then $q_A = 120 - 2q_B$ and $3q_B = 120$; confirming the result. Market price $p(40, 40) = 70$ and profit $\Pi_A = \Pi_B = 1{,}600$.

X17.21 **Calculate the profit maximizing output for *Blacks*, and hence the market-clearing price. Find the profits that each firm makes. Compare the outcome with the outcome of simultaneous quantity setting in Exercise X17.20.**
Blacks' best-reply function, $q_B(q_A) = 60 - 0.5q_A$; so with $q_A = 60$, $q_B = 30$ and the price $p(60, 30) = 60$. Aulds' profits $\Pi_A = (120 - q_A - q_B)q_A = 1{,}800$ and Blacks' profits $\Pi_B = (120 - q_A - q_B)q_B = 900$. We note that Aulds produces more, and makes more profit than with simultaneous quantity setting; but that Blacks produces less and makes less profit. Overall the total output of the firms is higher, so that the market price is lower, and total profits are less than with simultaneous quantity setting.

X17.22 **Sketch a diagram showing the firms' reaction functions, obtained in Exercise X17.20. Show the profit maximizing outputs from Exercises X17.20 and X17.21. What do you notice about *A*ulds' decision in X17.21?**
As before, in a diagram with the output of Aulds measured on the horizontal axis and the output of Blacks measured on the vertical axis, we draw in the reaction functions for the two firms as downward sloping lines. The reaction function for Aulds has gradient -2, whereas the reaction function for Blacks has gradient -0.5. The reaction function for Aulds meets the vertical axis at (0, 120) and the horizontal axis at (60, 0). The reaction function for Blacks meets the vertical axis at (0, 60) and the horizontal axis at (120, 0).
The profit maximizing outputs are at (60, 30). This is the same output for Aulds as if it were in a monopoly.

X17.23 **Show that in this situation with quantity leadership, *Aulds* chooses the same output that it would choose if it were to have a monopoly, but that it makes less profit than in a monopoly because *Blacks* does not quit the market.**
Given Blacks' reaction function, we are able to write Aulds' profit function in Expression 17.20a as $\Pi_A = ½(120 - q_A)q_A$. We note that were Aulds to have a monopoly, the expression for its profits would be exactly the same except for the factor ½. Since this is a constant factor in the expression, it does not affect the optimal output. So the leader produces the monopoly output, but achieves half of the monopoly profit.

X17.24 **Explain why *Aulds* cannot do worse in the sequential problem than in the simultaneous problem. [*Hint:* Remember that its objective is to maximize profits.]**
We can be certain that Blacks will choose its output so that the output pair will lie on its best-reply curve. With simultaneous quantity setting this is also true, so that if it were impossible to increase profits by using quantity leadership, Aulds would maintain its output.

X17.25 **Sketch a diagram showing the three segments of the firm's demand function. [*Hint:* You might find it useful to think about the relation between firm and market demand, as discussed in Chapter 12.]**
In a diagram with the firm's sales, q_A, on the horizontal axis and the price that it charges, p_A, on the vertical axis, then if price $p_A > p_B$, $q_A = 0$, and the demand curve runs along the vertical axis. If $p_A < p_B$., then the firm makes sales $q_A = \alpha - \beta p_A$. This is the equation of a straight line, which starts at the horizontal axis, passing through $(\alpha, 0)$ with gradient $\frac{dp}{dq} = -\frac{1}{\beta}$. The line segment approaches but does not reach the horizontal line, $p_A = p_B$.

There remains one point: when $p_A = p_B$, the demand curve is at the point $(\alpha - \beta p_B, p_B)$.

X17.26 **Using your diagram, confirm that: (a) after beginning with the situation where *Artful Roast* set a higher price than *Black Gold*, sales (and revenues) jump when p_A falls so that $p_A = p_B$; and (b) sales and revenues both jump again when p_A falls below p_B.**
We see from the diagram that there are two discontinuities in the demand curve. When a vanishingly small change in price leads to a large change in demand, revenue, as the product of demand and price, also increases substantially.

X17.27 **Confirm that *Artful Roast* cannot do better than choose p_A: $p_A > p_B$, if $p_B < c$. Briefly describe the outcome in this case.**
In this case, Artful Roast would make a loss by matching Black Gold's price. Setting a higher price than Black Gold, it avoids making a loss; but making no sales, generates no revenue. Breaking even is the best reply.

X17.28 **Define p_M as the price that *Artful Roast* would charge if it had a monopoly. By differentiating the expression for profit in Expression 17.23 and setting it to zero, find the profit maximizing price for the monopoly, and discuss when *Artful Roast* might choose this price.**
Define $\Pi_A = (p_A - c)(\alpha - \beta p_A)$; then differentiating with respect to p_A, $\frac{\partial \Pi_A}{\partial p_A} = \alpha - \beta p_A - \beta(p_A - c)$ *=* $\alpha + \beta c - 2\beta p_A$. *For the maximum of this function, we set the partial derivative to zero, so that* $p_A = \frac{\alpha + \beta c}{2\beta}$. *Artful Roast chooses this price when* $p_B = \frac{\alpha + \beta c}{2\beta}$

X17.29 **Suppose that *Black Gold* chooses price p_B: $c < p_B \leq p_M$. Explain why *Artful Roast* can do no better than just to undercut *Black Gold*. [*Hint:* Confirm that *Artful Roast*'s profits increase the closer it can set its price to p_B.]**
Black Gold has to choose a price $p_A \leq p_B \leq p_M$ in order to make sales. Matching price p_B, it shares the market. Setting price $p_A < p_B$, it obtains the whole market, but we know that $p_B \leq p_M$, so that the higher the value, p_A, the greater the firm's profits. Artful Roast sets out just to undercut Black Gold.

X17.30 **Suppose that *Black Gold* chooses price $p_B = c$. Confirm that *Artful Roast* will not try to undercut *Black Gold*, and that it is impossible for *Artful Roast* to make profits. Does it matter to *Artful Roast* whether or not it matches price p_B?**
With Black Gold setting price equal to marginal cost, Artful Roast makes losses by undercutting Black Gold. Matching price, Artful Roast obtains half the market, and makes no profit. Setting a higher price, it makes no sales, and so avoids both profit and loss.

X17.31 **Confirm each of the following statements:**

a) **There is no equilibrium outcome in which a café sets a price $p_f < c$.**
A café setting a price less than marginal cost makes losses. It can avoid these by setting a price higher than the other café's, making no sales, but but avoiding losses.

b) **There is no equilibrium outcome in which $p = \min(p_A, p_B) > c$.**
From X17.29, we know that in this case, the café setting the higher price can reduce the price that it sets so that it just undercuts its competitor, obtaining the whole market, and making positive profits.

c) **For an equilibrium, at least one café has to set a price $p_f = c$.**

If the lowest price set is the marginal cost, then neither café will want to set a lower price and incur losses; but once one sets price equal to marginal cost, the other will be indifferent between setting price equal to marginal cost or a higher price, since in either case, it will make zero profits.

d) **There is only one equilibrium outcome in which both firms make sales, and that is when both firms set price $p = c$, and so make zero profits.**
We have established that there is no equilibrium without one café setting price equal to marginal cost. But if its competitor sets a higher price, then the café setting price equal to marginal cost can increase its price, and make positive profits. So both cafés set price to marginal cost.

Chapter 18

X18.1 **What is *Aulds*' best reply to *Low*? What are *Blacks*' best replies to *High* and *Low*? What do we expect to happen?**
When Blacks chooses Low, Aulds receives a higher payoff from choosing High, so this is the best reply. For Blacks, the same argument applies. Whether Aulds chooses High or Low, Blacks makes higher profits by choosing High.

X18.2 **Suppose that *Aulds* and *Blacks* agreed to build small bakeries. What would happen to their profits? Why might such an outcome not occur?**
From the payoff table, we see that Aulds and Blacks increase their profits relative to the situation in which they choose to build large bakeries. We also note that both firms could do better still by agreeing to build a small bakery, but then building a large one. This suggests that they will have difficulty coordinating their behaviour in the way that we proposed here.

X18.3 **Describe the game fully, listing: (1) the players; (2) the action set of each player; (3) the set of action profiles (outcomes); and (4) each player's payoffs in terms of the outcomes.**
The players are Aulds and Blacks. Both players have the same pair of actions available, so that the action sets are $A_A = A_B = \{New, Old\}$. *Since each firm can choose one action, there are 2x2 = 4 action profiles:* $P = \{NN, NO, ON, OO\}$. *(Throughout, we adopt the convention that the first letter is for Aulds and the second for Blacks.) We write the payoffs for the action profiles,* $A = (a_A, a_B)$ *as the pairs* $[v_A(a_A, a_B), v_B(a_A, a_B)]$, *so that here they are* $[v_A(N, N), v_B(N, N)] = (50, 50)$; $[v_A(N, O), v_B(N, O)] = (120, 30)$; $[v_A(O, N), v_B(O, N)] = (30, 120)$; *and* $[v_A(O, O), v_B(O, O)] = (80, 80)$.

a) **Explain why *New* is a dominant strategy; and why the action profile (*New*, *New*) is the Nash equilibrium of the game.**
New is a dominant strategy because for both firms, the payoff to New is higher than the payoff to old, irrespective of what its competitor chooses to do. Since New is always both bakeries' best reply, (New, New) is the only Nash Equilibrium.

b) **Confirm that both bakeries would be better off in the action profile (*Old*, *Old*).**
Both bakeries obtain a payoff $v_F(O, O) = 80$. *This is higher than the payoff in equilibrium,* $v_F(N, N) = 50$.

X18.4 **Confirm that player *A*'s best reply to the conjecture *Right* is *Up*; and that player *B*'s best reply to the two conjectures *Up* and *Down* is *Left*.**
For conjecture, a_B^c = *Right, player A's payoffs are* $v_A(Up, Right) = 1$; *and* $v_A(Down, Right) = 0$. *So player A does better choosing Up, and this is the best reply.*
For conjecture, a_A^c = *Up, player B's payoffs are* $v_B(Up, Left) = 3$; *and* $v_B(Up, Right) = 2$. *So player B does better choosing Left, and this is the best reply.*
For conjecture, a_A^c = *Down, player B's payoffs are* $v_B(Down, Left) = 1$; *and* $v_B(Down, Right) = 0$. *So player B does better choosing Down, and this is the best reply.*

X18.5 **Confirm that (*Up*, *Left*) is the only action profile for which both players form consistent conjectures.**
Player A chooses Up, believing that B will choose Left. Player B chooses Left, believing that A will choose Up. If A believes B will choose Right, then A's beliefs will be proven wrong. If A's beliefs were to be right, B would do better to change action.

X18.6 **Show that neither player will wish to deviate from the profile (*Up*, *Left*).**
By definition, since Up is the best reply to Left, and Left is the best reply to Up, neither player wishes to deviate.

X18.7 **Confirm that in the example in Table 18.5, player *Y* will always choose *Right*, and that player *X* will then choose *Up*. Confirm that if player *Y* were to choose *Left*, then player *X* would choose *Down*.**
We see that for player Y, the payoff to choosing Right will be greater than the payoff to choosing Left, whether player X chooses Up or Down. Treating player Y's choice of Right as certain, player X obtains a higher payoff from choosing Up rather than Down.
If for some reason, player Y were to choose Left, then player X would receive a higher payoff from choosing Down rather than Up.

X18.8 **Confirm that the game shown in normal form in Table 18.3 is symmetric. Suppose that we reversed the order of the actions: would the game still be symmetric?**
We note that the action sets are identical: $A_A = A_B = \{N, O\}$. We also verify that payoff pairs satisfy the rule $v_A(a_1, a_2) = v_B(a_2, a_1)$. Since this property does not depend on the order in which actions are listed in the payoff table, the game will be symmetric if the order of actions is reversed.

X18.9 **Assume, as in Table 18.6, that players *X* and *Y* choose from the action set A = {*Left*, *Right*}, and that the game is symmetric.**

a) **What can you say about the payoffs in the top left and bottom right cells of the game?**
The payoffs to the players within each cell will be equal.

b) **What can you say about the payoffs in the bottom left and top right cells?**
The payoffs to the players within each cell will be the payoffs to the other player in the other cell.

X18.10 **Suppose that in the game, Coordination (1), for both players, the payoff to actions that form part of a Nash equilibrium is 2, and the payoff to the alternative action is 1. Complete the payoff table for Coordination (1), identify the best replies, and confirm that there are two Nash equilibria.**

Coordination (1)		*Player Y*	
		Left	*Right*
Player X	*Left*	*2, 2*	*1, 1*
	Right	*1, 1*	*2, 2*

Note that both players must receive a payoff of 2 when there is an equilibrium, and 1 whether there is not – and so from inspection, we see that where both believe the other will choose Left, and where both believe the other will choose Right, there will be consistent conjectures, with neither player wishing to change action. If they choose different actions, both would do better by changing their action.

X18.11 **Suppose that in a coordination (1) game with symmetric payoffs, players receive different payoffs in each action profile, say 3, 2, 1 and 0. Construct two different payoff tables in which the Nash equilibrium payoff pairs are (a) (3, 3) and (2, 2); and (b) (3, 3) and (1, 1).**

Coordination (1)		*Player Y*	
		Left	*Right*
Player X	*Left*	*2, 2*	*0, 1*
	Right	*1, 0*	*3, 3*

This is very straightforward; since the equilibrium payoffs are the two highest possible, it does not matter what the alternative payoffs are for given beliefs. For the second payoff table, though, players must have the payoff of zero to the alternative to the action giving an equilibrium payoff of 1.

Coordination (1)		Player Y	
		Left	Right
Player X	Left	1, 1	2, 0
	Right	0, 2	3, 3

X18.12 **Repeat X18.11, but for a coordination (2) game with symmetric payoffs, with payoffs to the action profiles 10, 7, 5, and 2. Construct two different payoff tables in which the Nash equilibrium payoff values are (a) 10 and 7; and (b) 10 and 5.**
This is almost the same problem as before. Note that we present the tables in a slightly different way, with the higher payoff associated with (Left, Left). For both players, choosing the same action as the other one is always the best outcome.

Coordination (1)		Player Y	
		Left	Right
Player X	Left	10, 10	5, 2
	Right	2, 5	7, 7

Coordination (1)		Player Y	
		Left	Right
Player X	Left	10, 10	2, 7
	Right	7, 2	5, 5

X18.13 **Confirm that in Table 18.7 there are two Nash equilbria: (*Lead*, *Follow*) and (*Follow*, *Lead*). Discuss how the Stackelberg approach might enable firms to avoid problems in such a situation.**

Quantity setting contest		Firm B	
		Lead	Follow
Firm A	Lead	(-100, -150)	(250, 100)
	Follow	(50, 200)	(150, 150)

Table 18.7: Competition in outputs

We see that firm B, if A chooses action a_A = Lead, v_B(Follow|Lead) = 100 > -150 = v_B(Lead|Lead). So a_B(Lead) = Follow. Applying the same type of argument, for firm B, a_B*(Lead) = Follow; and for firm A, a_A*(Lead) = Follow; and a_A*(Follow) = Lead.*

X18.14 **Suppose that firm *A* offers to make a payment to firm *B* of 50 after the game has been played, but conditional on firm *B* choosing *Follow*. Assuming that both firms make their decisions simultaneously, how might this help to resolve the coordination problem?**

Quantity setting contest		Firm B	
		Lead	Follow
Firm A	Lead	(-100, -150)	(200, 150)
	Follow	(50, 200)	(100, 200)

By inspection of the payoff table, we see that Firm A's offer means that v_B(Lead|Follow) = v_B(Follow|Follow) = 200; and this means that for firm B, Follow is a weakly dominant strategy. Firm B might then choose Follow (with certainty), so that firm A chooses Lead, and the Nash equilibrium (a_A, a_B*) = (Lead, Follow) emerges.*

X18.15 **Set out two payoff tables for this matching game, one showing the payments that Claudia and Dayna receive at the end of the game, and the other showing the payments net of the stake contributed. Confirm that in both payoff tables, there is no action profile that forms a Nash equilibrium. Using the concept of the best reply, explain why this should be.**

Matching game		*Dayna*	
		Left	*Right*
Claudia	*Left*	*200, 0*	*0, 200*
	Right	*0, 200*	*200, 0*

Total payments received

Matching game		*Dayna*	
		Left	*Right*
Claudia	*Left*	*100, -100*	*-100, 100*
	Right	*-100, 100*	*100, -100*

Net payments received

Since we apply exactly the same argument in both cases, this is simply the argument for the first version, with the total payments received.

The simplest statement of the argument is that Claudia wants to match Dayna's action, while Dayna wants to choose the alternative to Claudia's action. Suppose Dayna plans to choose Left; then Claudia will plan to choose Left. Alerted to this, Dayna will change her decision to Right, with Claudia also responding by changing her choice. This leads Dayna to return to choosing Left; and the cycle can continue indefinitely.

Formalizing the discussion, we begin by assuming that Dayna forms belief a_C^E *= Left, relating to Claudia's choice. Then Dayna expects to receive payoffs* v_D*(Left, Left) = 0 < 200 =* v_D*(Left, Right). So Dayna has a best reply* a_D**(Left) = Right.*

If Dayna forms belief a_C^E *= Right, she expects to receive payoffs* v_D*(Right, Left) = 200 > 0 =* v_D*(Right, Right). So Dayna has a best reply* a_D**(Right) = Left.*

In the same way, if Claudia forms belief a_D^E *= Left, she expects to receive payoffs* v_C*(Left, Left) = 200 > 0 =* v_C*(Right, Left). So Claudia has a best reply* a_C**(Left) = Left.*

Lastly, if Claudia forms belief a_D^E *= Right, she expects to receive payoffs* v_C*(Left, Right) = 0 < 200 =* v_C*(Right, Right). So Claudia has a best reply* a_C**(Right) = Right.*

We see that there is no pair of consistent conjectures and so we are unable to obtain an action profile that is a Nash equilibrium of the game.

X18.16 **Confirm that, for both players, *Confess* dominates *Silent*, and that the action profile (*Confess*, *Confess*) is the only Nash equilibrium.**

Prisoners' dilemma		*Prisoner B*	
		Confess	*Silent*
Prisoner A	*Confess*	*(-8, -6)*	*(-1, -9)*
	Silent	*(-12, -1)*	*(-2, -2)*

Table 18.9: Prisoners' dilemma

For prisoner A, forming belief a_B^E *= Confess, relating to prisoner B's choice, prisoner A expects to receive payoffs* v_A*(Confess, Confess) = -8 > -12 =* v_A*(Silent, Confess). So prisoner A has a best reply* a_A**(Confess) = Confess.*

To avoid repetition, we note that by this argument, we conclude that Confess is also prisoner A's best reply to B choosing the action, Silent; and that Confess is prisoner B's best reply to prisoner A's choice, irrespective of whether A chooses Silent or Confess.

Confess is therefore a dominant strategy, and the action pair (a_A**,* a_B**) = (Confess, Confess) is the only Nash equilibrium of this game.*

X18.17 **Were the prisoners able to talk to each other during a break in interrogation, how likely is it that this Nash equilibrium would arise? Would your answer change if the prisoners were then returned to separate interview rooms, and the offer of sentence reduction only made at that point?**

Simply being able to talk with each other seems unlikely to affect the outcome. A prisoner who intends to confess might easily lie to his accomplice at this point. We might expect the interrogating officer to present beliefs as facts, and to build up doubt over time, leaving the offer of sentence reduction as late as possible in the interview in order to make it seem credible when it is offered.

X18.18 **If the suspects were members of a criminal gang, how likely is it that the only concern of the suspects would be the length of time that they would spend in jail? [*Hint:* Might the reduction in time spent in jail from confessing be offset by sanctions imposed by the criminal organization?] Illustrate your answer by showing how the payoff matrix changes. Does *Confess* still dominate *Silent*?**

Prisoners' dilemma (with sanctions for confessing)		*Prisoner B*	
		Confess	*Silent*
Prisoner A	*Confess*	*(-28, -36)*	*(-61, -9)*
	Silent	*(-12, -51)*	*(-2, -2)*

We assume that the prisoners' associates have the ability to impose substantial sanctions on either, or both, should they confess. This changes the structure of the game, as in the payoff table from which we infer that Silent is a dominant strategy.

X18.19 **Suppose that the criminal enterprise has produced considerable income, which the partners have yet to divide up between them. Can you suggest any type of agreement that the suspects might have reached to ensure that they remain silent? Show how the payoff matrix changes under these circumstances, and explain how such changes might have similar effects on the equilibrium outcome as those found in Exercise X18.18.**

In these circumstances, we might assume that the partners in the crime have calculated that they are unlikely to be convicted of the major crime, and treat conviction of a minor offence as a necessary part of their activities. Anticipating the police behaviour, rather than sharing the proceeds of crime, they place it on deposit (somehow satisfying money laundering authorities that the funds are the proceeds of legal activity), in such a way that both partners need to be present when the funds are recovered. Incriminating their partner will then prevent access to the proceeds of crime, and this will be costly: if the cost is greater than that of spending a year in jail, then there will be a Nash equilibrium in which both remain silent (possibly as well as a Nash equilibrium in which both confess).

X18.20 **Suppose that the police did not have the evidence necessary to secure the minor conviction if both suspects chose *Silent*. How would this change the payoffs in Table 18.10? What difficulties would this cause the police?**

The police behaviour in this scenario is based on having the ability to make an offer to each prisoner separately that will undermine their agreement to cooperate and remain silent. In the absence of the evidence necessary to secure conviction on a minor offence, the payoff table will look something like this:

Prisoners' dilemma (with no evidence of minor crime)		*Prisoner B*	
		Confess	*Silent*
Prisoner A	*Confess*	*(-8, -6)*	*(0, -9)*
	Silent	*(-12, 0)*	*(0, 0)*

There are now two Nash equilibria, with Confess only being an optimal strategy where a prisoner believes that the other prisoner has chosen to confess. This weakens the strength of *the offer that the police can make very considerably, especially given the preceding arguments about other costs and rewards.*

X18.21 **Should the police be allowed to reward a suspect for providing information that leads to the conviction of an accomplice, where there is no other evidence that is admissible in court against either the suspect or the accomplice? Discuss this in the context of your answer to Exercise X18.20.**
Given the evidence presented here, it seems understandable that the police should find the ability to make payments or offer other rewards for evidence leading to conviction of an accomplice. We should note, though, that in many countries, criminal courts will require the disclosure of such agreements, and that the fact of payment may lead the court to question the reliability of the evidence provided.

X18.22 **Should the police be allowed to tell each suspect (separately) that it is likely that the other will confess?**
This is essential to the emergence of the dilemma. We assume that the police have good reasons to believe that the prisoners have committed a crime together, but they lack evidence in a form that would be admissible in court, and so do not believe that it will be possible to conclude the investigation without obtaining a confession from one or more prisoners. They therefore present the information that they do possess in such a way as to maximize the probability of obtaining a confession.

X18.23 **Consider a symmetric game in which both players, *X* and *Y*, choose between the actions *Left* and *Right*. If both choose *Left*, both obtain payoff *a*. If both choose *Right*, both obtain payoff *d*. If one chooses *Left* and the other chooses *Right*, the player choosing *Left* obtains payoff *b* while the player choosing *Right* obtains payoff *c*.**

a) **Sketch the payoff table in this case.**

Prisoners' dilemma (general case)		*Player Y*	
		Left	*Right*
Player X	*Left*	*(a, a)*	*(b, c)*
	Right	*(c, b)*	*(d, d)*

b) **Write down conditions that must hold in order for *Left* to be a dominant strategy.**
For Left to be a dominant strategy, it is a best reply whichever action the other player chooses. So, if player Y chooses Left, player X's best reply is Left if $a > c$; and if player Y chooses right, player X's best reply is Left if $b > d$. (It is easy to verify that the same conditions hold for Left to be a dominant strategy for player Y.)

c) **Write down a further condition that must hold for the dominant strategy to have the characteristics of a Prisoners' Dilemma.**
Left is the dominant strategy, and so the Nash equilibrium action profile (a_X^*, a_Y^*) = (Left, Left); but there is a prisoner's dilemma if both prefer (a_X, a_Y) = (Right, Right). This will emerge if $d > a$. We therefore have a ranking of the values in the payoff matrix: $b > d > a > c$.

X18.24 **Using the payoff matrix:**

a) **If *You* and your *Partner* agree that cooperation is better than defection, what can we conclude about the value of *b* relative to the value of *c*?**
Here, $b > c$.

b) **If *You* believe that your *Partner* will cooperate, but nonetheless you decide to defect, what can we conclude about the value of *b* relative to *a*?**

$b < a$.

c) **What do you expect your *Partner* to do, even when believing that *You* will cooperate?**
When you decide to cooperate, you expect your partner to defect.

d) **If your *Partner* believes that *You* will defect, and *also* defects, what can we conclude about the value of *c* relative to the value of *d*?**
$c > d$

e) **What do you expect *You* to do, believing that your *Partner* will defect?**
Your best reply is to choose Defect.

f) **Summarize the conditions required in this case for *Defect* to be a dominant strategy. Compare your answer with the conclusions of X18.23.**
If Defect is dominant, then $a > b$ and $c > d$. This ensures that the payoff to Defect is always greater than the payoff to Cooperate and is effectively the same condition as in X18.23.

X18.25 **Suppose that the payoff values $a > b > d > c$.**

a) **Confirm that the students will agree to cooperate.**
In advance of starting the assignment, cooperating is better than defecting and effectively abandoning the assignment.

b) **Confirm that if *You* believe that your *Partner* will defect, *You* will cooperate.**
Given that you believe that your Partner will defect, you can obtain payoff v_Y(Cooperate,Defect) = $d > c$ = v_Y(Defect, Defect). So Cooperate is the best reply to Defect.

c) **What will your *Partner* do, believing that *You* will defect?**
Applying the argument in part b), Cooperate is again a best reply to Defect.

d) **Show that the two action profiles in which one student cooperates and the other defects are both Nash equilibria.**
Given that we have shown that for both players Cooperate is the best reply to the other player choosing Defect, it is only necessary to demonstrate that v_P(Defect, Cooperate) > v_P(Cooperate, Cooperate), and this is true by assumption that $a > b$.

e) **Compare the relationship between the payoff values given here and those found for the Prisoners' Dilemma in X18.23 and X18.24. Explain how they differ.**
The relationships are very similar, except that we have changed the order of d and c.

X18.26 **Suppose that you find yourself in the situation described in the coordination game described in X18.25. How might you persuade your partner to choose *Cooperate*? [*Hint:* You may wish to encourage your partner to form beliefs about you that do not reflect your intentions!]**
If your partner forms the belief that you will defect, then your partner's best reply is to cooperate. To ensure cooperation, it is simply necessary to appear uncooperative. It seems unlikely that this is a strategy that could work well.

X18.27 **The situation described here might seem entirely artificial. Nonetheless, students often have to work together with colleagues, and there will be occasions in which cooperation has to be based on trust because students have to work separately. What actions might students take to make an agreed, cooperative, outcome more likely?**

We have assumed that all work will be completed separately; by sharing working files, it would be possible for partners to comment on progress and to verify that other members of the group are doing the work expected of them, encouraging cooperation to emerge, and possibly to last until the end of the process.

X18.28 **Suppose that both hunters value a share of the stag at 3 and a hare at 1, while returning home empty-handed has value 0.**

a) **Defining the actions available to the hunters as *Stag* and *Hare*, set out the game in normal form.**

Prisoners' dilemma (general case)		*Player H*	
		Stag	*Hare*
Player G	*Stag*	*(3, 3)*	*(0, 1)*
	Hare	*(1, 0)*	*(1, 1)*

b) **Identify the best reply to *Stag* and to *Hare*.**
Choosing Stag when the other hunter chooses Stag, both receive a payoff $v_G(S, S) = 3 = v_H(S, S)$. Choosing Hare, $v_G(H, S) = 1 = v_H(S, H)$. So Stag is the best reply to Stag.
Choosing Stag when the other hunter chooses Hare, both receive a payoff $v_G(S, H) = 0 = v_H(H, S)$. Choosing Hare, $v_G(H, H) = 1 = v_H(H, H)$. So Hare is the best reply to Hare.

c) **Identify action profiles that form pairs of consistent conjectures.**
We see that both hunters wish to choose the same action as their partners. (H, H) and (S, S) are both consistent conjectures, with neither player wanting to change choice when the other player's action is revealed.

d) **Discuss what you consider to be the most likely outcome in this case.**
We cannot really describe the outcome of the game with confidence. It is certainly true that when both hunters choose Stag, they obtain a higher payoff from cooperation than from defection. But it is also the case that when both choose Hare, the minimum payoff value is higher than if they were to choose Stag (taking the risk that the other will choose Hare). This maximin argument may cause both of them to choose Hare.

X18.29 **We assume that the hunters live in the same (small) village.**

a) **If one player hunts a hare, how might the other player impose social costs that reduce the benefits from deviation?**
In these circumstances, it is unlikely that either hunter will want to obtain a reputation for failing to cooperate, and this, together with a withdrawal of cooperation in other ways, might be sufficient to change the payoffs so that cooperation is more likely to emerge.

b) **Suppose that there is a kinship relation between the hunters. How might this affect their payoffs? Do you think that this would make cooperation more likely?**
We have set up the game so that the hunters do not know each other prior to going hunting. With a kinship relationship, we expect the hunters to have met each other many times, to have experienced opportunities for cooperation, and for there to be an expectation of cooperation.

c) **Suppose that each hunter has a sister, married to the other hunter. How might this affect the payoff to breaching cooperation?**
This is as close a kinship relationship as it is possible to imagine. Failure of either hunter to cooperate is likely to have relatively high costs for them; and it is possible that there would be

normally be sharing of the spoils of the hunt with the food prepared for a meal for the whole extended family, further reducing the benefits of defection from the plan.

X18.30 **If players believe that the other players in a game will behave in a certain way because that maximizes their payoff, is it reasonable to claim that they trust one another?**
This does seem to be a form of trust, but a very weak form based on rational calculation. Trust more generally might be expected to be based on reputation, acquired from a public history of choices made in similar situations.

Chapter 19

X19.1 **Given inverse market demand, $p = p(Q)$, where $Q = q_A + q_B$, and firm costs $c_F = c(q_F)$, where $F = A, B$, write down expressions for each firm's profits, and hence obtain their reaction functions in implicit form. (Remember to allow for the possibility that $q_G > q_G^0$.) Given that the inverse demand decreases as output q_B increases, demonstrate that so long as its marginal costs are not decreasing its output, the best reply $q_A^*(q_B)$ will be decreasing in q_B.**

Firm profits: $\Pi_A(q_A, q_B) = p(q_A + q_B)q_A - c(q_A)$; $\Pi_B(q_A, q_B) = p(q_A + q_B)q_B - c(q_B)$.
To obtain reaction function, we differentiate a firm's profit function (partially) with respect to its own output, setting this to zero. We also note that if the competitor is producing output greater than q_G^0, then the firm will set output $q_F^*(q_G^0) = 0$.
So reaction functions $\frac{\partial \Pi A}{\partial q_A} = \frac{\partial p}{\partial q_A} q_A + p(q_A + q_B) - \frac{\partial c}{\partial q_A} = 0$ *and* $\frac{\partial \Pi_B}{\partial q_B} = \frac{\partial p}{\partial q_B} q_B + p(q_A + q_B) - \frac{\partial c}{\partial q_B} = 0$.
We are interested in the values $q_A(q_B)$ and $q_B(q_A)$, which solve these expressions. The reaction functions take the form q_A^*: $q_A^*(q_B) = \max[q_A(q_B), 0]$, and q_B^*: $q_B^*(q_A) = \max[q_B(q_A), 0]$.

X19.2 **Confirm that a firm, F, will be able to make a profit if $p_F = \min(p_A, p_B) > c$.**
At this price, the firm undercuts its competitor, but is setting a price higher than marginal cost. It obtains the whole market, but is able to make a profit on each sale.

X19.3 **Suppose that there is some price p^M that firm F would set to maximize its profits if it had a monopoly. Describe firm F's best replies in each of the following situations:**

a) **Its competitor sets a price $p_G > p^M$.**
The best reply, $p_F^*(p_G) = p^M$. The firm sets price to achieve monopoly profits.

b) **Its competitor sets a price p_G: $c < p_G \leq p^M$.**
We cannot really define the best reply. We already know that the firm wishes to undercut its competitor by the smallest amount possible, but we cannot define such a price, without the further assumption that there is a smallest price interval.

c) **Its competitor sets a price $p_G = c$.**
The firm will be indifferent between all prices p_F: $p_F \geq c$, since it will then either share the market (setting $p_F = c$) and make no profit, or else make no sales, incur no costs, and make no profit.

d) **Its competitor sets a price $p_G < c$.**
$p_F^*(p_G) > p_G$. The firm avoids making sales and losses.

X19.4 **Sketch a graph of the reaction function for firm B.**
In a diagram with the price that firm A sets, p_A, measured on the horizontal axis, and the price that firm B sets measured on the vertical axis, we first draw in the line $p_A = p_B$, the line of gradient 1 passing through the origin.
In the interval in which $p_A < c$, firm B's best replies lie in the trapezium-like area bounded by the vertical axis, the line $p_A = p_B$ (although it does not include this line) and the vertical line $p_A = c$.
At $p_A = c$, the best reply starts from the point where $p_A = p_B$ and is the whole of the vertical line $p_A = c$ above that point.
In the interval in which $c < p_A \leq p^M$, we cannot really draw the best reply, since it is a price just below the price p_A.
Lastly, for $p_A > p^M$, the best reply is the horizontal line $p_B = p^M$, but defined only for values of p_A: $p_A > p^M$.

X19.5 **Add to your graph the graph of the reaction function for firm *A*, depicted in Figure 19.2. [*Note:* It may be useful to use different colours for the two reaction functions.]**
The reaction function for firm A is the mirror image of the reaction function for firm B, reflected in the line $p_A = p_B$.

X19.6 **Confirm that there cannot be a Nash equilibrium in regions I, III and IV. Confirm that there is a Nash equilibrium in region II, where $p_A = p_B = c$.**
For there to be a Nash equilibrium, in our diagram we require consistent best replies – in effect we are looking for action profiles of the game that lie in both best reply sets.
In region I, we know that firm A's reaction function lies below the line $p_A = p_B$, while firm B's reaction function lies above the line. The line $p_A = p_B$ lies between them, so there are no Nash equilibria in this region.
In region III, we know that best replies are not well-defined, so that there cannot be any intersection of them, and there are no Nash equilibria.
In region IV, we know that firm A's reaction function lies above the line $p_A = p_B$, while firm B's reaction function lies below the line. The line $p_A = p_B$ lies between the graphs of the best replies, so there are no Nash equilibria in this region.
This leaves us with region II. The two line segments have a single point of intersection, $p_A = p_B = c$, so that $(p_A, p_B) = (c, c)$ is the only price pair supporting consistent conjectures, and so the only Nash equilibrium.

X19.7 **What would you expect to happen in the market if the firms both faced constant marginal cost, c_F, lower for firm *A* than for firm *B*, so that $c_A < c_B$? Show that in the equilibrium, firm *A* will set a price so that it is able to obtain the whole market and still make profits.**
Firm A is now able to undercut firm B, acquire the whole market, and make profits, where firm B will set a price $p_B = c_B$.

X19.8 **Confirm that firm *B*'s strategy is to choose the opposite action from firm *A*, so that if firm *A* has chosen *High*, firm *B* will choose *Low*; but that if firm *A* has chosen *Low*, firm *B* will choose *High*.**
If firm B faces the situation where firm A has chosen High, then firm B can generate profits π_B(High, High) = -15, but profits π_B(High, Low) = 10. So Low is firm B's best reply to High.
If instead, firm A has chosen Low, then firm B can generate profits π_B(Low, High) = 20, but profits π_B(Low, Low) = 15. So High is firm B's best reply to Low.

X19.9 **Given firm *B*'s strategy in X19.8, find firm *A*'s strategy, and state the action profile that emerges in the equilibrium of this game.**
Firm A is able to predict firm B's strategy when making its choice. It therefore expects to receive a payoff of 25 to High and a payoff of 5 to Low. Its strategy, a_A = High. We expect to observe the action profile (a_A, a_B) = (High, Low).

X19.10 **Consider the example used in Section 17.2, where firms *A* and *B* face inverse market demand $p = 150 - q_A - q_B$ and marginal cost $c = 30$.**

a) **Assume that firm *A* has chosen output q_A. Write down an expression for the profit that firm *B* makes, and calculate its profit maximizing choice as a function of the output of firm *A*, q_A.**
Firm B's profit $\Pi_B = (120 - q_A - q_B)q_B$. To maximize profit, it set output q_B so that $\frac{\partial \Pi_B}{\partial pq_B} = 120 - q_A - 2q_B = 0$, which means that $q_B = 60 - 0.5q_A$.

b) Assume that firm *A* is able to predict how firm *B* will respond to its choice. Obtain an expression for the profit of firm *A* that does not involve firm *B*'s output. Hence, calculate firm *A*'s profit maximizing choice.

We can write firm A's profits as $\Pi_A = (120 - q_A - q_B)q_A$. With $q_B = 60 - 0.5q_A$, we can substitute for q_B in the expression for Π_A, obtaining $\Pi_A = ½(120 - q_A)q_A$. Differentiating this expression with respect to q_A, and setting the derivative to zero, we obtain $\frac{d\Pi_A}{dq_A} = 60 - q_A = 0$, so that we obtain the action profile in which both firms maximize their profits (q_A, q_B*) = (60, 30).*

X19.11 **Consider a more general form of the model used in X19.10. Firm *A* is the leader, and firm *B* the follower. Firms compete in quantities, with firm *F* producing output q_F, with inverse market demand, $p = a - b(q_A + q_B)$. There is a constant marginal cost, *c*.**

a) Write down expressions for each firm's profits.

Writing profit as the difference between revenue and cost, we obtain for firm A, profit Π_A: $\Pi_A(q_A, q_B) = p(q_A, q_B)q_A - cq_A = [a - c - b(q_A + q_B)]q_A$. Similarly, $\Pi_B(q_A, q_B) = [a - c - b(q_A + q_B)]q_B$.

b) Express the profit-maximizing output of firm *B* as a function of q_A, the output of firm *A*.

We differentiate the profit function (partially) with respect to firm B's output, obtaining $\frac{\partial \Pi_B}{\partial pq_B} = a - c - bq_A - 2bq_B$. Setting the partial derivative to zero for the profit maximum, we obtain $q_B^*(q_A) = \frac{a-c-bq_A}{2b}$.

c) Similarly, express the profit-maximizing output of firm *A*, q_A*, in terms of the parameters in this model, given your answer in part (b).

Anticipating firm B's response to its choice, firm A acts as if it considers its profit function to have the form $\Pi_A(q_A, q_B) = ½[a - c - bq_A]q_A$, taking advantage of its ability to influence firm B's output. It maximizes profit by setting output q_A: $\frac{d\Pi_A}{dq_A} = \frac{1}{2}(a - c - 2bq_A)$. Setting the derivative to zero, we obtain the profit maximizing output $q_A^* = \frac{a-c}{2b}$.*

d) Hence calculate the profit-maximizing output, q_B* and the firms' equilibrium profits.

Substituting firm A's output into firm B's best reply function, $q_B^*(q_A^*) = \frac{a-c}{4b}$.

e) Sketch a game tree similar to Figure 19.4 for this model, showing clearly the sub-game perfect equilibrium.

We show the action set for firm A as a triangle, representing a continuum of output choices, $q_A : 0 \le q_A \le \frac{a-c}{b}$. We also show the equilibrium choice, $q_A^* = \frac{a-c}{2b}$ as the midpoint of the choice set. We then show a second triangle, with its vertex at the end of the line showing firm A's output, representing the continuum of output choices, $q_B : 0 \le q_A \le \frac{a-c}{2b}$. Again, the equilibrium output is a line drawn from the vertex to the midpoint of the base of the triangle. We label the line $q_B^*(q_A) = \frac{a-c-bq_A}{2b}$, in order to emphasize that in the sub-game perfect equilibrium, firm B's strategy is defined for every possible level of output, q_A, and not just for firm A's profit maximizing output.

X19.12 **Sketch a diagram based on Figure 19.6 showing best-reply functions for two firms, and an agreed action profile for a cartel. Show the outcomes where: (a) the cartel is maintained; (b) firm *A* deviates; (c) firm *B* deviates; and (d) both firms deviate. Discuss the level of each firm's profits in these outcomes, compared with the competitive Nash equilbrium.**

There is a slight problem in completing this particular question – how much the firms should produce if they believe that their competitor will deviate. The first three parts of the question are straightforward.

Copying Figure 19.6, we have already shown the agreed cartel outputs ($q_A{}^M$, $q_B{}^M$) as point F on the diagram, and at point G ($q_A*[q_B{}^M]$, $q_B{}^M$), which lies on firm A's reaction function, we show the optimal output for firm A, if it believes that firm B will keep to the agreement. The analogous point – where firm B believes that firm A will keep to the agreement – lies on firm B's best reply curve, at point H ($q_A{}^M$, $q_B*[q_A{}^M]$), above point F.

It may seem obvious that if both firm's decide to defect, then they should production the output J ($q_A*[q_B{}^M]$, $q_B*[q_A{}^M]$), but the flaw in this claim is that this outcome will only arise if both firms are willing to defect, but believe that the other firm will not defect. It is perhaps more reasonable to argue that where each firm believes that the other firm will defect, it should produce at the point on its best-reply function where its output is the best reply to the defector's output. But if we allow both firms to anticipate defection by their supposed cartel partner, then we should allow for the same sort of argument that we used to argue for the emergence of the Nash equilibrium in the Cournot model, based on the requirement that conjectures must be consistent. This suggests that in the event of the cartel collapsing, output will revert to the Cournot-Nash equilibrium, point E in Figure 19.6.

X19.13 **Demonstrate that there is a single weakly dominant equilibrium in the two-player, two-action game illustrated in Table 19.2.**

Cartel		*Firm B*	
		Defect	*Cooperate*
Firm A	*Defect*	*(1, 2)*	*(5, 2)*
	Cooperate	*(1, 5)*	*(4, 4)*

Table 19.2 The cartel problem

From inspection of entries in the table, we see that when firm B chooses to defect, firm A is indifferent between defection and cooperation. Where firm B chooses to cooperate, firm A prefers to defect. The same is true for firm B: it is indifferent between cooperation and defection where firm A chooses to defect, but prefers to defect when firm A decides to cooperate. Defect is therefore the weakly dominant for both firms, and so (Defect, Defect) emerges as the only Nash equilibrium.

X19.14 **Explain why the situation is similar, but not identical, to the Prisoners' Dilemma discussed in Section 18.2.**

As in the Prisoners' Dilemma, there is a single Nash equilibrium, but both partners in the cartel would like to reach a non-equilibrium outcome. It is different from the pure prisoners' dilemma because of the property of weak dominance that is used to identify the equilibrium.

X19.15 **Consider the following argument. Both firms understand the decisions that the other firm is making. Firm *B* knows that firm *A*'s best reply is to increase output.**

a) **On your diagram, show firm *B*'s best reply to firm *A*'s best reply, $q_A(q_B{}^M)$.**
b) **Using a similar argument, show how firm *A* might anticipate *B*'s choice by changing its output choice, so that its output still lies on its best-reply line.**
c) **On your diagram, show that the agreement will collapse, with firms producing at the Cournot-Nash equilibrium.**

This is the argument developed in X19.13. We sketch in our diagram successive adjustments to best replies as conjectures change, showing that allowing firms to change beliefs means that they will converge on the Cournot-Nash outcome.

X19.16 **Suppose that there is no cartel agreement. Obtain the Cournot-Nash equilibrium, calculating the profits that the firms make.**

We write the profits for firm A, $\Pi_A(q_A, q_B) = (240 - q_A - q_B)q_A$, and so partially differentiating profit with respect to output, $\frac{\partial \Pi_A}{\partial p_A} = 240 - 2q_A - q_B$. Similarly, we obtain partial derivative $\frac{\partial \Pi_B}{\partial p_B} = 240 - q_A - 2q_B$. For the Cournot-Nash equilibrium, we set these partial derivatives to zero, and solve for the action profile (q_A^*, q_B^*). Then $q_A^* = 240 - 2q_B^*$, and $3q_B^* = 240$; we find that $(q_A^*, q_B^*) = (80, 80)$. Firms make profits $\Pi_A^* = \Pi_B^* = 6{,}400$.

X19.17 **With the cartel agreement in place, calculate the profit maximizing output, assuming that the firms produce equal quantities; and the increase in their profits compared with the situation where there is no cartel.**

We assume that both firms produce the same output, so $q_A = q_B = 60$. The joint profit is then $\Pi = 14, 400$, so that each firm makes profits $\Pi_A(60, 60) = \Pi_B(60, 60) = 7{,}200$.

X19.18 **Suppose that there is a cartel, but that the agreement specifies that the two firms produce different quantities, and that there is no profit sharing after sales have been made. Assume that $q_A < q_B$. Calculate the minimum value of q_A consistent with the firm entering into a cartel agreement.**

Without the cartel agreement, both firms can make a profit $\Pi_F^* = 6{,}400$. With total output Q = 120, firm A makes profit $\Pi_A = 120q_A$. So firm A will only make profit $\Pi_A > 6{,}400$ if $q_A > \frac{160}{3}$.

X19.19 **Confirm that in Table 19.3 there is no Nash equilibrium in which players' strategies are defined by the actions available to them.**

Heads or Tails		Player B	
		Heads	Tails
Player A	Heads	(1, -1)	(-1, 1)
	Tails	(-1, 1)	(1, -1)

We see from the payoff table that player A's best reply to B (choosing either action) is to match B's action; while player B's best reply is to choose the alternative to A's action. For each outcome, either player A or player B will want to change action, so that there cannot be a Nash equilibrium with strategies defined in terms of actions.

X19.20 **Explain why it would be very unusual for this game to have an extensive form in which player *B* could observe player *A*'s coin before simply choosing between *Heads* and *Tails*.**

This gives player B an important informational advantage; player B cannot lose, so player A would be unwilling to enter into the game.

X19.21 **Which concept of probability would you use in (a) matching pennies (Table 19.3) and (b) the matching game (Table 16.8)?**

In matching pennies, we can use an objective definition of probability, on the basis of the theory that a toss of a coin is a random event with two outcomes. For the matching game seen earlier, we use a subjective definition of probability, since this depends on participants randomizing.

X19.22 **Suppose that there are *n* outcomes to a trial. Defining the probability of an outcome as being the relative frequency with which it occurs, explain why the sum of probabilities must equal one. Explain also why, if the outcome of a trial is truly random, then, when there are *n* outcomes, the probability of any single outcome $Pr_i = \frac{1}{n}$.**

An event with a probability of one is certain to occur. In any trial, one event must occur, so it is certain that there will be an event, and the sum of probabilities of all events must be one. Our definition of an outcome is such that all outcomes are equally probable, so that with n outcomes, $n.Pr_i = 1$.

X19.23 **Suppose that there 12 outcomes to a trial. What is the probability of outcomes 3, 5, 7, and 11 occurring? What is the probability of an even-numbered outcome?**
With 12 outcomes in set X, four of which are the numbers, 3, 5, 7 and 11, the probability $Pr(X = 3, 5, 7, 11) = \frac{1}{3}$. Outcomes X = 2, 4, 6, 8, 10 and 12 are even numbered, so $Pr(X = 2, 4, 6, 8, 10, 12) = \frac{6}{12} = 0.5$.

X19.24 **Calculate player *A*'s expected payoff to (a) *Heads* and (b) *Tails*. What do you conclude about player *A*'s preference between *Heads* and *Tails*, given player *B*'s strategy? How will this change once player *B*'s choice is confirmed?**
Expected payoff for player A from choosing Heads, $E[v_A(H)] = 0.5*1 – 0.5*1 = 0$. Expected payoff for player A from choosing Tails, $E[v_A(T)] = 0.5*1 – 0.5*1 = 0$.
Once player A knows player B's choice, player A will wish either to have chosen Head or Tails in order to have obtained the payoff of one by matching B's choice.

X19.25 **What do you conclude about player *B*'s preferences between *Heads* and *Tails*?**
B must be completely indifferent between playing Heads and Tails.

X19.26 **Calculate the expected payoffs for players *A* and *B* from the strategy of choosing each action with probability *Pr* = ½, given that the other player is choosing the same strategy.**
We have already calculated the expected payoff to Heads and the expected payoff to Tails for player A. Suppose that A now plays Heads with probability $\pi_A = 0.5$ (and so Tails with probability $(1 - \pi_A) = 0.5$. Then $E[v_A(\pi)] = 0.5 E[v_A(H)] + 0.5 E[v_A(T)] = 0$. We might apply the same argument to player B. Neither player expects to win, or to lose, by participating in the game.

X19.27 **Confirm that in choosing *Right*, Dayna has expected payoff $E[V_D(R)] = (2\pi_C – 1).c$.**
Claudia plays Left with probability π_C and Right with probability $1 - \pi_C$. The expected payoff to Dayna for Right is then $E[V_D(R)] = \pi_C*1 + (1 - \pi_C)*(-1) = 2\pi_C – 1$.

X19.28 **Show that:**

a) **if $\pi_C > ½$, $E[V_D(R)] > 0 > E[V_D(L)]$;**
b) **if $\pi_C < ½$, $E[V_D(R)] < 0 < E[V_D(L)]$; and**
c) **if $\pi_C = ½$, $E[V_D(R)] = E[V_D(L)] = 0$.**

Calculating the expected payoff to Dayna of Left, Dayna receive payoff $V_D(Left, Left) = -1$ with probability π_C and payoff $V_D(Right, Left) = 1$ with probability $1 – \pi_C$. So the expected payoff, $E[V_D(L)] = \pi_C*(-1) + (1 - \pi_C)*(1) = 1 – 2\pi_C$.
The results follow immediately: $1 - 2\pi_C > 0$ if $\pi_C < ½$, and $2\pi_C – 1 > 0$ if $\pi_C > ½$. And when π_C = ½,
$1 - 2\pi_C = 2\pi_C – 1 = 0$.

X19.29 **Using the expected payoffs to the actions *Left* and *Right*, write down an expression for Claudia's expected payoff when Dayna has decided to follow the mixed strategy, π_D. By partial differentiation with respect to π_C, or otherwise, show that Claudia's expected payoff will be increasing (in π_C) if $\pi_D < ½$, decreasing if $\pi_D > ½$, and constant if $\pi_D = ½$. Interpret these results.**

Applying the argument above, since V_C(Left, Left) = V_C(Right, Right) = 1; and V_C(Left, Right) = V_C(Right, Left) = -1, when Claudia chooses Left with probability π_C and Dayna chooses Left with probability π_D, then the probability distribution across outcomes may be written Pr(Left, Left) = $\pi_C\pi_D$; Pr(Left, Right) = $\pi_C(1 - \pi_D)$; Pr(Right, Left) = $(1 - \pi_C)\pi_D$; Pr(Right, Right) = $(1 - \pi_C)(1 - \pi_D)$.
So, $E[V_C(\pi_C, \pi_D)] = \pi_C\pi_D*(1) + \pi_C(1 - \pi_D)*(-1) + (1 - \pi_C)\pi_D*(-1) + (1 - \pi_C)(1 - \pi_D)*(1) = \pi_C(2\pi_D - 1)$ +
$(1 - \pi_C)(1 - 2\pi_D) = (2\pi_C - 1)(2\pi_D - 1)$.
Partially differentiating with respect to π_C, $\frac{\partial E[V_C]}{\partial \pi_C} = 2(2\pi_D - 1)$. The conclusion follows immediately. Where Dayna is more likely to choose Right than Left, $\pi_D < ½$, and Claudia's expected payoff is decreasing in π_C. Claudia should choose the lowest possible value, $\pi_C = 0$; that is, she should choose Right, with certainty.
Where Dayna is more likely to choose Left than Right, $\pi_D > ½$, and Claudia's expected payoff is increasing in π_C. Claudia should choose the highest possible value, $\pi_C = 1$; that is, she should choose Left, with certainty.
Where Dayna is equally likely to choose Right and Left, $\pi_D = ½$, and Claudia's expected payoff is constant. Claudia is then indifferent between all possible mixed strategies, since they generate the same expected payoff.

X19.30 **Complete the analysis of the game, showing that there can only be consistent conjectures in this game, with neither player wishing to change their choice, at the Nash equilibrium in mixed strategies: (π_C*, π_D*) = (½, ½).**
If Dayna deviates from the proposed equilibrium, Claudia will choose either Left or Right with certainty. By the symmetry of the situation, we expect to find the same effect if Claudia deviates from the proposed equilibrium: Dayna will choose either Left or Right with certainty.

X19.31 **Confirm that if either Claudia or Dayna deviates from this equilibrium, then the other player's best reply will offer a higher expected payoff than in the equilibrium found in X19.30. Explain this result.**
In the equilibrium, (π_C, π_D*) = (½, ½), $E[V_C] = E[V_D] = 0$. Suppose that Dayna chooses some other best reply (say $\pi_D = 0.6$, so that she is a little more likely to choose Left than Right). Then Claudia will always choose Left, and her expected value, $E[V_C(1, 0.6)] = 0.6*1 + 0.4*(-1) = 0.2 > 0$.*

X19.32 **Confirm that (*Stag*, *Stag*) and (*Hare*, *Hare*) are Nash equilibria in pure strategies.**

Stag Hunt		*Player B*	
		Stag	*Hare*
Player A	*Stag*	*(5, 5)*	*(0, 2)*
	Hare	*(2, 0)*	*(2, 2)*

Since this is a symmetric game, we consider only the behaviour of player A. Believing that B will choose Stag, A's best reply is Stag; but believing B will choose Hare, A's best reply is Hare. The players will try to coordinate actions.

X19.33 **Assume that player *A* believes player *B* follows the strategy: Pr(*Stag*) = p_B. Confirm that for player *A*:**

a) **the expected payoff to *Stag*, v_A(*Stag*) = 5p_B;**
$E[V_A(Stag, p_B)] = 5*p_B + 0*(1 - p_B) = 5p_B$.

b) **the expected payoff to *Hare*, v_A(*Hare*) = 2;**
$E[V_A(Hare, p_B)] = 2*p_B + 2*(1 - p_B) = 2$.

c) player *A* is indifferent between choosing *Hare* and *Stag* if p_B = 0.4.
We require $E[V_A(Stag, p_B)] = E[V_A(Hare, p_B)]$; so $5p_B = 2$. The result follows immediately.

X19.34 **Confirm that there is a Nash equilibrium in mixed strategies $(p_A^*, p_B^*) = (0.4, 0.4)$. Calculate the probability of each outcome occurring, and hence confirm the expected payoffs, $v_A(p_A^*, p_B^*) = v_B(p_A^*, p_B^*) = 2$.**
If $p_B = 0.4$, then $E[V_A(p_A, 0.4)] = p_A E[V_A(Stag, 0.4)] + (1 – p_A) E[V_A(Hare, 0.4)] = 2p_A + 2(1 – p_A) = 2$. Player A is indifferent between all strategies. So player A does not wish to deviate from the equilibrium; the same argument applies for player B.

X19.35 **Show that player *A*'s best-reply function may be written $p_A^*(p_B) = \begin{cases} 1, \text{if } p_B > 0.4 \\ [0,1], \text{if } p_B = 0.4 \\ 0, \text{if } p_B < 0.4 \end{cases}$. Sketch this best-reply function, the equivalent for player *B*, and confirm that there are three Nash equilibria in this strategic game, two in pure strategies, and one in mixed strategies. How might we define trust between the players in this context?**
Given that $E[V_A(Stag, p_B)] = 5p_B$ and $E[V_A(Hare, p_B)] = 2$, then if $p_B > 0.4$, $E[V_A(Stag, p_B)] > E[V_A(Hare, p_B)]$, so player A should adopt the strategy $p_A = 1$, or 'Stag.'
Similarly if $p_B < 0.4$, $E[V_A(Stag, p_B)] < E[V_A(Hare, p_B)]$, so player A should adopt the strategy $p_A = 0$, or 'Hare.'
We have also verified that if $p_B = 0.4$, then $E[V_A(Stag, 0.4)] = E[V_A(Hare, 0.4)]$, and that all mixed strategies, p_A, therefore have the same expected value.
We can think of trust as being a belief that the probability of hunting a stag is high enough that a player decided to hunt the stag, rather than the hare.

X19.36 **We define the *maximin strategy* as choosing the action that maximizes the minimum possible payoff. Confirm that the maximin strategy here does not support the optimal level of cooperation.**
In this case the maximin strategy is Hare; this prevents the loss coming from defection if the other hunter chooses Hare, when choosing Stag. But we have seen that the strategy, Stag, results from cooperation, so the maximin strategy is worse than the Nash equilibrium strategy.