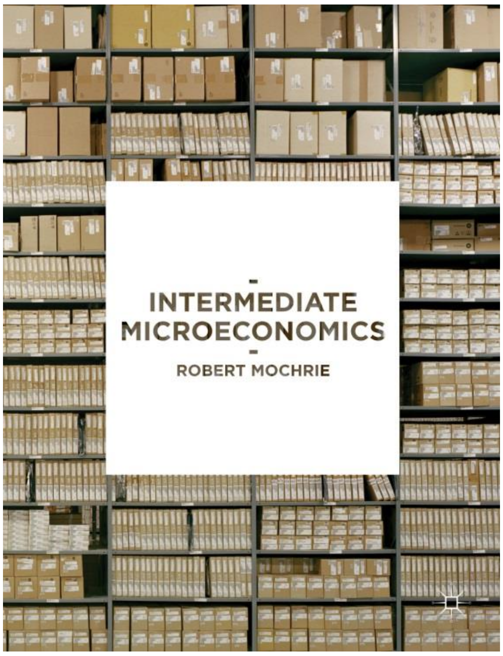
Solutions Manual: Part VI

Behaviour

Summary answers to the 'By yourself' questions



Chapter 24

X24.1 **We have mentioned very direct interventions in the market here. How might a government in a poor country intervene indirectly in the market, using *supply-side* policies to improve market efficiency so that food prices will fall?**

Supply-side policies are designed to increase the productivity of factor inputs, so that the productive capacity of the economy increases. We might think of them as removing barriers to the attainment of production on the production possibility frontier, or measures to shift the frontier outwards.

In this context, any policy which leads to an increase in agricultural productivity, such as land reform, or the improved management of common property resources, or the establishment of agricultural extension centres, or more effective credit markets (for example, through micro-credit programmes enabling farmers to diversify production), or infrastructure investment (such as improved roads, to reduce the costs of distribution) has a supply-side effect.

X24.2 **We define consistency of choices as follows: if a consumer chooses consumption bundle *X* before a reduction in the price of a good or before an increase in the money available for consumption, and consumption bundle *Y* afterwards, then the choice is consistent if *Y* was not in the original affordable set.**

a) **Confirm that the choices in Figure 24.1 exhibit consistency.**

We see that if A is chosen, both B and C are unaffordable; that if B is chosen, A is affordable, but C is unaffordable, while if C is chosen, then B is unaffordable. There is no evidence of consumers changing their choice after a price change to a bundle that would have been affordable before the price change.

b) **Indicate the income and substitution effects into which the total price effect might be decomposed. Classify beef and cassava as being normal or inferior goods, and state whether you consider them to be gross complements or gross substitutes.**

Suppose that the consumer is indifferent between bundles B and C. Then we may draw a single indifference curve, which passes through both points B and C in our diagram. Then the income effect is the shift from A to C and the substitution effect is from C to B.

c) **In Figure 24.1, a reduction in price p_c leads to a reduction in demand, c. Explain this outcome in terms of the income and substitution effects.**

We know that for a normal good, the income effect and the substitution effect complement each other, with both leading to an increase in demand after a price decrease. For an inferior good, as income increases, demand decreases, and so the income effect associated with a price change has the same sign as the price change (and the opposite sign to the substitution effect). For there to be a negative total effect after a price reduction, the income effect must be larger in size, and opposite in sign, to the substitution effect.

X24.3 **We have drawn the subsistence constraint as a straight line, but identified most-preferred, affordable consumption bundles at B and C.**

a) **On what basis might we argue that beef and cassava are perfect substitutes on the subsistence constraint?**

We treat subsistence as a minimum caloric input, below which the person will die. We assume that beef and cassava both have constant caloric values, so that one can be substituted for the other at a constant rate.

b) Why is the preceding argument consistent with beef and cassava being imperfect substitutes when the subsistence constraint is not binding?
Without a binding calorific constraint, we expect to see that there will be a diminishing marginal rate of substitution – that an appropriate mixture of beef and cassava is preferable to a bundle that is almost all beef or almost all cassava.

c) Suppose that it is possible to become sated with cassava at relatively low levels of consumption. Sketch a preference map showing this property and the subsistence constraint.
In a diagram with consumption of beef shown on the horizontal axis and consumption of cassava on the vertical axis, the line showing the minimum calorific value is the lowest indifference curve. We show the indifference curves as being bunched together close to the horizontal axis, but fanning out and becoming steeper as we move up and to the left along them. For consumption bundles where the consumer reaches satiation, the indifference curves become vertical. Beyond that point, they will be upward sloping (and concave).

X24.4 **In our modelling, we have assumed that all economic agents have perfect information. Were that to include knowledge of everyone else's wealth, would there be any reason to engage in conspicuous consumption?**
No. If all consumers have perfect information, then everyone's wealth is common knowledge, so that consumers have no reason to engage in an activity designed to display their wealth.

X24.5 **Might any good be used in displays of conspicuous consumption? If not, what characteristics do you think are likely to make particular goods well suited to that purpose?**
We expect any good being used in conspicuous consumption to have little 'usefulness' apart from its value as a signal, but also to be difficult to mistake, or indeed to copy.

X24.6 **We have suggested that conspicuous consumption might be wasteful. Suppose that people of type *L* demonstrate their wealth by burning money in public.**

a) Explain why this is wasteful.
Burning money reduces the ability to consume; and it has very little caloric value.

b) Why do you think that people of type *W* would not engage in it?
People of type W lack sufficient resources to want to demonstrate their wealth.

c) What activities seem to you to be the closest approximation to the wilful destruction of property in order to demonstrate wealth?
This is difficult to say, but we might consider the selling proposition of many luxury brands to be that they allow the demonstration of wealth.

X24.7 **In a well-known account that sought to bring many of Veblen's insights into much more standard economic modelling, Harvey Leibenstein identified *snob* and *bandwagon* effects. How do these additional effects relate to Veblen's initial treatment of conspicuous consumption?**
We might model the snob effect as being that people of type L will tend to reduce consumption of goods that are increasingly used by people of type W. The bandwagon effect is that people of type L will increase their consumption of goods that are used by other people of type L. This suggests quite complex dynamic adjustment processes based on imitation and avoidance, with the goods chosen for display changing frequently.

X24.8 **Using the Slutsky decomposition, confirm that $w_2(p_1, p_2, m_W)$ is decreasing in p_2.**
In the diagram, we decompose the price fall using equivalent variation. When the price of good 2 falls, the income effect is the shift from point X to point Z in the diagram, while the substitution effect is the shift from point Z to point Y. The total effect is a shift from Z to X. We note that the total effect is positive, and that with demand for both goods increasing as a result of the income effect, this is consistent with both goods being normal. Writing the Slutsky substitution in the form $\frac{\partial w_2{}^M}{\partial p_2} = \frac{\partial w_2{}^M}{\partial p_2} + w_2 \frac{dw^M}{\partial m}$, the first term on the right, $\frac{\partial w_2{}^H}{\partial p_2}$, is the substitution effect and the second term, $w_2 \frac{dw^M}{\partial m}$, is the income effect. From the diagram, we propose that $\frac{\partial w_2{}^H}{\partial p_2} < 0$, which is consistent with preferences being well behaved; and that $\frac{\partial w_2{}^M}{\partial m} > 0$. Then both factors in the second term are positive, and both terms on the right hand side of the decomposition take negative values. The total effect, $\frac{\partial w_2{}^M}{\partial p_2} < 0$.

X24.9 **Suppose that for consumer *W*, demand for good 2 is price-elastic. What can you say about expenditure on this good following a price rise? How does this affect the budget constraint of consumer *L*?**
Following an increase in the price of good 2, consumers of type W reduce consumption by a larger proportion, so that their expenditure on the good falls. This means that for consumers of type L, the cost of consuming a given multiple of type W expenditure falls. Following an increase in the price of the good, the effective price of the consumption multiple has fallen, and treating the consumption multiple as a normal good, we predict an increase in its value.

X24.10 **Adapt Figure 24.2 so that the effect on the consumption relative of an increase in the price of good 2 is zero. Confirm that the income effect is then negative. Explain why this might happen and interpret the results**
We draw the diagram so that the points X and Y lie on the same flat line. Since the demand for the consumption relative remains constant, the income effect must be equal and opposite to the substitution effect. We would expect an outcome like this when demand for good 2 among type-W consumers is price inelastic. Then an increase in price leads to a relatively small reduction in demand among these consumers, and it becomes more difficult to use positional consumption to signal wealth.

X24.11 **Suppose that consumer *L* has a Cobb-Douglas utility function. Adapt Figure 24.2 for this case, confirming that point Y would then lie directly above point X, and that consumption of good 2 would definitely increase following a price increase.**
This result is a direct result of the demands associated with a Cobb-Douglas utility function, where the demand for any good is a function of its own price and the money available to finance consumption only. Following a change in the price of the consumption relative, there is no change in demand for good 1. We use the previous logic. Following a price increase, the price of the consumption relative falls, so demand for it, $\frac{l_2}{w_2}$, increases. With demand for good 1, l_1, remaining constant, it follows that demand for good 2, l_2, must increase.

X24.12 **Initially, consumer *L* chooses consumption bundle W($l_1{}^0$, $l_2{}^0$) . When price p_2 increases, *L*'s demand for good 2, L_2, falls. What are the likely effects on:**

a) **the total utility of bundle W;**
The total utility will increase because the value of the consumption relative, $\frac{l_2{}^0}{w_2}$, has increased.

b) the marginal utilities, $MU_1(l_1^0, l_2^0)$ and $MU_2(l_1^0, l_2^0)$;
The marginal utility of good 1 seems likely to increase – assuming that there are interaction effects, so that marginal utility for each good depends on the quantities of all goods in the consumption bundle. Since the consumption relative is likely to increase, we propose that the marginal utility of consumption, MU_2 will also increase.

c) the marginal rate of substitution $MRS = -\frac{MU_1}{MU_2}$?
Assuming that the increase in utility follows primarily from changes in the utility derived from the consumption relative, we expect the marginal rate of substitution, MRS, to increase (that is approach zero) as the value of the consumption relative falls.

X24.13 **How does the change in the marginal rate of substitution affect the indifference curve through V? Suppose that bundle V was affordable after the price change. How would you expect *A*'s demands to change?**
From X24.12, we expect the price change to lead to the indifference curve through V to become flatter. This means that at any given price ratio, V is no longer the utility maximizing choice – with the new set of indifference curves, we expect to see substitution towards the consumption relative $\frac{l_2}{w_2}$, away from good 1.

X24.14 **In Chapters 3 and 4, we argued that for well-behaved preferences the assumptions of convexity and monotonicity have to be satisfied. Discuss the implications of these assumptions for the utility function $U = U(b, c, l)$ in Expression 24.4.**
We see that utility is neither increasing, nor convex in l. Formally, we cannot rely on the methods that we have developed to obtain a maximum, without some reinterpretation of the function.

X24.15 **Consider again the assumption that there should be neither utility nor disutility associated with labour time. How reasonable does this seem to be?**
Work requires effort, often has to be carried out at a time of someone else's choosing, and may also take a form that is unpleasant, repetitive, and dangerous. There are many reasons for expecting there to be disutility associated with many types of work. Alternatively, work can also be a social activity, involving substantial creativity and problem-solving. This second set of reasons might lead us to believe that there are sources of utility in work. Our assumption is effectively that we cannot determine which of these is more important; but these effects are not particularly important to the arguments that we develop here.

X24.16 **Define $U = U(g, l)$ the utility generated from consumption of goods, g, and leisure time, l. For $U(g,l)=\frac{gl}{g+l}$, maximized subject to the feasibility constraint $w(T - l) \geq pg$, calculate the utility-maximizing combination (g^*, l^*). Suppose that the wage rate increases. Calculate the effects on leisure and purchase of goods.**
We recognize the functional form here as a CES utility function with elasticity of substitution $\sigma = 0.5$, so that consumption and leisure are complements. Rather than rely on standard results, we derive the demands by forming the Lagrangean, Λ:

$$\Lambda(g,l,\lambda)=\frac{gl}{g+l}+\lambda[w(T-l)-pg].$$

We obtain first-order conditions:

$$\frac{\partial\Lambda}{\partial g}=\frac{l}{g+l}-\frac{gl}{(g+l)^2}-\lambda p=0\ ;\ \frac{\partial\Lambda}{\partial l}=\frac{g}{g+l}-\frac{gl}{(g+l)^2}-\lambda w=0\ ;\ \text{and}\ \frac{\partial\Lambda}{\partial\lambda}=w(T-l)-pg=0.$$

From the first two expressions, we see that $\frac{l^2}{p} = \frac{g^2}{w} = \lambda(g+l)^2$. *So,* $g = l\sqrt{\frac{w}{p}}$. *Substituting into the third condition, we obtain* $w(T-l) = l\sqrt{wp}$ *; and on rearranging, we obtain the optimal consumption bundle,* $[l^*(w,p), g^*(w,p)] = \left[\frac{w}{p^{0.5}(w^{0.5}+p^{0.5})}T, \frac{w^{0.5}}{w^{0.5}+p^{0.5}}T\right]$.

Now suppose that the wage, w, increases. Partially differentiating, we obtain

$$\frac{\partial l^*}{\partial w} = \left(\frac{0.5w^{-0.5}}{w^{0.5}+p^{0.5}} - \frac{0.5w^{-0.5}w^{0.5}}{(w^{0.5}+p^{0.5})^2}\right)T = \frac{T}{2}\left(\frac{p}{w}\right)^{0.5}(w^{0.5}+p^{0.5})^{-2} > 0$$ *; and*

$$\frac{\partial g^*}{\partial w} = \left(\frac{1}{w^{0.5}+p^{0.5}} - \frac{0.5w^{-0.5}w}{(w^{0.5}+p^{0.5})^2}\right)\frac{T}{p^{0.5}} = \frac{T}{2}\left(\frac{2p^{0.5}+w^{0.5}}{p^{0.5}}\right)(w^{0.5}+p^{0.5})^{-2} > 0.$$

With an increase in the wage rate, the consumer reduces hours of work, increasing leisure, but taking advantage of the higher wage rate to increase income and expenditure, even while reducing hours of work.

X24.17 **Suppose instead the utility function *U*(*g*, *l*) = *g* + *u*(*l*), where *u* is increasing, but concave. Describe optimal choice (*g**, *l**), and explain how this changes with the wage rate.**

We can write the consumer's problem as $\max_{g,l} g + u(l) : w(T-l) \geq pg$. *We characterize the solution of the problem by the first-order condition,* $MRS = \frac{du}{dl} \Big/ \frac{dg}{dg} = \frac{w}{p}$. *We require* $\left.\frac{du}{dl}\right|_{l=l^*} = \frac{w}{p}$.

The optimal level of leisure is chosen so that the marginal utility of leisure is the real wage. As w increases, $\frac{w}{p}$, *the marginal utility in the optimal bundle also increases, so leisure falls.*

X24.18 **Exemptions from the European Union's Working Time Directive are generally available to senior managers, many of whom voluntarily work much longer hours than they can require their colleagues to work. How might we explain this observation?**

This may be further evidence of people obtaining utility from work, whether directly from the element of problem solving or from some other source, such as its contribution to personal identity.

X24.19 **While senior managers have considerable freedom to choose their hours of work, other workers will often have to work fixed hours. Sketch diagrams showing the situation in which a fixed contract might lead to workers (i) working longer hours than they would ideally prefer, given the wage rate; and (ii) working shorter hours. If workers are required to be present at work for longer hours than they themselves require, how do you think they might respond?**

In a diagram with hours of leisure measured on the horizontal axis and consumption of goods measured on the vertical axis, in case (i) we think of the minimum hours to which someone is contracted, L_0*, as giving the maximum leisure available to that person, so that* $l \leq T - L_0$*. The feasible set of leisure-consumption profiles is bounded by the line* $l = T - L_0$ *and the line* $pg + wl = wT$*. We assume that* $MRS\ (T - L_0) > \frac{w}{p}$*. The indifference curve through the endpoint of the downward line segment forming the boundary of the feasible set of leisure-consumption profiles is steeper than the boundary. The first-order condition is not satisfied, and this person would prefer to give up consumption, reduce working hours and increase leisure.*

In case (ii), we think of the opposite problem. People cannot work more than L_0 *hours, so that leisure* $l \geq T - L_0$*. The feasible set is then bounded by the vertical line* $l = T - L_0$*, and the line segment* $wT = wl + pg$*. We now assume that at the vertex of the feasible set,* $MRS\ (T - L_0) < \frac{w}{p}$*. The indifference curve through the endpoint of the downward line segment forming the boundary of the feasible set of leisure-consumption profiles is flatter than the boundary. The* *first-order condition is not satisfied, and this person would prefer to increase consumption, increase working hours and reduce leisure.*

X24.20 **In order to elicit greater labour supply, firms may choose to offer their workers overtime payments. Sketch a diagram showing the effect of an increase in the wage from w_0 to w_1 after labour supply reaches L_0. Discuss how workers' labour supply decisions are likely to be affected by the marginal, rather than the average, wage. In terms of managing firms' costs, what do you conclude about offering overtime payments?**
In a diagram with hours of leisure measured on the horizontal axis and consumption of goods measured on the vertical axis, we have until now defined the boundary of the feasible set as $pg + wl = wT$. We now assume that for $l < l_0$, the wage increases to kw, where $k > 1$. At $l = l_0$, the constraint is kinked, and we have two sections to the boundary of the feasible set, with the left hand one steeper than the right hand one. Note that this means that the feasible set is no longer convex, and so we cannot necessarily apply our usual solution techniques.
We therefore assume that the first-order condition for the optimal leisure-consumption profile applies in the left hand segment, where $MRS(l^*, g^*) = \frac{kw}{p}$. (We also assume that this is the only leisure-consumption profile for which this indifference curve lies within the feasible set, so that the indifference curve lies above the right hand segment of the boundary of the feasible set.)
We note that if the firm were to offer wage kl for all hours worked, then the feasible set would expand, and we might expect workers to be able to attain a more strongly preferred leisure-consumption profile, (l_1, g_1), in which the worker would obtain more leisure, but also be able to finance a higher level of consumption from wages.

X24.21 **We have assumed that the endowment consists solely of time. Suppose that we relax this to allow for the payment of social benefits, many of which are means-tested and are therefore gradually withdrawn as income rises. Many commentators have noted the presence of 'benefits traps' in which the tapering of benefits occurs alongside the imposition of income taxes, with reports (in extreme cases) of marginal tax and benefit reduction rates in excess of 100%. Explain how such benefits structures might lead people to decide not to enter employment.**
Payment of benefits, b, means that the bundle (T, b) is affordable. That is, without working, it is now possible for someone to finance some level of consumption. The benefit trap, as defined above, means that the boundary of the feasible set is upward sloping across some range of values of l: increasing working hours reduces consumption. The indifference curve passing through (T, b) may therefore not pass through any other in the feasible set, and we have a corner solution, in which people choose not to work.

X24.22 **In the preceding paragraph we assume that people buy personal services only after they have withdrawn entirely from the labour market. How would you explain the purchase of such services by people who are engaged in paid work?**
We can think of time spent on personal services as not part of leisure time. Suppose that someone is paid wage w_1, the cost of personal services is p_S per hour, where $w_1 > p_S$. Then working less than an hour, this person can purchase personal services, and increase leisure and total utility.

X24.23 **Rework the example of household allocation, but with these alternative assumptions:**

a) **The decision maker's payoff $U = \min[u_I(z_I, z_J), u_J(z_I, z_J)]$.**
b) **Individual utilities $u_m = z_m^a$, where $1 > a > 0$.**

c) **The quantity of Z goods available to each member of the household, when offering L_m labour hours, $z_I = (T - L_I)^{0.5b}(T - L_J)^{0.5(1 - b)}(w_I L_I + w_J L_J)^{0.5}$; $z_J = (T - L_I)^{0.5(1 - b)}(T - L_J)^{0.5b}(w_I L_I + w_J L_J)^{0.5}$.**

We note that the decision maker considers utilities to be perfect complements, and that utilities only depend upon individual household members' consumption. It follows that the decision maker should want utilities to be equal, so that $(T - L_I)^b(T - L_J)^{(1-b)} = (T - L_I)^{(1-b)}(T - L_J)^b$. Multiplying through, we see that $T - L_i = T - L_j$, so that leisure must be the same for both people.

Then to maximize $z = (T - L)^{0.5}[(w_I + w_J)L]^{0.5}$, we differentiate with respect to L, obtaining $\frac{\partial z}{\partial L} = 0.5(w_I + w_J)^{0.5}\left\{(T-L)^{0.5}L^{-0.5} - (T-L)^{-0.5}L^{0.5}\right\} = 0$, or that $T = 2L$. The decision maker directs both members of the household to divide their time equally between labour and leisure.

X24.24 **Rework the example of household allocation, but with these alternative assumptions:**

a) **The decision maker's payoff $U = U_I U_J$**

b) **Individual utilities $u_m = z_m^a$, where $1 > a > 0$.**

c) **The quantity of Z goods available to each member of the household, when offering L_m labour hours, $z_I = (T - L_I)^{0.5b}(T - L_J)^{0.5(1 - b)}(w_I L_I + w_J L_J)^{0.5}$; $z_J = (T - L_I)^{0.5(1 - b)}(T - L_J)^{0.5b}(w_I L_I + w_J L_J)^{0.5}$.**

We note that the decision maker wishes to maximize the product of utilities, and we are able to simplify this expression to $U(L_I, L_J) = (T - L_I)^{0.5a}(T - L_J)^{0.5a}(w_I L_I + w_J L_J)^a$.

Then to first-order conditions for the maximum of U, we obtain the partial derivatives with respect to L_I and L_J, setting these to zero.

$\frac{\partial U}{\partial L_I} = (T - L_J)^{0.5a}\left\{a w_I (w_I L_I + w_J L_J)^{a-1}(T - L_I)^{0.5a} - 0.5a(w_I L_I + w_J L_J)^a (T - L_I)^{0.5a-1}\right\} = 0$. *We can extract a common factor, $a(w_I L_I + w_J L_J)^a(T - L_I)^{0.5a}$, so that $2w_I(T - L_I) = w_I L_I + w_J L_J$ and on rearranging, this condition becomes $3w_I L_I + w_J L_J = 2w_I T$. We omit the calculations, but we also obtain the equivalent condition for $\frac{\partial U}{\partial L_J} = 0$: that $w_I L_I + 3w_J L_J = 2w_J T$.*

Multiplying the first of these equations by 3, and subtracting the second one, we obtain $8w_I L_I = (6w_I - 2w_J)T$; or that $L_I = \left(\frac{3w_I - w_J}{4w_J}\right)T$. Substituting, we obtain $L_J = \left(\frac{3w_J - w_I}{4w_I}\right)T$.

X24.25 **Explain why it is not possible to include price as a characteristic of a good.**

Price reflects the balance between market WTA and WTP; it is not a measure of intrinsic value based on, say, the cost of producing the good.

X24.26 **List the possible characteristics of an apple that you consider relevant to this analysis. Choosing any two, sketch a diagram showing a variety of possible combinations of characteristics. Explain which ones might be chosen by growers. Indicate the preferences of a consumer over these characteristics, and identify the utility-maximizing combination.**

We could make a list including bitterness, sweetness, crispness of flesh, colour, smoothness of skin, toughness of flesh, resistance to infection and hardiness in transport. All of these will affect the consumer's willingness to pay. In a diagram with bitterness on the horizontal axis and sweetness on the vertical axis, we choose several points. We indicate preferences by the indifference curves passing through these points; only those that are not dominated by other points might be chosen by the grower; and among these the most preferred by the consumer lies on the highest indifference curve.

X24.27 **Through experimentation and selective breeding, it is possible for farmers to create new breeds of apple. Discuss how such processes might change the analysis.**

Farmers will seek to find a combination of characteristics that has not previously been produced, and which will be preferred by the consumer.

X24.28 **We have concentrated our discussion on characteristics of apples in consumption. In considering production and distribution, we might also expect there to be characteristics that affect the varieties chosen, such as the length of time for which each might be stored, or the robustness of the fruit in the mechanized harvesting process. What effects might such characteristics have on the choice available to the consumer?**
These characteristics relate to the ease of production; these will reduce the WTA, and so the market price.

X24.29 **How would you explain the fact that supermarkets continue to stock several apple varieties? On what criteria would you expect the decision to stock any particular variety to be based?**
Two points: firstly, different uses require different apples (bitter for cooking, sweet for eating), so people like variety; and secondly, people differ in their tastes.

Chapter 25

X25.1 **Given the expenditure constraint, Expression 25.1, calculate:**

a) The opportunity cost of current consumption.

Writing $c_1 = m_1 + (1+r)(m_0 - c_0)$, the opportunity cost, $\frac{dc_1}{dc_0} = -(1+r)$.

b) The opportunity cost of future consumption.

Writing $c_0 = m_0 + (1+r)^{-1}(m_1 - c_1)$, the opportunity cost, $\frac{dc_0}{dc_1} = -\frac{1}{(1+r)}$.

c) The level of consumption in the current period (t = 0) if consumption in the future (t = 1) is set to zero.

When $c_1 = 0$, $c_0 = m_0 + (1+r)^{-1}m_1$.

d) The level of consumption in the future if consumption in the current period is set to zero.

When $c_0 = 0$, $c_1 = m_1 + (1+r)m_0$.

X25.2 **Suppose that the interest rate increases. Adapt Figure 25.2 to show how this will affect the affordable set.**

Firstly, we note that at point E in the diagram, this consumer spends the endowment as it is received: $(c_0, c_1) = (m_0, m_1)$. This consumption profile is independent of the interest rate, so irrespective of the interest rate, E is part of the affordable set.

Secondly, we note that the boundary of the affordable set is linear, irrespective of the interest rate: the opportunity cost of current consumption is constant.

Thirdly, we note that in the consumption profile $(0, (1+r)m_0 + m_1)$, $\frac{\partial c_1}{\partial r} = m_0$. *Since the intersection of the affordability constraint with the vertical axis moves outwards, and point E is invariant, then the constraint pivots clockwise around E when the interest rate increases.*

X25.3 **We have assumed that financial markets are perfect, so that consumers can borrow and lend at the same interest rate. In reality, the interest rate paid by borrowing consumer is usually higher than the interest rate received by savers. Adapt Figure 25.2 to show this effect.**

Borrowing occurs to the right of E in the affordability set, while saving occurs above E. If the interest charged on loans is greater than the interest paid on savings, then the boundary of the affordable set to the right of E will be steeper than the boundary above E.

X25.4 **Suppose that there are two individuals, whose utility functions may be written $u_A = c_0^a c_1^{1-a}$, and $u_B = c_0^a + c_1^a$. Their income profiles are $M_A = (m_A, 0)$ and $M_B = (0, m_B)$. For each individual, write down the relationship that must exist between the level of consumption in the current period and that in the future period, and obtain the consumption profile. Indicate clearly the amount that individual *A* saves, and the amount that individual *B* borrows in the current period (t = 0).**

For individual A, the utility maximization problem may be written as:

$\max_{c_0, c_1} c_0^a c_1^{1-a} : (1+r)m_0 \geq (1+r)c_0 + c_1$. *Then applying the usual first-order condition, that the marginal rate of substitution equals the price ratio, we obtain,*

$$MRS = \frac{\partial u_A}{\partial c_0} \Big/ \frac{\partial u_A}{\partial c_1} = \frac{a\left(\frac{c_1}{c_0}\right)^{1-a}}{(1-a)\left(\frac{c_0}{c_1}\right)^{a}} = \frac{ac_1}{(1-a)c_0} = 1+r.$$

We can rewrite the first-order condition as $ac_1 = (1 - a)(1 + r)c_0$. Substituting for c_1 in the affordability constraint, we obtain $m_0 = c_0 + \frac{1-a}{a}c_0$, so that $c_0 = am_0$, and $c_1 = (1 - a)(1 + r)c_0$.
Note: this result follows from the fact that with Cobb-Douglas utility functions, there are constant expenditure shares on each good.
For individual B, the utility maximization problem may be written as:
$\max_{c_0, c_1} c_0^{\,a} + c_1^{\,a} : m_B \geq (1+r)c_0 + c_1$. *Then applying the usual first-order condition, that the marginal rate of substitution equals the price ratio, we obtain,*

$$MRS = \frac{\partial u_A}{\partial c_0} \bigg/ \frac{\partial u_A}{\partial c_1} = \frac{ac_0^{\,a-1}}{ac_1^{\,a-1}} = \left(\frac{c_1}{c_0}\right)^{1-a} = 1 + r\,.$$

We can rewrite the first-order condition as $c_1^{\,1-a} = (1 + r)c_0^{\,1-a}$. Substituting for c_1 in the affordability constraint, we obtain $m_B = (1+r)c_0 + (1+r)^{\frac{1}{1-a}}c_0$. We can collect together the terms in c_0, writing $c_0 = \frac{m_B}{(1+r)\left[1+(1+r)^{\frac{a}{1-a}}\right]}$; and then substituting, we obtain $c_1 = \frac{(1+r)^{\frac{a}{1-a}}m_B}{\left[1+(1+r)^{\frac{a}{1-a}}\right]}$

X25.5 **Assuming perfect capital markets, explain how the consumption profiles would change if the income profiles became $M_A' = (0, (1 + r)m_A)$ and $M_B = (\frac{m_B}{1+r}, 0)$. Explain why the consumption profile will always stay the same if the future (or present) value of the income profile remains constant.**
Since the present value (or indeed the future value) of the income profiles do not change, the value of the consumers' endowments m_A and m_B do not vary with the change in the payment flows. A and B still face the same affordability constraints as in X25.4 and so there is no change in their most-preferred, affordable consumption profiles.

X25.6 **Suppose that there is a minimum level of consumption that must be reached in both periods, $c_{\min}$. By sketching an indifference map of the utility function $u = (c_0 - c_{\min})^a(c_1 - c_{\min})^{1-a}$, explain how this characteristic is captured.**
The preference map will be similar to that for a standard Cobb-Douglas utility function. However, in a diagram with consumption on the current period shown on the horizontal axis and consumption in the future period shown on the vertical axis, the minimum consumption constraints are indicated by the vertical line $c_0 = c_{min}$ (for the current period) and the horizontal line $c_1 = c_{min}$ for the future period. These lines form the asymptotes for the indifference curves, which are of course smooth, downward-sloping and convex.

X25.7 **In Expression 25.7, explain in words the term $(1 + \pi)c_1$. Confirm that the consumer foregoing consumption in the future period is not affected by the increase in inflation, while a consumer who abstains from current consumption cannot purchase as large a bundle after an increase in inflation. Hence, show in a diagram the effect of an increase in the value of the inflation rate, π, on the affordability constraint.**
The term $(1 + \pi)c_1$ is the monetary cost of achieving a consumption level, c_1, measured in the value of goods. Since inflation only affects purchase cost in the future period, a consumer who makes no purchases in future is able to afford the same bundle as before the change in inflation. A consumer who defers purchases, and then faces unexpectedly high inflation has to reduce purchases. In a diagram with consumption in the current period on the horizontal axis and consumption in the future period on the vertical axis, the change in inflation has the effect of a price increase on future consumption, and so the affordability constraint pivots anti-clockwise around its intersection with the horizontal axis. Note that the real value of the endowment, E, falls as a result of this change.

X25.8 **Show that the opportunity cost of current consumption, $\frac{dc_1}{dc_0} = -\frac{(1+r)}{(1+\pi)}$. Confirm that this opportunity cost falls as inflation, π, increases.**

Given endowment M: (m_1, m_2), with future value $FV(M) = (1 + r)m_0 + m_1$; and consumption profile C: (c_0, c_1), with future value of the acquisition cost $FV(C) = (1 + r)c_0 + (1 + \pi)c_1$, we write the affordability constraint, A: $FV(M) - FV(C) = 0$, or that $(1 + r)m_0 + m_1 - [(1 + r)c_0 + (1 + \pi)c_1] = 0$.

Partially differentiating the constraint with respect to c_0 and c_1, $\frac{\partial A}{\partial c_0} = -(1+r)$, and $\frac{\partial A}{\partial c_1} = -(1+\pi)$. The discount rate, $\delta = MRS = -\frac{\partial A}{\partial c_0}\Big/\frac{\partial A}{\partial c_1} = -\frac{(1+r)}{(1+\pi)}$.

Differentiating δ with respect to the inflation rate, π, $\frac{\partial \delta}{\partial \pi} = \frac{(1+r)}{(1+\pi)^2} > 0$.

X25.9 **When the interest rate, r, and the inflation rate, π, are small, we approximate the *real interest rate* by the term $r - \pi$. Show that if the real interest rate is negative, then the consumer can purchase more goods with a fixed sum of money in the current period ($t = 0$) than in the future period ($t = 1$).**

We now write the constraint as $(1 + r - \pi)(m_0 - c_0) = m_1 - c_1$. The real interest rate $r - \pi < 0$ if $r < \pi$. The repayment cost of borrowing $(m_0 - c_0)$ is less than the loan principal.

X25.10 **Assume that consumption at time t is a normal good. Sketch a diagram indicating clearly the decomposition of the total effect on the consumption profile of an increase in inflation through (a) the real wealth effect, associated with a shift in the consumer's affordability constraint; and (b) the substitution effect, associated with the change in the opportunity cost of current consumption.**

In a diagram showing consumption in the current time period, c_0, on the horizontal axis, and consumption at time, c_1, on the vertical axis, we show the affordable set as a right-angled triangle, formed by the axes and the affordability constraint,
A: $(1 + r)m_0 + m_1 - [(1 + r)c_0 + (1 + \pi)c_1] = 0$. We indicate an increase in inflation by rotating the constraint anti-clockwise around its intersection with the horizontal axis.
We demonstrate the consumer's choice of consumption profile at the two inflation levels as points X and Z on the affordability constraints before and after the change in inflation. We then sketch indifference curves passing through these points, and then a downward sloping line that is parallel to the original affordability constraint, but which is tangent to the affordability constraint after the increase in inflation at point Y. (Note that for consumption in both periods to be normal, $c_t^Y < c_t^X$.) The shift from X to Y represents the income effect, and from Y to Z is the substitution effect.

X25.11 **Calculate the total utility of the consumption profiles:**

a) **(100, 100), with discount factor 0.95;**
U(100, 100) = 10(1 + 0.95) = 19.5.

b) **(121, 144), with discount factor 0.9;**
*U(121, 144) = 11 + 0.9*12 = 21.8.*

c) **(64, 256), with discount factor 0.5.**
*U(64, 256) = 8 + 0.5*16 = 16.*

X25.12 **Calculate the marginal rate of substitution, $\left.\frac{dc_1}{dc_0}\right|_{u(c_0,c_1)=u_0} = -\frac{\partial u}{\partial c_0}\Big/\frac{\partial u}{\partial c_1}$, for a consumer with utility function of the form in Expression 25.8. Confirm that as the discount factor, δ, increases, the marginal rate of substitution also increases.**

MRS = $\left.\frac{dc_1}{dc_0}\right|_{u(c_0,c_1)=u_0} = -\frac{\partial u}{\partial c_0}\Big/\frac{\partial u}{\partial c_1} = -\frac{0.5c_0^{-0.5}}{0.5\delta c_1^{-0.5}} = -\frac{1}{\delta}\left(\frac{c_1}{c_0}\right)^{0.5}$. Partially differentiating with respect to δ, we obtain $\frac{\partial MRS}{\partial \delta} = \delta^{-2}\left(\frac{c_1}{c_0}\right)^{0.5} > 0$.

X25.13 **Using the results of X25.12, sketch an indifference map for the utility function $u = c_0^{0.5} + \delta c_1^{0.5}$, where the discount factor $\delta \approx 0$. Interpret the diagram in terms of the consumer's present valuation of current and future consumption.**

In a diagram with consumption in the current period, c_0, measured on the horizontal axis, and consumption in the future period, c_1, measured on the vertical axis, when $\delta \approx 0$, MRS $\to \infty$. The present value of future consumption is very small. Indifference curves are very steep (and in the limit, $\delta = 0$, would be vertical). We expect that at all points on the affordability constraint, the indifference curves will be steeper than the constraint, which means that the first-order condition for maximum utility cannot be satisfied, and we obtain the corner solution, with income profile $(c_0^*, c_1^*) = (PV(M), 0)$, and all consumption taking place in the current period.

X25.14 **Assume that the consumer has an income profile, $M = (m_0, m_1)$, with price level $(p_0, p_1) = (1, 1 + \pi)$, and can borrow or save any amount of money on a perfect capital market, so that the interest rate on loans and savings is r.**

a) **Write down an expression for the consumer's affordability constraint over consumption profiles, $C = (c_0, c_1)$.**

The consumer cannot spend more than the value of the endowment, so we write the constraint (in terms of future values): $(1 + r)(m_0 - c_0) = -(m_1 - (1 + \pi)c_1)$.

b) **Write down an expression for the opportunity cost of current consumption, $\left.\frac{dc_1}{dc_0}\right|_M$.**

We have already confirmed that the opportunity cost, $\left.\frac{dc_1}{dc_0}\right|_M = -\frac{1+r}{1+\pi}$.

c) **Find the relation between current and future consumption on the consumer's wealth expansion path, associated with changes in the income profile.**

We have confirmed that the marginal rate of substitution, MRS = $\left.\frac{dc_1}{dc_0}\right|_{u_0} = -\frac{1}{\delta}\left(\frac{c_1}{c_0}\right)^{0.5}$. The first-order condition for an optimum is $\left.\frac{dc_1}{dc_0}\right|_M = \left.\frac{dc_1}{dc_0}\right|_{u_0}$, so that the constraint forms a tangent to the indifference curve. Then $-\frac{1+r}{1+\pi} = -\delta^{-1}\left(\frac{c_1}{c_0}\right)^{0.5}$, and squaring both sides of this equation,

$$c_1 = \frac{\delta^2(1+r)^2}{(1+\pi)^2}c_0$$

d) **Find the consumer's optimal consumption profile.**

Substituting the first-order condition into the consumer's expenditure constraint, we see that $(1+r)m_0 + m_1 = (1+r)c_0 + \frac{\delta^2(1+r)^2}{(1+\pi)}c_0$, so that $\frac{(1+r)}{(1+\pi)}\left(1+\pi+\delta^2(1+r)\right)c_0 = (1+r)m_0 + m_1$. Then

$c_0 = \frac{1+\pi}{(1+r)}\left[\frac{m_1+(1+r)m_0}{(1+\pi)+\delta^2(1+r)}\right]$, *so that* $c_1 = \frac{\delta^2(1+r)}{1+\pi}\left[\frac{m_1+(1+r)m_0}{(1+\pi)+\delta^2(1+r)}\right]$

X25.15 **Use the optimal consumption profile stated in Expression 25.16:**

$$C=(c_0,c_1)=\left[M\left(\frac{1+\pi}{1+r}\right)\left(\frac{1}{1+\pi+\delta^2(1+r)}\right),\delta^2 M\left(\frac{1+r}{1+\pi}\right)\left(\frac{1}{1+\pi+\delta^2(1+r)}\right)\right]$$

a) **to confirm the optimal consumption profile in Exercise 25.14;**
This follows directly from substitution.

b) **to find the optimal consumption profile for the parameter values $\delta = 0.95$, $r = 0.1$, $\pi = 0$;**
With these parameter values, $c_0 = \frac{M}{1.1}\left(\frac{1}{1+0.95^2(1.1)}\right) \approx 0.456M$, *while* $c_1 \approx 0.498M$.

c) **to find the optimal consumption profile for the parameter values $\delta = 0.6$, $r = 0.5$, $\pi = 0.4$.**
We obtain $(c_0, c_1) \approx (0.481M, 0.199M)$

X25.16 **In X25.12, we found that for the utility function, $u(c_0, c_1) = c_0^{0.5} + \delta c_1^{0.5}$, the marginal rate of substitution, $\left.\frac{\partial c_1}{\partial c_0}\right|_{U=U_0} = -\frac{c_1^{0.5}}{\delta c_0^{0.5}}$.**

a) **Setting utility $u = u_0$, write an expression for the level of future consumption, c_1, in terms of the utility obtained and the level of current consumption, c_0.**
We can rewrite the expression as $\delta c_1^{0.5} = u_0 - c_0^{0.5}$, *and so rearranging further,* $c_1 = \delta^{-2}(u_0 - c_0^{0.5})^2$.

b) **Using the expression found in part (a), rewrite the marginal rate of substitution in terms of the utility, $u = u_0$ and the current consumption, c_0.**
MRS = $-\frac{c_1^{0.5}}{\delta c_0^{0.5}} = -\frac{(u_0 - c_0^{0.5})}{\delta^2 c_0^{0.5}}$.

c) **Confirm that for a given value of current consumption, c_0, and utility, u_0, as the value of the discount factor, δ, decreases, both future consumption, c_1, and the marginal rate of substitution decrease.**
This is easy to confirm by partial differentiation with respect to the discount factor, δ.
$\frac{\partial c_1}{\partial \delta} = -\frac{2(u_0 - c_0^{0.5})^2}{\delta^3}$ < 0; *and so future consumption is decreasing in the discount factor.*
$\frac{\partial MRS}{\partial \delta} = \frac{2(u_0 - c_0^{0.5})}{\delta^3 c_0^{0.5}}$ > 0, *and so the marginal rate of substitution of the optimal profile increases with the discount rate.*

X25.17 **Sketch a diagram showing the optimal consumption profile for a consumer with this utility function. Indicate the effect of a reduction in the discount factor on the marginal rate of substitution, and explain how the consumption profile will change, assuming that the affordability constraint remains the same.**
We recognize the utility function as having a constant elasticity of substitution form, with different weights applying to consumption in different periods. From previous examples (see Chapter 9 in particular), we know that the indifference curves for this type of utility function are smooth, concave, and have tangents where they meet the axes. The effect of reducing the discount factor is to reduce the weight placed on future consumption, so that indifference curves are stretched out vertically; at any consumption bundle (c_0, c_1), *a reduction in the value of* δ *would lead to the marginal rate of substitution increasing.*

X25.18 **Throughout our discussion we have adopted an additive CES utility function, choosing parameter values that have the effect of turning current and future consumption into**

substitute goods. How reasonable do the assumptions of additivity and inter-temporal substitutability seem?

Additivity: this implies that utility is derived directly from consumption in each period. There are no roles for inter-temporal effects, such as habit formation, where previous consumption affects current utility.

Substitutability: this suggests that demands in each period will be sensitive to changes in the price level. We might argue that an increase in interest rates would be likely to reduce consumption in both periods, rather than having a substantial degree of inter-temporal substitution, in which case, consumption at different times would be complements, rather than substitutes.

X25.19 **Replace the utility function in Expression 25.8 with the Cobb-Douglas utility function:**

$$u(c_0, c_1) = c_0^{0.5} c_1^{0.5\delta}$$

Repeat X25.14 using this utility function, obtaining the optimal consumption profile. Explain how the optimal consumption profile will change as the value of δ increases.

As before, the consumer cannot spend more than the value of the endowment:

$(1 + r)(m_0 - c_0) = (1 + \pi)c_1 - m_1$; so that the opportunity cost, $\left.\frac{dc_1}{dc_0}\right|_M = -\frac{1+r}{1+\pi}$. We obtain the marginal rate of substitution, MRS = $\left.\frac{dc_1}{dc_0}\right|_{u_0} - \frac{\partial u}{\partial c_0} \Big/ \frac{\partial u}{\partial c_1} = -\frac{0.5c_0^{-0.5}c_1^{0.5\delta}}{0.5\delta c_0^{0.5}c_1^{0.5\delta-1}} = -\frac{c_1}{\delta c_0}$. The first-order condition for an optimum is $\left.\frac{dc_1}{dc_0}\right|_M = \left.\frac{dc_1}{dc_0}\right|_{u_0}$, so that the constraint forms a tangent to the indifference curve. Then $-\frac{1+r}{1+\pi} = -\frac{c_1}{\delta c_0}$, and cross-multiplying, $(1+\pi)c_1 = \delta(1+r)c_0$.

Substituting into the constraint, $(1 + r)(m_0 - c_0) = m_1 - \delta(1 + r)c_0$; so that $c_0 = \frac{M}{(1+r)(1+\delta)}$, where $M = (1 + r)m_0 + m_1$ is the future value of the endowment. Substituting from the first-order condition, $c_1 = \frac{\delta M}{(1+\pi)(1+\delta)}$.

X25.20 **By defining s_1 firstly in terms of s_0 and c_0, and then by defining s_0 in terms of m_0 and c_0, show that the affordability constraint can be written in terms of future values (at $t = 2$) as $(1 + r)^2 m_0 = c_2 + (1 + r)c_1 + (1 + r)^2 c_0$. Write the constraint in terms of present values as well.**

Savings are the difference between the amount of money available to spend in any period, and the consumption in any period. We define $s_0 = m_0 - c_0$; then income in period $t = 1$ may be written, $m_1 = (1 + r)s_0$, so $s_1 = (1 + r)s_0 - c_1$; and similarly, we require for the endowment to be spend by the end of period $t = 2$, $(1 + r)s_1 - c_2 = s_2 = 0$.

Substituting recursively, $0 = (1 + r)[(1 + r)s_0 - c_1] - c_2 = (1 + r)^2(m_0 - c_0) - (1 + r)c_1 - c_2$. Rearranging, we obtain the required expression.

X25.21 **Repeat the argument of X25.20, but for the case where there are three future periods, so that $t = 0, 1, 2, 3$.**

We build on the answer to the previous question. We continue to define $s_2 = (1 + r)s_1 - c_2$, but now allow $s_2 > 0$. Then by the end of period $t = 3$, when the endowment is spent, $s_3 = (1 + r)s_2 - c_3 = 0$. Then, we substitute recursively, beginning with $(1 + r)[(1 + r)s_1 - c_2] - c_3 = (1 + r)^2(m_1 - c_1) - (1 + r)c_2 - c_3$. Repeating the process by substitution for s_1 and then s_0, and then rearranging the expression, we obtain the required result:

$$\sum_{t=0}^{3} (1+r)^{3-t} c_t = (1+r)^3 m_0$$

X25.22 **Using the results of X25.20 and X25.21, show that the affordability constraint when there are T future periods may be written: $(1+r)^T m_0 \geq \sum_{t=0}^{T}(1+r)^{T-t} c_t$, or equivalently as:**

$$m_0 \geq \sum_{t=0}^{T}(1+r)^{-t} c_t \text{ .}$$

This is an example of proof by the method of induction. We have shown that the expenditure constraint takes the form in the case where T = 1, 2, and 3. We now demonstrate that if it holds for some value $T = T_0$, then it must also hold when $T = T_0 + 1$. For simplicity, we concentrate on the case where there is equality.

We assume that $(1+r)^{T_0} m_0 = \sum_{t=0}^{T_0}(1+r)^{T_0-t} c_t$ *. Now suppose that consumption is extended to $T_0 + 1$ periods. In period T_0, it must be true that* $(1+r)^{T_0} m_0 \geq \sum_{t=0}^{T_0}(1+r)^{T_0-t} c_t$*, so that saving*

$$s_{T_0} = (1+r)^{T_0} m_0 - \sum_{t=0}^{T_0}(1+r)^{T_0-t} c_t \text{ .}$$

Then in period $T_0 + 1$,

$$s_{T_0+1} = (1+r)\left[(1+r)^{T_0} m_0 - \sum_{t=0}^{T_0}(1+r)^{T_0-t} c_t\right] - c_{T_0+1} = (1+r)^{T_0+1} m_0 - \sum_{t=0}^{T_0+1}(1+r)^{T_0+1-t} c_t \text{ .}$$

If the required relationship holds when $T = T_0$, then it also holds when $T = T_0+1$. Knowing that it holds when T = 1, 2, 3, we infer that it holds for all values of T > 1.

X25.23 **How do the expressions found in X25.22 change if instead of an endowment received in the current period, m_0, the consumer receives an endowment m_T in the final period ($t = T$).**
We have written the constraint using the future values. The future value of the consumption profile does not change; all that happens is that we replace the future value of the initial endowment with the value of the endowment received in the final period, so that

$$m_T = \sum_{t=0}^{T_0}(1+r)^{T_0-t} c_t \text{ .}$$

X25.24 **Now suppose that the consumer receives income in every period, so that the income profile $M = (m_0, m_1, \ldots, m_T)$. Show that the affordability constraint can be written:**

$$\sum_{t=0}^{T}(1+r)^{-t}(m_t - c_t) \geq 0 \qquad \textbf{[25.18]}$$

We note here that the expression in the summation sign is the difference between income and consumption in every period; so we can write $m_t - c_t = s_t$, where s_t is here the current savings. The factor $(1 + r)^{-t}$ is the discount factor required to obtain the present value of savings s_t which accrue t periods from the current period, t = 0. So, expression [25.17] has the simple interpretation that the present value of current savings across the saving profile cannot be negative. Facing perfect capital markets, the present value of expenditure cannot exceed the present value of income.

X25.25 **Confirm that we can write the present value of consumption profile C_T as:**

$$U(C_T) = \sum_{t=0}^{T}\delta^t u(c_t) \qquad \textbf{[25.22]}$$

Writing the present value of consumption, $U(C_1) = c_0 + \delta u(c_1)$, we also denote $U_t(C_T)$ as the present value of the consumption profile calculated in period t, with consumption continuing until period T.
We can then write $U(C_2) = u(c_0) + \delta U_1(C_2)$, where $U_1(C_2) = u(c_1) + \delta u(c_2)$.
So, $U(C_2) = u(c_0) + \delta[u(c_1) + \delta u(c_2)] = u(c_0) + \delta u(c_1) + \delta^2 u(c_2)$, as required. By iteration, we can write $U(C_T) = u(c_0) + \delta U_1(C_T)$, where $U_1(C_T)$ is the present value of future utilities received between t = 1 and t = T; and $U(C_T) = u(c_0) + \delta[u(c_1) + \delta U_2(C_T)] = u(c_0) + \delta u(c_1) + \delta^2 U_2(c_T)$. By successive iterations, we obtain the required result.

X25.26 **Using Expressions 25.18 and 25.22, write down the constrained maximization problem facing the consumer in terms of present values.**
The problem can be written as a constrained optimization problem:

$$\max_{C_T} \sum_{t=0}^{T} \delta^t u(c_t) : \sum_{t=0}^{T} \frac{m_t - c_t}{(1+r)^t} \geq 0 .$$

X25.27 **Consider the maximization problem:**

$$\max_{c_0, c_1, c_2} \left[u(c_0) + \delta u(c_1) + \delta^2 u(c_2)\right] : c_0 + \frac{c_1}{1+r} + \frac{c_2}{(1+r)^2} \leq m_0 + \frac{m_1}{1+r} + \frac{m_2}{(1+r)^2} \qquad \textbf{[25.24]}$$

Form the Lagrangean, as in Expression 25.23. By partially differentiating with respect to the levels of consumption, c_0, c_1, and c_2, and setting the derivatives to zero, confirm that the optimal consumption profile C_2* will satisfy the condition:

$$\lambda = \frac{\partial u}{\partial c_0} = \delta(1+r)\frac{\partial u}{\partial c_1} = \delta^2 (1+r)^2 \frac{\partial u}{\partial c_2} \qquad \textbf{[25.25]}$$

We write the Lagrangean,

$$\Lambda(c_0, c_1, c_2, \lambda) = u(c_0) + \delta u(c_1) + \delta^2 u(c_2) + \lambda \left\{ m_0 - c_0 + \frac{m_1 - c_1}{1+r} + \frac{m_2 - c_2}{(1+r)^2} \geq 0 \right\}.$$

Then partially differentiating with respect to the consumption variables,
$\frac{\partial \Lambda}{\partial c_0} = \frac{\partial u}{\partial c_0} - \lambda = 0; \frac{\partial \Lambda}{\partial c_1} = \delta \frac{\partial u}{\partial c_1} - \frac{\lambda}{1+r} = 0; and \frac{\partial \Lambda}{\partial c_2} = \delta^2 \frac{\partial u}{\partial c_2} - \frac{\lambda}{(1+r)^2} = 0$. *On rearranging, we obtain the relation between the partial utilities:* $\lambda = \frac{\partial u}{\partial c_0} = \delta(1+r)\frac{\partial u}{\partial c_1} = \delta^2 (1+r)^2 \frac{\partial u}{\partial c_2}$.

X25.28 **Suppose that $\delta = (1 + r)^{-1}$. Explain what this means in terms of personal and market discount rates. What do you infer about the optimal level of consumption in the three time periods in X25.27?**
When $\delta(1 + r) = 1$, the personal and the market discount rates are equal. In the optimization problem, consumers choose a consumption profile in which the marginal utility of consumption is constant over time, so the consumption profile also remains constant.

X25.29 **Suppose that $\delta(1 + r) < 1$. What do you infer about the optimal consumption profile?**
When $\delta(1 + r) < 1$, the personal discount rate is less than the market discount rate. In the optimization problem, consumers choose a consumption profile in which the marginal utility of consumption increases over time, and assuming a diminishing marginal utility from consumption levels, this means that consumption decreases over time. We consider this person to be impatient, preferring higher consumption now and lower future consumption to a constant level of consumption.

X25.30 **Suppose that the advertised rate of interest on a loan is 12%, so that $r = 0.12$. Calculate the cumulative applied rate when compounding takes place (a) quarterly (every 3 months), (b) monthly, (c) weekly, and (d) daily. Using Expression M25.4, calculate the cumulative applied rate of interest when there is continuous compounding.**

We write the compounded interest rate, r_n, where n is the number of applications of compounding within the one year period. Then with $r_n = \left(1+\frac{r}{n}\right)^n$, we obtain the results $r_4 = 1.03^4 - 1 \approx 12.55\%$; $r_{12} = 1.01^{12} - 1 \approx 12.68\%$; $r_{52} = \left(1+\frac{0.12}{52}\right)^{52} \approx 12.73\%$; and $r_{365} = \left(1+\frac{0.12}{365}\right)^{365} \approx 12.75\%$.

X25.31 **For many centuries, it was standard for the monthly charging rate of interest on loans to be set at 1%. Much more recently, payday lending has emerged, and in this market interest accrues at a rate of up to 1% per day, with the loan and interest being repaid in full when it falls due. Compare the cumulative applied rates of interest in these cases:**

a) **a loan of £1,500, taken from a conventional lender, with a monthly charging rate of 1%; repaid in a year;**
We calculated CAR = $r_{12} \approx 12.68\%$ in X25.30.

b) **a loan of £400 taken from a payday lender, which sets a charging rate of 1% per day, and repaid after 15 days (you may assume that compounding would occur every 15 days);**
In this case, there is no compounding over 15 days, so that the charging rate of 1% per day becomes 365% per year; and there are n = $\frac{365}{15}$ compounding periods.

$$CAR = \left(1+\frac{3.65}{\frac{365}{15}}\right)^{\frac{365}{15}} \approx 2{,}99\%$$

c) **a loan of £300 on which there is daily compounding of the interest of 0.5% per day, with repayment in full after 15 days.**
We apply the standard formula, with n = 365: CAR = $\left(1+\frac{1.825}{365}\right)^{365} \approx 6{,}175\%$

X25.32 **Given the principal, L_0, the charging rate of interest, $\frac{r}{n}$, and the loan term, T, obtain expressions for:**

a) **the number of repayment instalments;**
There are nT instalments

b) **the amount outstanding on the loan after one, two, and three repayments.**
Writing the amount outstanding after t repayments as L_t, we can write $L_1 = L_0\left(1+\frac{r}{n}\right) - R$; *then by iteration,* $L_2 = L_1\left(1+\frac{r}{n}\right) - R = \left[L_0\left(1+\frac{r}{n}\right) - R\right]\left(1+\frac{r}{n}\right) - R = L_0\left(1+\frac{r}{n}\right)^2 - R\left[\left(1+\frac{r}{n}\right)+1\right]$ *and*
$L_3 = L_2\left(1+\frac{r}{n}\right) - R = \left\{L_0\left(1+\frac{r}{n}\right)^2 - R\left[\left(1+\frac{r}{n}\right)+1\right]\right\}\left(1+\frac{r}{n}\right) - R = L_0\left(1+\frac{r}{n}\right)^3 - R\left[\left(1+\frac{r}{n}\right)^2 + \left(1+\frac{r}{n}\right)+1\right]$

X25.33 **Confirm that we can write the amount outstanding after *t* repayments as:**

$$L_t = \left(1+\frac{r}{n}\right)^t L_0 - R\sum_{s=0}^{t-1}\left(1+\frac{r}{n}\right)^s \qquad \textbf{[25.28]}$$

We see from X25.32 that the formula is correct for the cases t = 1, 2, 3. Now suppose that it is correct for some general value, t. Then
$L_{t+1} = L_t\left(1+\frac{r}{n}\right) - R = \left\{\left(1+\frac{r}{n}\right)^t L_0 - R\sum_{s=0}^{t-1}\left(1+\frac{r}{n}\right)^s\right\}\left(1+\frac{r}{n}\right) - R$. *Expanding the brackets,*
$L_{t+1} = \left(1+\frac{r}{n}\right)^{t+1} L_0 - R\sum_{s=1}^{t}\left(1+\frac{r}{n}\right)^s - R = \left(1+\frac{r}{n}\right)^{t+1} L_0 - R\sum_{s=0}^{t}\left(1+\frac{r}{n}\right)^s$. *If the expression holds at time t, it also holds at time t + 1.*

X25.34 **Applying the formula for the sum of a geometric sequence, Expression M25.6, show that the loan repayment, *R*, may be written:**

$$R = \frac{\frac{r}{n} L_0}{1 - \left(1 + \frac{r}{n}\right)^{-nT}} \qquad \textbf{[25.29]}$$

We require R to be chosen so that $L_T = 0$, so that $\left(1+\frac{r}{n}\right)^{nT} L_0 = R\sum_{s=0}^{T-1}\left(1+\frac{r}{n}\right)^s$. The expression on the right hand side is a geometric series, with first term R and common ratio $\left(1+\frac{r}{n}\right)$. We can therefore rewrite it: $R\sum_{s=0}^{nT-1}\left(1+\frac{r}{n}\right)^s = R\frac{\left(\left(1+\frac{r}{n}\right)^{nT}-1\right)}{1+\frac{r}{n}-1}$. Then substituting for the sum, $\frac{r}{n}\left(1+\frac{r}{n}\right)^{nT} L_0 = R\left(\left(1+\frac{r}{n}\right)^{nT}-1\right)$, and the result follows immediately.

X25.35 **Using Expression 25.29, calculate the monthly repayment on:**

a) **A loan of £160,000, repayable over 10 years in 120 monthly instalments, when the advertised simple interest rate is 3.5% p.a.**

We write out the expression for the repayment as $R = \frac{\frac{0.035}{12}.160{,}000}{1-\left(1+\frac{0.035}{12}\right)^{-120}} \approx 1{,}582.17.$

b) **A loan of £280,000, repayable over 25 years in 100 quarterly instalments, when the advertised simple rate of interest is 6.5% *per annum.***

We write out the expression for the repayment as $R = \frac{\frac{0.065}{4}.280{,}000}{1-\left(1+\frac{0.065}{4}\right)^{-100}} \approx 5{,}683.95$

X25.36 **For each of the loans in X25.35, calculate the effect on the monthly repayment of an increase in the interest rate of (a) 1% p.a. and (b) 3% p.a.**

a) For the 1% increase in interest rates, we see that the new payments can be calculated as $R_{4.5} = \frac{\frac{0.045}{12}.160{,}000}{1-\left(1+\frac{0.045}{12}\right)^{-120}} \approx 1{,}658.21.$ The increase in payments from 1,582.17 is approximately 76.04.

Similarly, $R_{7.5} = \frac{\frac{0.075}{4}.280{,}000}{1-\left(1+\frac{0.075}{4}\right)^{-100}} \approx 6{,}220.68$, an increase of 536.73 over the original payments of 5,683.95.

b) For the 3% increase in interest rates, we see that the new payments can be calculated as $R_{6.5} = \frac{\frac{0.065}{12}.160{,}000}{1-\left(1+\frac{0.065}{12}\right)^{-120}} \approx 1{,}816.77.$ The increase in payments from 1,582.17 is approximately 234.59.

Similarly, $R_{9.5} = \frac{\frac{0.095}{4}.280{,}000}{1-\left(1+\frac{0.095}{4}\right)^{-100}} \approx 7{,}353.21$, an increase of 1,669.26 over the original payments of 5,683.95.

X25.37 **Confirm that the present value of the bond, as given in Expression 25.31, may be written:**

$$PV(V,T,C) = \frac{C}{r}\left(1-(1+r)^{-T}\right) + V(1+r)^{-T} \qquad \textbf{[25.32]}$$

We can write the present value as $PV=\sum_{t=1}^{T}\frac{C}{(1+r)^t}+\frac{V}{(1+r)^T}=\frac{C}{1+r}\sum_{t=0}^{T-1}\frac{1}{(1+r)^t}+\frac{V}{(1+r)^T}$. *Then, applying the formula for the sum of a geometric series, with first term* $\frac{C}{1+r}$ *and common ratio* $(1+r)^{-1}$,

$$PV=\frac{C}{1+r}\left\{\frac{1-(1+r)^{-T}}{1-\frac{1}{1+r}}\right\}+\frac{V}{(1+r)^T}=\frac{C}{r}\left[1-(1+r)^{-T}\right]+V(1+r)^{-T}.$$

X25.38 **Perpetual, or undated, bonds offer their holders an indefinite income stream from the coupon, so that there is no redemption date: the issuer in effect promises to continue to make payments forever.**

a) **Show that we can write the present value of a perpetual bond $PV(C)$:**

$$PV(C)=\lim_{T\to\infty}PV(V,T,C)=\frac{C}{r} \qquad \textbf{[25.33]}$$

Since there is no redemption value, for a perpetual bond, $PV(C)=\frac{C}{r}\left[1-(1+r)^{-T}\right]$*, and as* $T\to\infty$, $(1+r)^{-T}\to 0$*, so that the result follows immediately.*

b) **Confirm that if $C = rV$, then the present value of the perpetual bond in Expression 25.33 is equal to the present value of the redeemable bond in Expression 25.32.**

When $C = rV$, $PV=\frac{rV}{r}\left[1-(1+r)^{-T}\right]+V(1+r)^{-T}=V$ $= PV(C)$. *The redemption payment is the future value of an infinite stream of payments of amount rV per period.*

X25.39 **We have defined a bond in terms of the stream of payments associated with it, all of which are certain. Why might the present value of a bond change over time? [*Hint:* Think about both the nominal value and the real value of the bond.]**

Two factors that might lead to changes in (market) value: changes in beliefs about future interest rates, so that the value of r used in calculating future values changes; and changes in beliefs about future inflation, so that beliefs about the real interest rate change. Since the monetary value of the coupon is fixed, its real value will change.

X25.40 **The UK government last issued perpetual bonds in 1946. Why do you think that few governments are willing to issue such securities to fund expenditure? [*Hint:* Think about why investors might prefer to hold a sequence of redeemable bonds, rather than a perpetual bond.]**

Perpetual bonds never have to be redeemed, and so investors have greater exposure to inflation and interest rate changes holding a perpetual bond than a short-term bond. For the government, there is the question of the replacement cost of these bonds. They were issued at a time when interest rates were historically very low. The cost of refinance is therefore likely to be greater than their nominal value.

X25.41 **Most large companies will finance their activities through a mixture of debt finance (bond issue) and equity finance (share issue). We have defined bonds in terms of the flow of income associated with them. Shares in a company can similarly be defined as a claim on a share of the profits that the company will make in future, distributed through a sequence of dividends, agreed by shareholders, usually at the company's annual meeting. Explain why it is more difficult to calculate the present value of the equity of a company compared with the present value of its debt.**

Dividend is formally a share in profits, so will be agreed by shareholders after profits have been made. Interest is a contractual requirement, which the company must meet. So dividends do not have a fixed nominal value.

X25.42 **We sometimes refer to a *speculative motive* for holding money.**

a) **Under what circumstances might an investor prefer to hold cash assets in a bank account rather than in (perpetual) bonds?**

Suppose that the only choice of investment was cash (paying interest) and bonds (paying a coupon, but with the possibility of capital gains and losses). The capital value of a bond increases if expectations of the market interest rate fall: the future value of coupon payments then increases. It is rational to forego a remote possibility of capital gains where it is much more likely that there will be capital losses; and this means that an investor believes that other investors will come to believe that the interest rate will rise, so that the value of bonds will fall.

b) **What might cause such an investor to withdraw savings from a bank, using the funds to purchase bonds?**

Such an investor must change beliefs, expecting investor sentiment to change so that the market expectation will be of an interest rate fall, meaning that the price of bonds will increase. Note that investors have to be forward-looking, and able to anticipate interest rate changes.

X25.43 **Explain why we might expect there to be a wage differential between people of types *A* and *B* in future, if neither complete a university degree.**

We have argued that wages should equal the value of marginal product in competitive industries. We now extend that argument, saying that we expect people to be able to affect their own marginal product, and to associate the wage with the individual's marginal product, allowing for differences in productivity.

X25.44 **Write down an expression for the present value of income received over the current and future periods, less any education costs, for:**

a) **someone of type *A* who does not attend university;**

We denote the present value, $PV_{AN} = W_0 + \frac{W_1^N}{1+r}$ *; that is the sum of the wage received in the current period plus the discounted value of the wage received in the next period.*

b) **someone of type *A* who attends university;**

Adapting our notation, $PV_{AU} = -F + \frac{(1+g)W_1^N}{1+r}$

c) **someone of type *B* who does not attend university;**

Using the same process, $PV_{BN} = W_0 + \frac{W_1^S}{1+r}$ *;*

d) **someone of type *B* who attends university.**

and $PV_{BU} = -F + \frac{(1+g)W_1^S}{1+r}$

X25.45 **Confirm that if $gw_1^S < (1 + r)(w_0 + F)$, people of type *B* would choose to go straight into employment, rather than going to university; while if $gw_1^N > (1 + r)(w_0 + F)$, people of type *A* would choose to go to university, rather than going straight into employment.**

For someone of type B to prefer to go straight into employment, $PV_{BU} – PV_{BN} = \frac{gW_1^S}{1+r} - (W_0 + F) < 0$*; so that the condition is satisfied. We do not repeat the calculations, but the equivalent condition for people of type A follows by a similar argument.*

X25.46 **Using the results of X25.45, explain the circumstances under which low-productivity people go straight into employment, and high-productivity people go to university. Compare the present value of lifetime earnings of high- and low-productivity people.**
We have a very simple model here in which the returns to education are positively related to the underlying productivity of workers. Education increases the (already greater) productivity of type A people more than it increases the productivity of type B people. So, since there is selection into education, the earnings differential between people of the two types will widen.

X25.47 **Suppose that the public authorities in a country pay tuition fees directly to a university, and offer a study stipend to students. How do you think that this would affect participation in education? Think carefully about how the government would raise the money necessary to pay these fees.**
There are many valid points that might be raised here. Firstly, we note that subsidies should reduce the costs borne directly by students, increasing the incentive to participate in education. But if the government levies taxes on graduates (other than through proportional taxes on income or expenditure), this suggests that graduates defer, rather than avoid, paying the costs of education. Assuming that capital markets are perfectly efficient, and that there is full information, we do not expect to see any substantial effects on participation as a result of funding reforms.

Chapter 26

X26.1 **In commercial law, the principle of *caveat emptor* (let the buyer beware) is generally honoured. Why might it not be appropriate to apply this in the case of bank deposits? What undesirable effects might a universal guarantee, supported by public authorities, have on: (a) banks' behaviour; and (b) depositors' behaviour?**

Bank deposits are not purchases. They are the storage of assets for use (in consumption) later. If a belief forms that it will not be possible to withdraw funds held on deposit because the bank may no longer be solvent, then it is logical for depositors to seek to withdraw all their funds as soon as possible. Banks do not simply hold onto deposits for safe keeping: they lend them to customers, at interest, keeping such deposits on hand as they expect to need in day-to-day business. It is therefore possible that a solvent bank will experience a problem of liquidity, with more customers seeking to withdraw their funds than the bank is able to pay. The main argument in favour of some form of deposit insurance is that it reduces the probability of banks suffering such speculative 'runs.'

The other point to note about the principle of caveat emptor is that placing funds on deposit involves neither an active relationship, nor a transaction that is completed instantaneously. A bank's balance sheet might deteriorate during the period that funds are held, with the depositor unable to scrutinize the bank's management accounting data. For smaller depositors, in particular, the loss of deposits may have a substantial effect on their total wealth.

Lastly, we note the counter-argument: that insurance means that depositors do not consider the risk of bank failure when choosing where to place deposits, and that knowing that deposits are insured, banks may not adopt a prudent lending policy, but instead seek to generate larger returns from a portfolio of loan advances that are of rather poor quality.

X26.2 **Some surgeons, prior to operating on a patient, will require the patient to alter a risky behaviour, perhaps changing diet to reduce weight and blood pressure, stopping smoking, or reducing alcohol consumption. Explain how the surgeon might consider this to be part of a risk-management or risk-mitigation strategy.**

All of the interventions defined above might be considered risk factors, both during the period of the operation and during the subsequent period of recovery. We define risk management as actions taken to reduce the probability of complications occurring, while risk mitigation consists of action taken to reduce the costs of these complications; and so all of these interventions seem likely to have effects both in terms of risk management and risk mitigation.

X26.3 **Governments in all developed countries have had to recognize that demographic change – people living longer (about 2.5 extra years per decade), with populations stabilizing as fertility rates fall below 2 live births per woman – will require careful management. How might governments encourage people to engage in financial planning for a lengthy period of retirement?**

This is a very large problem for many advanced economies. The proportion of the population who are in work is expected to fall for many years, so that it is important for people to be sufficiently productive during the period when they are working that they can accumulate the savings that will generate the income needed to live well in old age.

There has been a tendency to move away from simple schemes of social insurance, and instead to develop schemes in which individual contributions are matched by contributions from both government and employers, with savings deducted at source by employers.

X26.4 **Confirm that when the pot is shared, each player receives an expected payment of £10, so that the value of participation in the game, net of the initial contribution, is zero.**
*We define expected winnings at the end of the game, E[W]: E[W] = 0.5*20 + 0.5*0 = 10. So, the expected payout is the stake, S, placed to play the game, and the net expected value of participation, E[W] – S = 0.*

X26.5 **The *gambler's fallacy* is a belief that if successive trials are truly random, then if one result occurs more frequently than expected in a sequence of trials, in subsequent trials it is likely to occur less frequently. (The most famous example of this occurred at a casino in Monte Carlo in 1913: the ball landed on 'black' in a roulette wheel in 26 successive trials, and gamblers, convinced that the sequence was highly improbable, behaved as if they believed that with each successive spin of the wheel, the probability of the ball landing on 'red' was increasing.) Explain why the fallacy is inconsistent with the belief that successive trials are independent.**
The fallacy confuses the probability of such a sequence occurring in a single trial, which is approximately 10^{-8} *or* $\frac{1}{100{,}000{,}000}$ *with the probability of separate events (where the probability is approximately 0.5). While it is inevitable that the sequence will end, it is as likely that a ball will land on red after landing on black on 25 successive occasions as in any other circumstance.*

X26.6 **Consider an experiment in which we throw four coins in the air simultaneously.**

a) **By listing possible results, confirm that there are 16 in total.**
The results are: HHHH; HHHT; HHTH; HTHH; THHH; HHTT; HTHT; HTTH; THHT; THTH; TTHH; HTTT; THTT; TTHT; TTTH; TTTT.

b) **If we assume that for all four coins, Pr(*H*) = Pr(*T*) = 0.5, explain why the results *HTHH*, *HHHT* and *HHHH* are equally probable. What do you conclude about the probability of every result?**
Consider each toss of the coin as a separate event. Then there are two possible outcomes, only one of which occurs. So the probability of each outcome is 0.5. Treating the experiment as a compound event, with four coin tosses, there are, as noted above, 16 possible results, each of which is equally probable.

c) **Define the events n_H = 0, 1, 2, 3 and 4, where n_H is the number of coins facing head up. Using the listing in part (a), calculate the probabilities Pr(n_H = 0), Pr(n_H = 1), Pr(n_H = 2), Pr(n_H = 3), and Pr(n_H = 4).**

n_H	0	1	2	3	4
$Pr(n_H)$	$\frac{1}{16}$	¼	$\frac{3}{8}$	¼	$\frac{1}{16}$

X26.7 **For an experiment in which we toss two dice at the same time, summing together the numbers showing on the upward faces:**

a) **Confirm that there are 36 possible results, which can be classified into 11 events, according to the value of the sum, *s*. Calculate the probability of each event.**

Score, s		Die 2					
		1	2	3	4	5	6
Die 1	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11

	6	7	8	9	10	11	12

We see that there are 11 possible results of the experiment, and simply by counting the frequency, f(s), of these, we obtain the probability distribution, $\pi(s)$

s	2	3	4	5	6	7	8	9	10	11	12
f(s)	1	2	3	4	5	6	5	4	3	2	1
$\pi(s)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

b) Hence calculate the probabilities of the value of *s* being:

i. 11 or 12;

$Pr(s \geq 11) = Pr(s = 11) + Pr(s = 12) = \frac{1}{12}$

ii. 7 or more; and

$$Pr(s \geq 7) = \sum_{s=7}^{12} \pi(s) = \frac{21}{36} = \frac{7}{12}$$

iii. a multiple of 4.

Where k = 1, 2, 3, ..., $Pr(s = 4k) = Pr(s = 4) + Pr(s = 8) + Pr(s = 12) = \frac{4+5+1}{36} = \frac{5}{18}$

X26.8 **We carry out a very similar experiment, throwing two dice. On this occasion, we define the associated events as the product, π, rather than the sum, of the numbers showing on the upward faces of the dice.**

a) In this case, $1 \leq \pi \leq 36$. Calculate the probability of each value. [*Note:* There are several events for which the probability is zero.]

Product π		Die 2					
		1	2	3	4	5	6
Die 1	1	1	2	3	4	5	6
	2	2	4	6	8	10	12
	3	3	6	9	12	15	18
	4	4	8	12	16	20	24
	5	5	10	15	20	25	30
	6	6	12	18	24	30	36

Denoting the number of outcomes $f(\pi)$ and the probability distribution, $\phi(\pi)$, by counting results, we obtain:

π	1	2	3	4	5	6	8	9	10	12	15	16	18	20	24	25	30	36
$f(\pi)$	1	2	2	3	2	4	2	1	2	4	2	1	2	2	2	1	2	1
$\phi(\pi)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{36}$

b) Calculate the probability of the following events:

i. obtaining a multiple of three;

$Pr(\pi = 3k) = \frac{5}{9}$

ii. obtaining a multiple of four;

$Pr(\pi = 4k) = \frac{5}{12}$

iii. obtaining a number greater than or equal to 24; and

$Pr(\pi \geq 24) = \frac{1}{6}$

iv. obtaining a number less than or equal to 8.

$Pr(\pi \leq 8) = \frac{4}{9}$

X26.9 **Calculate the expected value of the experiments in X26.6–X26.8.**

$E[n_H] = \sum_{n_H=0}^{4} n.Pr(n_H) = 0*\frac{1}{16} + 1*¼ + 2*\frac{3}{8} + 3*¼ + 4*\frac{1}{16} = 2$

In a similar manner, we obtain $E[s] = 7$; and $E[\pi] \approx 12.25$.

X26.10 **Working with the experiments in X26.6–X26.8, calculate the expected change in wealth, $E[w_1] - w_0$:**

a) **The experiment consists of four coin tosses. A participant in the experiment is required to pay an initial stake of £4. In return, the participant receives a payout $v(r) = 2^r$, where *r*, the result of the experiment, is the number of heads in the sequence of four coin tosses.**

Define wealth after participation in the experiment, w_1 and wealth before participation, w_0. We can write the expected value of participation in the experiment, $E[w_1] - w_0 = \sum_{r=0}^{4} 2^r.Pr(n_H = r) - 4$.

Then evaluating the expression, we obtain $E[w_1] - w_0 = 1.0625$.

b) **When summing the values on two dice, say that the participant receives the value if the total is an even number, but has to pay the value if the total is an odd number.**

We now write $E[w_1] - w_0 = \sum_{r=2}^{12} (-1)^r r.Pr(s = r)$. Evaluating this expression, we obtain $E[w_1] - w_0 = 0$.

c) **When multiplying the values on two dice, the participant has to pay a stake of £10 at the start of the experiment, but receives a payout of £25 if the product is greater than 12.**

We obtain $E[w_1] - w_0 \approx -0.97$.

X26.11 **For the sequence of coin tosses described above:**

a) **Calculate the probability of the first *Head* occurring on the t^{th} toss, assuming that the coin is fair, and that each toss forms a distinct experiment (so that successive coin tosses are independent of each other).**

Define the result $H_1 = t$ when the experiment produces a sequence of $t - 1$ tails followed by a head. Then $Pr(H_1 = 1) = Pr(H) = ½$; $Pr(H_1 = 2) = Pr(TH) = ¼$; $Pr(H_1 = 3) = Pr(TTH) = \frac{1}{8}$; ...

We see that there are two parts to the result: the requirement that each sequence consists of tails (which occurs with probability ½ on every coin toss) followed by a single head (which also occurs with probability ½). All coin tosses are independent, so $Pr(H_1 = t) = Pr(T...TH) = 2^{-t}$.

b) **Suppose that the casino offers a payment of 2^{t-1} at the end of each sequence, up to a maximum length *T* = 20. (This means that after a sequence of 20 *Tails*, the casino ends the game, and the gambler loses the initial stake.) Calculate the expected value of the payment the casino would make.**

We demonstrate the argument using this table:

t	1	2	3	t	20
$Pr(H_1 = t)$	2^{-1}	2^{-2}	2^{-3}	2^{-t}	2^{-20}
$p(t)$	2^0	2^1	2^2	2^{t-1}	2^{19}

$p(t).Pr(H_1 = t)$	0.5	0.5	0.5	0.5	0.5

The contribution to the expected payoff from every length of sequence is the product of the probability $Pr(H_1 = t)$ and the payment, $p(t)$, made when $H_1 = t$. For all values of t, this contribution takes the value ½. Paying out on sequences for which $t \leq 20$, the casino's expected payout $E[p(H_1)] = 10$.

c) **Suppose that the casino offers a payment of 2^{t-1}, irrespective of the length of the sequence. Confirm that the expected value of the casino's payout is infinite.**

It is possible that the sequence will never end: that the coin will turn up tails time after time, but the probability of a sequence ending remains constant throughout the sequence, so it is an improbable outcome. We now obtain $E[p(H_1)] = \sum_{t=1}^{\infty} Pr(H_1 = t).P(t) = \sum_{t=1}^{\infty} 0.5$*, which is infinite.*

X26.12 **Assume that $U(W_0) = 0$ and that there is no stake money paid to take part in the gamble. Calculate the utility $U(W(t))$ obtained when the casino pays out at the end of a sequence of length, t. Defining the expected utility as a probability-weighted sum of the utilities obtained from sequences of coin tosses of all possible lengths, write out the first terms as the product of the probability of a sequence occurring and its payoff. Hence confirm that the expected utility may be written:**

$$E[U(W)] = \tfrac{1}{2}\sum_{t=1}^{\infty} 2^{-\frac{t-1}{2}} = 1 + 2^{-\frac{1}{2}} \qquad \textbf{[26.6]}$$

We now adapt the table from X26.11, showing the contribution to utility at the end of each sequence. Instead of using the payoff, we use its utility, $U[p(t)]$.

t	1	2	3		t
$Pr(H_1 = t)$	2^{-1}	2^{-2}	2^{-3}		2^{-t}
$p(t)$	2^{0}	2^{1}	2^{2}		2^{t-1}
$U[p(t)]$	2^{0}	$2^{0.5}$	2^{1}		$2^{\frac{t-1}{2}}$
$U[p(t)].Pr(H_1 = t)$	2^{-1}	$2^{-1.5}$	2^{-2}		$2^{-\left(\frac{t+1}{2}\right)}$

We obtain expected utility $E[U[p(t)]] = \sum_{t=1}^{\infty} 2^{-\left(\frac{1+t}{2}\right)}$. *We recognize this as an infinite geometric series, with initial term ½, and common ratio $\frac{1}{\sqrt{2}}$. Applying the usual formula,* $E[U[p(t)]] = \frac{1}{2}\left(\frac{1}{1-2^{-0.5}}\right) = \frac{\sqrt{2}}{2}\left(\frac{1}{\sqrt{2}-1}\right)$. *Multiplying the numerator and denominator by the factor $1+\sqrt{2}$, the result follows.*

X26.13 **In our analysis, we will assume (1) that participation in a lottery is not a source of utility; (2) that the probability distribution across outcomes is objective, and known; and (3) that all people choosing whether or not to participate in the lottery have the same information available to them. To what extent might we appeal to deviations from such assumptions in seeking to explain why people actively assume risks? [*Note:* You may wish to distinguish between three types of activity: (a) buying lottery tickets; (b) playing a card game, such as poker or bridge; and (c) betting on the outcome of sporting contests, such as horse races.]**

All of these extensions of the model are possible, and should be considered. In a simple gamble, such as buying a lottery ticket, there is no possibility of the exercise of skill or judgment. Within a utilitarian definition, participation may well be pleasurable. In contrast, in most card games, all players have incomplete information. Indeed, the term 'poker-faced' refers to the ability not to transmit signals inadvertently that might contain information of *use to opponents in a game. The strategy of gamblers should depend on the beliefs that they form given the information that they have available to them. In sporting contests, we might argue that in principle all information is available to all observers. Differences in judgment might then not reflect objective factors, but are inferences about the outcome given beliefs: for example, supporters of a team may be more optimistic about its prospects than other people.*

X26.14 **Sketch a diagram showing a utility of wealth function, which is convex. Demonstrate that the expected utility, E[*U*(*W*)], from participation in the lottery, $L = (W_1, W_2, \pi)$ is greater than the utility of certain wealth, *U*(E[*W*]). Compare the differences in attitude to risk between two people, one with concave, and the other with convex, utility of wealth functions.**

On a diagram with wealth on the horizontal axis and the utility of wealth on the vertical axis, the utility of wealth curve will be upward sloping and, moving along it from left to right, becomes steadily steeper, indicating an increasing marginal utility of wealth. Drawing a line connecting two points on the curve, we note that it lies above the curve. There is some value of probability for which every point on the line represents the expected wealth and the expected utility of wealth following participation in a gamble. Since the line lies above the utility of wealth curve, then in every gamble, expected utility exceeds the utility of expected wealth. This person would prefer to take part in a gamble, rather than have a certain wealth, so wishes to assume risk: we might describe this person as risk-loving, comparing their behaviour with the risk-averse behaviour of people who prefer certain outcomes to taking part in gambles with expected outcome equal to the outcome under certainty.

X26.15 **Demonstrate that for the utility function, *U*: $U(W) = a + bW$, then the expected utility, E[*U*(*W*)], from participation in the lottery, $L = (W_0, W_1\ \pi)$, equals the utility of the expected wealth, *U*(E[*W*]).**

We write the expected utility: $E[U(W)] = \pi(a + bW_0) + (1 - \pi)(a + bW_1) = a + b[\pi W_0 + (1 - \pi)W_1]$. *We write expected wealth* $E[W] = \pi W_0 + (1 - \pi)W_1$, *and the required result follows directly, since* $U(E[W]) = a + bE[W]$.

X26.16 **Write down the terms of the contract between *A* and *B* that would ensure that: (a) *A* obtains expected wealth E[*W*] – *c* with certainty; (b) *A* is indifferent between entering into the contract and bearing the risk; and (c) *B*, who loves risk, would be willing to enter into the contract.**

A pays B the full risk premium, $\delta = W_1 - W_\delta$ *when state* $S = 1$ *occurs, but B pays A an amount* $W_\delta - W_0$, *when state* $S = 0$ *occurs.*

X26.17 **For the following situations, calculate the risk premium (the maximum insurance premium that an individual would pay).**

a) **Wealth $W_0 = 1{,}000$, loss $D = 1{,}000$; probability of loss, $\pi = 0.1$; utility *U*: $U(W) = W^{\frac{1}{3}}$.**

The expected utility, $E[U(W)] = \pi U(W_0 – D) + (1 - \pi)U(W_0) = 0.9 * 1{,}000^{\frac{1}{3}}$. *So* $E[U(W)] = 9$. *This person can be certain of obtaining utility* $U(W_0 - \delta) = 9$ *if* $W_0 - \delta = 1{,}000 - \delta = 9^3 = 729$; *so* $\delta = 271$.

b) **$W_0 = 200$; $D = 100$; $\pi = 0.05$; $U(W) = W^{½}$.**

Applying the same argument, the expected utility, $E[U(W)] = \pi U(W_0 – D) + (1 - \pi)U(W_0)$ $= 0.05(200 – 100)^{0.5} + 0.95(200)^{0.5} \approx 13.93$. *This person can be certain of obtaining utility* $U(W_0 - \delta) \approx 13.93$ *if* $W_0 - \delta = 200 - \delta = 13.93\text{^}2 \approx 194.19$; *so* $\delta \approx 5.81$.

c) **$W_0 = 500$; $D = 200$; $\pi = 0.2$; $U(W) = \ln(1 + W)$.**
The expected utility, $E[U(W)] = \pi U(W_0 - D) + (1 - \pi)U(W_0) = 0.2\ln(301) + 0.8\ln(501) \approx 6.11$. So $E[U(W)] \approx 6.11$. This person can be certain of obtaining utility $U(W_0 - \delta) = 6.11$ if $W_0 - \delta = 500 - \delta = e^{E[U(W)]} - 1 \approx 451.4$; so $\delta = 48.6$.

X26.18 **Using diagrams, demonstrate that no one who loves risk will want to buy insurance.**
This follows directly from the argument of X26.14. On a diagram with wealth on the horizontal axis and the utility of wealth on the vertical axis, the utility of wealth curve will be upward sloping and, moving along it from left to right, becomes steadily steeper, indicating an increasing marginal utility of wealth. Drawing a line connecting two points on the curve, we note that it lies above the curve. There is some value of probability for which every point on the line represents the expected wealth and the expected utility of wealth following participation in a gamble. Since the line lies above the utility of wealth curve, then in every gamble, expected utility exceeds the utility of expected wealth. Insuring therefore leads to a reduction in expected utility.

X26.19 **Explain why most health insurance schemes give the general population limited control over whether or not they participate.**
The people who are most likely to wish to obtain health insurance coverage are the people who are most likely to make a claim. During the introduction of the Affordable Care Program in the United States (sometimes called Obamacare), there were claims made that there was a risk of the program entering a 'death spiral' in which healthy people would refuse to participate, pushing up premiums. This would lead to people choosing to leave the system, so that eventually the only people willing to pay the premiums would be so ill that their costs of care would bankrupt the system. To avoid this possibility, most schemes rely on some form of compulsion or very strong incentives, treating scheme membership as an employment benefit, to ensure that a broad cross-section of the population becomes members.

X26.20 **Why might it be more difficult for a market to exist for flood insurance than for fire insurance?**
In our modelling of risks, we have assumed that they are all equal and independent. A house built on top of a hill may face a high risk of subsidence; houses built on a flood plain are likely to suffer losses at the same time. Indeed, since large areas of a country tend to flood at the same time, even if we identify houses that might be at higher risk of flooding because of their location, then we expect insurance claims to occur at the same time. There is therefore limited pooling of risk through insurance.

X26.21 **The European Court of Justice has held that there should be no differences in the motor insurance premiums offered to men and women. On what basis might there be differences? How reasonable is it to make the practice unlawful? Given that equalities legislation recognizes other protected characteristics, such as age, how might insurers determine risk classes without relying on this characteristic?**
We might expect men and women to take different risks. It is arguable that genetically transmitted characteristics are a cause of this difference, but the courts have declined to accept that argument. Rather than conditioning the probability of experiencing an accident on gender, insurers must now rely on other information about the proposer of the policy, relating to behaviour (for example, monitoring devices placed in the car to record performance). To the extent that such information is correlated with gender, it will have the same effect.

X26.22 **Within analysis of bank deposit insurance, we sometimes refer to *double moral hazard*. Explain why the existence of insurance might cause (a) the depositors and (b) the managers of an insolvent bank to take more extreme risks.**

See the answer to X26.1. Customers may seek higher returns, not being concerned how the bank is able to offer them, while banks may deliberately make risky loans, knowing that if they fail they do not have to meet the costs of failure.

X26.23 **Why might a casino offering a game with the structure of the one in the St Petersburg Paradox find it difficult to buy insurance against that risk?**

There is no limit to the potential loss; and so for an insurer only interested in making a profit, the necessary premium rate (as a percentage of the stake) would be very large.

X26.24 **Many people who insure their home contents, their car and even their lives will happily gamble. How would you explain the seeming contradictions in this behaviour?**

There have been many explanations advanced for this. People might simply obtain utility from small gambles, while being averse to large risks. The act of gambling then becomes similar to a form of consumption. A more modern explanation (see Chapter 27) is based on the principle that people are averse to losses, but embrace gains, so that there is an asymmetric response, varying according to the description of the choice environment.

X26.25 **Economists sometimes argue that in many poor countries, by enabling people to pay for high-value goods at the time they need them, informal, often community-based, lending channels are effectively a form of insurance. Explain why people might need to rely on such methods of insurance.**

When people in advanced economies need to borrow money to finance the purchase of durable goods, they have a wide variety of channels – among them bank loans, credit cards, store credit, and hire-purchase agreements. Where people have no formal access to such facilities, informal savings groups emerge, within which there is mutual lending – a well-known form being the tontine or rotating savings and credit association – in which members make regular payments, but take turns to borrow, usually according to their needs.

X26.26 **Consider the utility of wealth function, $U(W) = W^{0.75}$.**

a) **Confirm that the function is concave.**

For concavity, we require $\frac{dU}{dW} > 0$ *and* $\frac{d^2U}{dW^2} < 0$. *Here,* $\frac{dU}{dW} = 0.75W^{-0.25} > 0$ *and* $\frac{d^2U}{dW^2} = -0.1875W^{-1.25} < 0$*, so the conditions are satisfied.*

b) **Confirm that any linear transformation, $V_1 = a + bU$, is also concave.**

Differentiating, $\frac{d^2V_1}{dW^2} = -0.1875bW^{-1.25} < 0$*, so the function is concave in W.*

c) **Confirm that the transformation $V_2 = U^2$ is (i) monotonically increasing; and (ii) convex.**

Differentiating, $\frac{dV_2}{dW} = \frac{dV_2}{dU}\frac{dU}{dW} = 2U * 0.75W^{-0.25} = 1.5W^{0.5} > 0$*, so* $\frac{d^2V_2}{dW^2} = 0.75W^{-0.5} > 0$.

X26.27 **Alex believes that at the end of this time period his wealth, W, will either be 16 or 64. He believes that $\Pr(W = 16) = \Pr(W = 64) = ½$. He considers his utility to take the value $U(W) = W^{0.75}$.**

a) **Calculate his expected wealth, E[*W*], at the end of this time period.**

Expected wealth E[W] = ½[16 + 64] = 40.

b) Calculate his expected utility of wealth, E[*U*(*W*)] at the end of the period, confirming that *U*(E[*W*]) > E[*U*(*W*)].
E[U(W)] = $\frac{1}{2}[16^{3/4} + 64^{3/4}] \approx 15.31$. *U(E[W]) =* $40^{3/4} \approx 15.90$.

c) Suppose that there is a change of circumstances. Alex now believes that his wealth at the end of the period will be *W* = 24, or else *W* = 56, with probabilities as before, Pr(*W* = 24) = Pr(*W* = 56) = ½. Confirm that Alex's utility of expected wealth, U(E[*W*]), does not change, but that his expected utility of wealth, E(*U*[*W*]) increases (from approximately 15.3 to 15.7).
Expected wealth E[W] = ½[24 + 56] = 40, while expected utility, E[U(W)] = $\frac{1}{2}[24^{3/4} + 56^{3/4}] \approx 15.66$. *As before, U(E[W]) =* $40^{3/4}$. *We note that, maintaining the expected wealth, as the outcomes come closer together the expected utility increases.*

X26.28 **Alex's friend Bianca faces almost exactly the same situation as Alex, except that her utility of wealth, *V*: $V(W) = 0.05[U(V)]^2$. Calculate Bianca's expected utility from participation in the lotteries in X26.27b and X26.27c. Confirm that Bianca's expected utility, E[*U*(*W*)], decreases (from 14.4 to approximately 13.4), and that the utility of her expected wealth, *U*[E(*W*)] ≈ 12.6.**
We know that Bianca's and Alex's expected wealth will always be the same, so E[W] = 40, and U(E[W]) = $0.05*40^{1.5} \approx 12.65$. *In the first gamble, Bianca's expected utility, E[U(W)] =* $\frac{1}{2}[0.05*16^{1.5} + 0.05*64^{1.5}] = \frac{1}{40}(4^3 + 8^3) = 14.4$, *while in the second gamble, E[U(W)] =* $\frac{1}{2}[0.05*24^{1.5} + 0.05*56^{1.5}] \approx 13.42$.

X26.29 **Consider the two utility of wealth functions, *U* = *U*(*W*) = $W^{3/4}$, and *V* = $V(W) = 0.05[U(V)]^2$.**

a) Confirm that *V* is a monotonically increasing transformation of *U*.
Differentiating, $\frac{dV}{dU} = 0.1U(W) > 0$, *since U > 0.*

b) Show that the marginal utility, $\frac{dU}{dW}$, is decreasing in *W*, but that $\frac{dV}{dW}$ is increasing in *W*.
The marginal utilities are the first derivatives of the utility functions, U and V, with respect to wealth, W. Differentiating, we obtain the marginal utility, $\frac{dU}{dW} = 0.75W^{-0.25}$, *and*
$\frac{dV}{dW} = \frac{dV}{dU} \cdot \frac{dU}{dW} = 0.1U(W) * 0.75W^{-0.25} = 0.075W^{0.5}$.
Differentiating these marginal utilities a second time, we obtain $\frac{dU}{dW} = -0.1875W^{-1.25} < 0$, *so that the marginal utility is decreasing, while* $\frac{d^2V}{dW^2} = 0.0375W^{-0.5} > 0$, *so that the marginal utility is increasing.*

c) For the lottery, *L* = (*W* – δ, *W* + δ, ½), confirm that expected wealth E[*W*] = *W*. Show that the expected utility E[*U*(*W*)] is decreasing in δ, while the expected utility E[*V*(*W*)] is increasing in δ. Relate these results to the underlying attitude to risk implied by the utility function.
Expected wealth E[W] = ½(W - δ + W + δ) = W.
Expected utility, E[U(W)] = $\frac{1}{2}[(W - \delta)^{0.75} + (W + \delta)^{0.75}]$. *Partially differentiating this expression with respect to δ, we see that*
$\frac{\partial E[U(W)]}{\partial \delta} = \frac{1}{2}\left[0.75(W+\delta)^{-0.25} - 0.75(W-\delta)^{-0.25}\right] = \frac{3}{8}\left[(W+\delta)^{-0.25} - (W-\delta)^{-0.25}\right] < 0$, *since W + δ > W - δ, so that as δ increases, the expected utility decreases.*
Expected utility, E[V(W)] = $\frac{1}{2}[0.05(W - \delta)^{1.5} + 0.05(W + \delta)^{1.5}]$. *Partially differentiating this expression with respect to δ, we see that*

$\frac{\partial E[U(W)]}{\partial \delta} = \frac{1}{40}\left[1.5(W+\delta)^{0.5} - 0.75(W-\delta)^{0.5}\right] = \frac{3}{80}\left[(W+\delta)^{0.5} - (W-\delta)^{0.5}\right] > 0$, *and expected utility increases in* δ. *We associate a concave utility function with risk aversion, so that the greater the spread of the possible outcomes, the lower the expected utility will be; while we associate a convex utility function with risk-loving behaviour, with the expected utility increasing with the spread of possible outcomes.*

X26.30 **Show that where indifference curves are convex, the person is risk-averse.**
Here, we simply consider a diagram, with wealth received on the horizontal axis and utility on the vertical axis. For a concave utility function, which is increasing, but at a decreasing rate, the graph of utility of wealth is an upward-sloping curve, which becomes flatter as we move from left to right. Note that a line joining two points on the utility of wealth curve lies entirely below the curve. For every point on this line, we can find some probability distribution, ϕ, *for which the point represents the expected wealth and the expected utility from participation in the lottery* (W_0, W_1, ϕ, $1 - \phi$). *We see that the utility of expected wealth is then greater than the expected utility of wealth.*
An alternative argument would be to carry out the exercise that we have just performed. We consider a second lottery ($W_0 + \delta$, $W_1 - \delta$, ϕ, $1 - \phi$). *This lottery has the same expected wealth, but the line connecting the two points on the utility of wealth curve lies above the first one. The expected utility has increased. Reducing the risk associated with the lottery, the expected utility of wealth increases.*

X26.31 **Suppose that the risk of loss increases, so that the probability of the better outcome, π, falls from π_0 to π_1. On a diagram, illustrate the impact of this change on: (a) the slope of the fair-odds line; and (b) the gradient of the indifference curves where they intersect the certainty line.**
As the probability π *increases,* $1 - \pi$ *decreases, so that the ratio* $\frac{\pi}{1-\pi}$ *increases; the fair odds line becomes steeper, as must the indifference curve that is tangent to the fair odds line, where they meet on the certainty line.*

X26.32 **Explain why no insurer would offer an insurance policy with premium rate ρ: $\rho < 1 - \pi$, the risk of loss. Discuss whether or not it is reasonable to expect an insurer to offer a policy for which the premium rate, ρ: $\rho > 1 - \pi$.**
A premium rate less than the risk of loss would expect to lose money, and so would not be offered. A premium rate higher than the risk of loss would only emerge if the insurer had some degree of market power.

X26.33 **Suppose that the involuntary lottery, *L* = (400, 100, 0.9), and that the utility of wealth, $U(W) = W^{½}$.**

a) **Calculate the expected utility after participation.**
The expected utility $E[U(W)] = 0.9*400^{0.5} + 0.1*100^{0.5} = 19$.

b) **Calculate the risk premium, δ, the certainty equivalent, $W_0 - \delta$, and the wealth when it is possible to buy full, fair insurance.**
If it were possible to have certain wealth, $W = 400 - \delta$, *offering utility* $U(W) = 19$, *then* $W = 361$, *which is the certainty equivalent. With fair, full insurance, it should be possible to pay a premium,* $p = 30$, *and obtain certain wealth* $W = E[W] = 370$. *Defining the risk premium as the difference between the expected wealth and the wealth offering the same utility, we obtain a risk premium,* $\delta = 9$.

c) **Obtain the equation of the fair-odds line.**

Suppose that there is fair insurance. The insured wealth profile (W_1, W_2) satisfies the condition $0.1(W_2 - 100) + 0.9(W_1 - 400) = 0$; so that the premium paid is equal to the payment made when state 2 occurs. Then $9W_1 + W_2 = 3,700$

d) Confirm that the gradient of the indifference curve passing through the intersection of the certainty line and the fair-odds line is the slope of the fair-odds line.

The gradient of the fair-odds line, $\frac{dW_2}{dW_1} = -9$.

Writing the expected utility, $E[U(W)] = 0.9U(W_1) + 0.1U(W_2)$, then taking the partial derivatives with respect to W_1 and W_2, $\frac{\partial E[U(W)]}{\partial W_1} = 0.9\frac{dU}{dW_1}$ *; and* $\frac{\partial E[U(W)]}{\partial W_2} = 0.1\frac{dU}{dW_2}$*, so that we obtain the marginal rate of substitution, MRS =* $-\left(\frac{\partial E[U(W)]}{\partial W_1} \Big/ \frac{\partial E[U(W)]}{\partial W_2}\right) = -\frac{0.9\frac{dU}{dW_1}}{0.1\frac{dU}{dW_2}} = -9\frac{\frac{dU}{dW_1}}{\frac{dU}{dW_2}}$*. Then when $W_1 = W_2$,* $\frac{dU}{dW_1} = \frac{dU}{dW_2}$ *and the result follows.*

e) Assume that it is possible to purchase an insurance policy for any sum, *B*, at the fair premium rate, $\rho = 1 - \pi$. Write down an expression for the expected utility on purchasing insurance, *B*. Confirm that the purchase of full insurance maximizes utility.

The expected utility, $E[U(W)] = 0.9(400 - 0.1B)^{0.5} + 0.1(100 + 0.9B)^{0.5}$. Partially differentiating the expected utility with respect to the value, B,

$\frac{\partial E[U(W)]}{\partial B} = -0.045(400 - 0.1B)^{-0.5} + 0.045(100 + 0.9B)^{-0.5} = 0$*, for a maximum. Then $100 + 0.9B = 400 - 0.1B$, and $B = 300$. Then paying a premium, $\pi = 0.1B$, we can confirm that in both states, wealth $W = 370$, and there is full insurance.*

f) Suppose that the premium rate, r > 1 – p. Rework the calculations in part (e) and explain when it will be optimal to purchase partial insurance. Illustrate your conclusions.

With a premium rate of 15% of the sum insured, the expected utility, $E[U(W)] = 0.9(400 - 0.15B)^{0.5} + 0.1(100 + 0.85B)^{0.5}$. Partially differentiating the expected utility with respect to the value, B,

$\frac{\partial E[U(W)]}{\partial B} = -0.0675(400 - 0.15B)^{-0.5} + 0.0425(100 + 0.85B)^{-0.5} = 0$*, for a maximum. Then, $17(400 - 0.15B)^{0.5} = 27(100 + 0.85B)^{0.5}$ and $289(400 - 0.15B) = 729(100 + 0.85B)$; we can rewrite this as $663B = 42,700$, so that $B = 64.4$, which is much less than the full insurance sum.*

We illustrate the best market odds line as being flatter than the fair odds line, as in Figure 26.5.

X26.34 **Suppose that competition in the insurance industry means that every insurance company expects to make zero profits. Sketch a diagram illustrating the fair-odds lines where the probability of loss: (1) for the lowest-risk person is 0.05; (2) for the average person is 0.2; and (3) for the highest-risk person is 0.5. Given that they all can buy insurance at the same premium rate, explain how the coverage they purchase will differ.**

In a diagram with wealth in the good state on the horizontal axis and wealth in the bad state on the vertical axis, we draw in the fair odds lines for each of these consumers as having different slopes. For (1), the slope of the line, $-\frac{\pi}{1-\pi} = -19$*, where π is the probability of the good state, and $1 - \pi$ is the probability of loss. For (2), the slope of the fair odds line will be -4; and for (3) it will have slope -1. Assuming that there is a single policy marketed, which is fair for a person of type (2), we expect a person of type (1) to buy partial insurance, a person of type (2) to buy full insurance, and a person of type (3) to overinsure against loss, ending up better off when there is a loss than when there is not, assuming that the insurance policy allows for this.*

Chapter 27

X27.1 **Suppose that Anthony is choosing a new car. (We shall assume that he has made the decision to buy a car, but has still to choose between the variety of models available.) To what extent might we expect the choice to be a result of conscious consideration? To the extent that information is costly to acquire and to process, how might this affect his decision?**

A car is a high-value consumer durable good, from which Anthony will wish to derive value over a long period of time. Not only will the costs of making the wrong decision be high, but they will persist over time. We argue, though, throughout this chapter that even though we might expect this to be a choice where Anthony will seek to use conscious consideration, there will be many factors that limit its application.

X27.2 **Suppose that Britney smokes regularly. What differences might there be between making the decision: (a) to have the first cigarette of a particular morning; and (b) to stop smoking?**
We can think of choosing to have the first cigarette of a day as simply a matter of timing. 'I will stop smoking,' is an attempt to impose a rule on behaviour. Effectively, we set participation in this activity to zero, when it may have taken a positive value every day for many years. This suggests that something has changed to make stopping smoking a rational choice; yet if that were to be the case, it should not be a difficult decision to make, unless we rely on some account that involves habituation, or, in activities such as smoking, addiction.

X27.3 **We introduced Clive, a large white rat, to the Skinner box yesterday. He now happily presses the lever, gobbling up his rewards. Explain how you think his behaviour will change – that is, what he will learn to do – as the reinforcement (reward) changes in the ways suggested above.**
We expect Clive to be able to understand such changes to the system quite well. He will begin to press the lever more frequently when the green light is on; or press the lever and immediately retrieve the pellet; or press the lever several times in succession (but reduce the frequency with which he engages in the activity because the reward rate has reduced), or press the lever in bursts. (The last one, because there is a negative feedback loop so that the rate of reinforcement falls with activity, represents something like habituation, which can lead to very high levels of activity that are self-defeating.)

X27.4 **We have only discussed positive reinforcers in the example above (the delivery of food). Within the analysis of operant conditioning, there are also roles for negative reinforcers, effectively punishments for behaviour. Giving possible examples of these, explain their role.**
One variant of the Skinner box is to place a metal grid in the base, through which an electric current might be passed. Animals learn to avoid the behaviour that leads to their experiencing such negative reinforcement or reward.

X27.5 **In a *variable ratio* reinforcement schedule, reinforcement occurs after a randomly determined number of actions, with the expected frequency of reinforcements set in advance. How might conditioned subjects behave in this case?**
With VR reinforcement, we expect animals to behave as if they understand that there is a fixed probability of receiving reinforcement with each action, so that the expected payoff to the action is reduced. Assuming that animals will require the situation to satisfy some sort of incentive compatibility constraint, we assume that the expected reward from a push is

greater than its cost; and that the greater the expected surplus, the more the animal will engage in the behaviour, so the greater the frequency of pushes. This suggests that the frequency of engagement in an activity where there is a VR reward will depend upon both the size and certainty of the reward.

X27.6 **How might we expect experimental subjects to respond during the *extinction* phase of an experiment, when they are still in the box, but reinforcements are no longer available?**
Thinking of Clive, who is now accustomed to pushing a lever in order to obtain food, when the food stops being delivered, we do not expect him simply to stop pressing the lever. Just as it takes time for him to learn about the environment, it takes time for him to adapt to changes in the reinforcement schedule. Stopping the food is just a severe form of VR (or the related variable interval ratio, where food will not be delivered for a set time after each delivery) where the ratio and interval extend towards infinity.

X27.7 **Consider 'Example: petrol stations in a city' (Section 1.2.4). There is substantial competition in supply, and across petrol stations fuel is a perfect substitute. The standard model predicts a single market price.**

a) Use a price comparison website (e.g. http://www.fleetnews.co.uk/static/tools/compare-fuel-prices/) to find out the extent of variation that occurs in the city in which you live, or one nearby.
I carried out this exercise repeatedly while writing this book, finding that in Edinburgh, the range of prices was generally ±2% of the median price; or, given that about 60% of the price of fuel in the UK is simply taxes, roughly ±5% of the median price received by retailers.

b) Discuss how our analysis might change if we treat buying fuel as a learned behaviour.
If we treat buying fuel as a learned behaviour, then we should start to think of it as being rather more like the foraging activity that we have described for animals. People will tend to go back to a site where they have found the result of their activity to be satisfying. They are likely to avoid conscious consideration. For example, the difference in price of filling my own car's fuel tank between any pair of petrol stations in Edinburgh is very unlikely to be more than 4% of the total price that I would normally pay – rather less than £2.00. Say that I value my time at £30 per hour; then driving for four minutes would have greater cost than the saving that I would make from cheaper fuel. This simple cost-benefit analysis suggests that I am likely to buy my fuel at the most convenient place, rather than the cheapest source.

c) We have suggested that people are likely to treat the choice environment as relatively stable. Suppose in this case that they will only carry out local experiments (buying from another petrol station) if there is some relevant change to the environment. Consider the possible effects on behaviour of: (i) an increase in the price charged by the most frequently used petrol station; (ii) a decrease in that petrol station's price; (iii) an increase in the price charged by a competing petrol station; and (iv) a decrease in the competing petrol station's price.
An increase in the price charged at a frequently used petrol station is likely to be noticed relatively quickly, and since it is an additional cost it seems most likely to trigger search activity. A reduction in price seems likely to have a weaker effect because it is a benefit to the purchaser. In the same way, an increase in price at a petrol station that is rarely used seems likely to have a minimal effect on search, since it is unlikely to be noticed. While much the same argument might be made about a price reduction at a rarely used petrol station, since the reduction might be a way of attracting more customers it is possible that the price cutter will advertise the fact in order to stimulate search activity.

d) **What might we conclude about the price elasticity of demand when there are: (i) price increases; and (ii) price decreases? What does this suggest about the marginal revenue function around the current level of sales?**
A price increase is likely to have a greater effect on demand than a price decrease. Demand is more elastic when the price rises than when it falls. Defining the marginal revenue, MR, in terms of the inverse demand p(q) and the price elasticity, ε_p, we write $MR(q) = p(q)\left[1 + \frac{1}{\varepsilon}\right]$, so that if the elasticity of demand is different for a price increase than for a price cut, the marginal revenue function will have a discontinuity, being greater for the price increase than for a decrease.

X27.8 **Suppose that the petrol station faces (a) an increase in its costs and (b) a small decrease in its costs, so that $MC_1 < MC(Q_0) < MC_0$. Describe its profit-maximizing behaviour in each case.**
In both cases, the condition $MR(Q_0) = MC(Q_0)$ is satisfied. The firm does not change its output, and so prices are sticky, and they do not change in response to small changes in input costs.

X27.9 **For a person with utility, $U : U(B, C) = B^{0.5}C^{0.5}$, where utility is derived from consumption bundle, (B, C), and where choice is defined by the affordability constraint $B + C = 20$, and the payoff constraint $U(B, C) \geq 8$, sketch a diagram indicating the set of acceptable combinations.**
On a diagram with consumption of good B measured on the horizontal axis and consumption of good C measured on the vertical axis, the affordability constraint, C = 20 – B is a downward-sloping line with gradient 1, and intersects the vertical axis at (0, 20) and the horizontal axis at (20, 0). The indifference curve, which forms the acceptability constraint, has equation BC = 64, and so is a rectangular hyperbola, passing through (8, 8) with MRS(8, 8) = -1, and with the axes forming asymptotes to the curve. We omit the calculations, but it is easy to check that the consumption bundles (4, 16) and (16, 4) lie on both the acceptability and the affordability constraints. The lens with these end points defines the set within which local experimentation will stop.

X27.10 **Using the example in X27.9, suppose that this person initially chooses the consumption bundle (B, C), for which the affordability and payoff constraints are met, but do not bind. Making a sequence of choices, this person gradually adjusts the consumption bundle.**

a) **Given an adjustment cost of 0.05, in terms of payoff, with consumption of good *B* increasing by 0.1 units on every occasion, explain why this consumer might not reach the affordability constraint.**
We define the marginal utility of consumption, $MU_B = \frac{\partial U}{\partial B} = \frac{1}{2}\left(\frac{C}{B}\right)^{\frac{1}{2}}$. Suppose that we start from a consumption bundle in the acceptable, affordable set for which $C = \frac{B}{3}$. Then $\frac{\partial U}{\partial B} = \frac{1}{2\sqrt{3}}$, and for an increase in consumption of good B of dB = 0.1 units, $dU \approx dB\frac{\partial U}{\partial B} \approx \frac{1}{20\sqrt{3}} \approx 0.028 < 0.05$. The cost of the experiment exceeds the benefits, and the search stops.
More generally, we may note that if C = cB, and $\frac{\partial U}{\partial B} = \frac{\sqrt{c}}{2}$, then the change in utility, dU, from an increase in consumption, dB = 0.1, will be $dU \approx 0.1\frac{\partial U}{\partial B} = \frac{\sqrt{c}}{20}$, and so if c > 1, dU > 0.05, and the experiment continues. In effect, for any initial choice, $C_0 < 10$, the experiment is likely to stop short of the boundary.

b) **Repeat the argument of part (a) for consumption of good *C*.**

The argument is almost identical, and so we conclude that if the initial choice $B_0 < 10$, then experimentation will stop before reaching the boundary.

c) Given an adjustment cost of 0.1, the consumer experiments by increasing consumption of both goods by one unit. Discuss whether or not this consumer will continue experimenting until income is exhausted.

The increase in utility as the result of a one unit increase in consumption of both goods, $dU \approx \frac{\partial U}{\partial B} + \frac{\partial U}{\partial C} = \frac{1}{2}\left[\left(\frac{C}{B}\right)^{\frac{1}{2}} + \left(\frac{B}{C}\right)^{\frac{1}{2}}\right] = \frac{B+C}{(BC)^{0.5}}$. If we write $C = cB$, then $dU \approx \frac{1+c}{\sqrt{c}} = c^{-0.5} + c^{0.5}$, and we can verify that dU is minimized when c = 1, so that $dU \geq 2$. We can therefore be certain that the benefit from the experiment will exceed its cost, so that the consumer will reach the affordability constraint.

X27.11 **Discuss the range of outcomes that might emerge in this example. Define the range of initial choices that will lead to the consumer reaching (at least approximately) the optimum, B^*, and those that might lead (approximately) to the local maximum, B_1.**

We note that permitting local examples, there are two values, B^* and B_1, which are local optima, but that B^* supports the global optimum. However, allowing only local experiments, we expect anyone initially choosing $B > B_0$ is likely to end up at B_1, which is the inferior outcome.

X27.12 **What elements of habituation does this model possess, which would not emerge in the standard model?**

We allow only local experiments, so that migration is to a local optimum, rather than a global optimum.

X27.13 **Standard leases over a residential property run for one year, but can then be extended on a monthly basis without a further agreement being signed.**

a) Discuss the extent to which conscious consideration might guide choice: (i) when first entering into a lease; and (ii) when deciding whether or not to terminate a lease on its anniversary.

When first entering into a lease, we have to make a conscious choice among the various alternatives that are open to us. Most people will view several properties, but we do not have the chance to experience what it is like to live in the house before signing the lease (see Chapter 28 for further discussion of associated problems that result from this). We are likely to be guided by fashion, by what friends have chosen to do, even by advice from parents and relatives, and so while we may have a perfectly good rationale for what we do, it seems very likely that we will conclude with some sort of satisficing behaviour.

Once the lease is signed, we require a conscious decision to terminate it. The status-quo is to allow the lease to continue, and this becomes the basis for our comparison of alternatives.

b) What do you consider to be the transaction costs associated with moving house? Estimate their size, relative to the monthly rental that you would have to pay.

Moving house has quite substantial costs. We might expect that even moving between furnished apartments, it will take two days to pack up personal belongings, transport them and arrange the house as we wish. Then there will be other arrangements – transferring utility accounts, insurance and paying local taxes, cleaning the property that we are leaving, and of course the business of selecting the new apartment. The costs might be considered to be in the order of £1,000 or higher, and so will be a substantial proportion of any saving in rental from moving between apartments.

X27.14 **Pensions are long-term saving contracts. Governments in many countries wish to encourage greater private provision. Discuss measures that might be taken to make increased participation become the default option. Explain the importance to these of status-quo bias.**

In recent years, this has been the subject of much interest among policymakers, especially through the work of Richard Thaler, who with Cass Sunstein has written 'Nudge,' which is a relatively accessible introduction to research in this area. Thaler's thesis is that government and policymakers should shape the choice environment that people face in order to guide them towards what is considered to be the socially desirable outcome. In the case of pensions this can be done, for example, through a mixture of compulsion and subsidy, such as requiring employers to enrol all staff in pension schemes, with the government paying some of these costs. As long-term savings contracts, pension schemes have many other complications. For example: should it be possible for people to have access to pension savings before retiring? How do we encourage people to choose an investment profile that will actually allow them to generate the income in retirement that they claim to wish? And what should be the balance between private and public provision of retirement benefits?

X27.15 **In the following examples, discuss whether you think that there is evidence of dissonant cognitions, and how you think people might resolve them.**

a) **(i) 'Smoking kills.' (ii) 'I enjoy smoking.'**

Smoking tobacco is probably the most widely practised risky behaviour globally at the present time, with regular users sacrificing up to 15 years of expected good health (quality-adjusted life-years). The remoteness and uncertainty of the consequences from the practice, and the distribution of consumption over time allows smokers to ignore the consequences and to defer taking action.

b) **(i) 'Theft is wrong.' (ii) 'I could use these office supplies at home.'**

Given the former belief, we need to persuade ourselves that the use of office supplies is not theft, either because of the low value, or by arguing that it is an implicit part of workplace compensation.

c) **(i) 'We need to reduce our carbon footprint.' (ii) 'I could never be a vegetarian.'**

Given the former belief, it is desirable to ignore the requirement that substantial quantities of energy are produced in the production of meat, and that switching from meat consumption to other sources of nutrition might reduce our carbon footprint.

d) **(i) 'You can never be too cautious.' (ii) 'Wait until I win the lottery.'**

We might argue that the first statement is meaningless. We will introduce a theory later in the chapter based on loss aversion, in which people are willing to enter into gambles where they consider that there will only be gains.

X27.16 **For each question, estimate the smallest and largest reasonable values for the answers, and also a single point estimate.**

a) **In 1603, Queen Elizabeth of England was succeeded by James VI, King of Scotland. During Elizabeth's reign, Spain assembled a large fleet, the Armada, with which it intended to invade England. In what year did the Armada sail?**

1588.

b) **Between February 1931 and September 1937, Malcolm Campbell broke the world land speed record five times. He reached a speed of 246 mph in February 1931. What speed did he achieve in September 1937?**

301mph (he was the first person to reach an average speed in excess of 300mph (483km/h)).

c) **The Star of Africa, which weighs 530 carats, was for many years the largest cut diamond in the world. It is only one of 104 diamonds cut from the Cullinan diamond. What is the weight (in carats) of the second largest diamond?**
317 carats

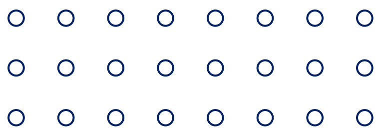
d) **The orbital period of Mercury (the planet closest to the sun) is 88 days. What is the orbital period of Venus (the planet second closest to the sun)?**
225 years

e) **In 1066, William I won the Battle of Hastings and secured the English throne. William undertook a comprehensive survey of landholdings, recorded in what is known as the Domesday Book. In what year was the survey completed?**
1086

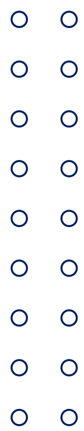
f) **At its completion in 1311, Lincoln Cathedral had the highest spire in the world. After its collapse in 1549 and partial rebuilding, the spire is now 83 m high. What was its original height?**
It is commonly estimated to have been 160m; but this is not clearly documented.

g) **The Greek philosopher, Socrates, was executed in 399 BC. When was his pupil, Plato, born?**
Unknown, but the traditional estimate is 428 BC.

X27.17 **Consider these two structures, *A* and *B*. A path in a structure consists of one element in each row, from the top to the bottom.**



Structure A



Structure B

Without attempting calculations, state whether you believe the number of paths to be greater in structure *A* or in structure *B*; and then estimate the number of paths in each structure.
In every row in structure A, there are $8 = 2^3$ possible choices, so that there are $(2^3)^3 = 512$ paths. In structure B, in every row, there are two possible choices, so that with 9 rows, there are $2^9 = 512$ possible choices. The appearance of the diagram means that most people incorrectly conclude that there are more paths in structure A than in structure B.

X27.18 **There are 10 intermediate stops on a bus route between two cities. Services can stop at any combination of *r* out of the 10 stops.**

START	1	2	3	4	5	6	7	8	9	10	TERMINUS

For r = 2, 3, 4, 5, 6, 7 and 8, without carrying out any calculations, estimate the number of service patterns that are possible.

There is a simple formula for N_r the number of service combinations with r stops:

$N_r = \frac{10!}{r!(10-r)!}$. *In this expression n! = 1*2*3* ...*n, the product of the first n natural numbers.*

We obtain

r	2	3	4	5	6	7	8
N_r	45	120	210	252	210	120	45

It is probably easiest to estimate the number of alternatives when r = 2; so people tend to underestimate these values.

X27.19 **Consider structure *C*, below. We again define a path as consisting of one element from each row.**

X	O	X	X	X	X
X	X	X	X	O	X
X	X	O	X	X	X
X	X	X	O	X	X
X	X	X	X	X	O
O	X	X	X	X	X

Structure *C*

Estimate, without any calculations, for r = 0, 1, 2, 3, 4, 5, and 6, the percentage of paths that pass through r 'O' nodes and 6 – r 'X' nodes. [*Note:* The sum of your estimates should be 100.]

In each row there is a single 'O' node. In each row, the probability of choosing an 'O' node is $\frac{1}{6}$. The probability of choosing a path with r of these 'O' nodes is then the probability of any path having exactly r such nodes multiplied by the number of these paths. There is a standard formula that we can apply in this case: Pr(N(O) = r) = $\frac{6!5^{(6-r)}}{r!(6-r)!6^6}$. Applying this, we obtain the following percentage distribution of paths:

r	0	1	2	3	4	5	6
Pr(N(O) = r)	33.49%	40.19%	20.09%	5.36%	0.80%	0.06%	0.00%

If we think of the probability of an 'O' as being the probability of obtaining a 6 on rolling a single die, then we are counting the number of sixes when rolling six dice simultaneously.

X27.20 **Now consider a situation in which 6 players are each dealt a single card from a deck in which $\frac{1}{6}$ of the cards are marked 'O' and $\frac{5}{6}$ are marked 'X'. In a long sequence of deals, estimate, without any calculations, for r = 0, 1, 2, 3, 4, 5 and 6, the percentage in which r players receive 'O' cards and 6 – r receive 'X' cards. [*Note:* Once again, the sum of your estimates should be 100.]**

We note that this is formally the same problem but, being described so that attention is drawn to the probabilities, people tend to analyse the problem in terms of these probabilities.

X27.21 **Suppose that we consider a sequence of nine tosses of a fair coin. Rank the following outcomes by probability of occurrence:**

a) H H H H H H H H H

b) H H H H H H T T T T
c) T H T H T H T H T
d) T H T T T H T H H

On every coin toss, there is a probability Pr(H) = Pr(T) = 0.5. So all of these occurrences are equally probable (and the probability, π, of any occurrence is then $\pi = 2^{-9}$).

X27.22 **Consider the following pairs of lotteries. [*Note:* Remember our definition of a lottery in terms of a set of outcomes, and a probability distribution over these.]**

- **A: L_{A1} = (2,500, 2,400, 0; 0.33, 0.66, 0.01) and L_{A2} = (2,400; 1).**
- **B: L_{B1} = (2,500, 0; 0.33, 0.67) and L_{B2} = (2,400, 0; 0.34, 0.66).**

Decide whether you would prefer the opportunity to participate in lottery A1 or in lottery A2. Repeat this for pair B. Briefly explain your choice.

It is important to remember that in these questions, there is no right answer. We argue below that it is consistent with the standard von Neumann-Morgenstern axiomization for people who prefer L_{A1} to L_{A2} to prefer L_{B1} to L_{B2} (or to reverse both preferences).

X27.23 **Review the von Neumann-Morgenstern axiomization in Section 26.3.**

a) Calculate the expected value of participation in lotteries, L_{A1}, L_{A2}, L_{B1}, and L_{B2}.

*$E[v(L_{A1})]$ = 2,500*0.33 + 2,400*0.66 = 2,409; $E[v(L_{A2})]$ = 2,400.*
*$E[v(L_{B1})]$ = 2,500*0.33 = 825; $E[v(L_{B2})]$ = 816.*
In both cases, we see that the expected value of lottery 1 is 9 greater than the expected value of lottery 2.

b) Assume that $u(0) = 0$.

i) Suppose that lottery L_{A2} is preferred to lottery L_{A1}. What condition must hold between u(2,500) and u(2,400)?

We require $E[u(L_{A1})]$ = 0.33u(2,500) + 0.66u(2,400) < u(2,400), or that 0.33u(2,500) < 0.34u(2,400).

ii) Repeat part (i), given that lottery L_{B1} is preferred to lottery L_{B2}.

We require $E[u(L_{B1})]$ = 0.33u(2,500) > 0.34u(2,400) = $E[u(L_{B2})]$.

c) Discuss the differences between lotteries L_{A1} and L_{B1}; and between L_{A2} and L_{B2}.

There is a probability Pr(W_{A1} = 2,500) = 0.33; and a probability Pr(W_{A1} = 0) = 0.01 in lottery L_{A1}, where these events have zero probability in lottery L_{A2}.
There is a probability Pr(W_{B1} = 2,500) = 0.33; and a probability Pr(W_{B1} = 0) = 0.67 in lottery L_{B1}, where the former event has zero probability in lottery L_{A2}, and the probability Pr(W_{B2} = 0) = 0.66. The changes in the probability distributions across lotteries have the same pattern.

d) Many people prefer L_{A2} to L_{A1}, but L_{B1} to L_{B2}. Discuss whether or not this is likely to be consistent with the axiom of the independence of irrelevant alternatives.

In both of the lotteries in case A, we can omit the probability of 0.66 of obtaining the payoff of 2,400 as a common element. Lottery L_{A1} simplifies to a gamble between receiving 2,500 with probability $\frac{33}{34}$ and 0 with probability $\frac{1}{34}$. Lottery L_{A2} remains receipt of 2,400 with certainty.
We can similarly subtract the probability of 0.66 of receiving zero from both lotteries in case B, confirming that the differences between these pairs of lotteries are identical, so that by the axiom of IIA, preferences across the pair should also be identical.

X27.24 **Consider the following situations:**

- **C: L_{C1} = (4,000, 0; 0.8), L_{C2} = (3,000; 1); and D: L_{D1} = (4,000, 0; 0.2), L_{D2} = (3,000, 0; 0.25).**

- **–C: $L_{–C1}$ = (–4,000, 0; 0.8), $L_{–C2}$ = (–3,000; 1); and –D: $L_{–D1}$ = (–4,000, 0; 0.2), $L_{–D2}$ = (–3,000, 0; 0.25). [*Note:* Here the lotteries necessarily offer losses.]**
- **E: L_{E1} = (6,000, 0; 0.45), L_{E2} = (3,000, 0; 0.9); and F: L_{F1} = (6,000, 0; 0.002), L_{F2} = (3,000, 0; 0.001).**
- **–E: $L_{–E1}$ = (–6,000, 0; 0.45), $L_{–E2}$ = (–3,000, 0; 0.9); and –F: $L_{–F1}$ = (–6,000, 0; 0.002), $L_{–F2}$ = (–3,000, 0; 0.001).**

a) **Without any formal evaluation, decide whether or not you would prefer to participate in lottery 1 or lottery 2. Set out your reasoning.**
Again, we do not have a right or a wrong answer. We note that there is a much higher probability of receiving a positive payoff in both lotteries of type C than of type D; or of losing more in lotteries of type C than of type D. We also note that the expected payoffs across lotteries in situation E and (separately) in situation F are equal.
It may seem tempting to choose the certainty in L_{C2}, but to try to win the larger prize with the slightly smaller probability in L_{D1}. Facing losses, the reverse might be the case.

b) **What do you consider to be the best anchor for making a decision in each situation?**
If we take the anchor as being the most probable outcome, then in situation C the anchor is the certain payoff of 3,000, but in situation D the anchor is zero. Note that this means thinking of lottery L_{C1} as a choice between a loss and a gain, but the lotteries in situation D as involving only gains (and no losses). The reverse might be the case in situations –C and –D. In situation E, the gain of 3,000 might form an anchor, but in situation F, the payout of zero seems likely to take on that role.

c) **Comparing situations C and D, what combinations of lotteries are inconsistent with maximizing expected utilities?**
In all of these cases, people should either prefer the first lottery to the second in both situations C and D or the second to the first (or be indifferent between them). We would also expect consistency of choices across –C and –D.

d) **Comparing situations E and F, what can you say about the preferences of someone who is risk-neutral?**
Anyone who is risk-neutral should be indifferent between each pair of lotteries – both in the case of gain and in the case of losses.

e) **Considering your own answers to part (a), what evidence can you now see of: (i) preference reversals; and (ii) differences between valuations of gains and losses of the same size?**
See the text for discussion of typical patterns.

X27.25 **In situations A–F, H and J which lotteries are regular?**
Lotteries A and B only involve gains, so are not regular (as written). Note that we can think of lottery L_{A1} as regular if we treat a gain of 2,400 as the anchor value. The lotteries in C, D, E and F only have two outcomes, so cannot be regular. Like the lotteries in A, the lotteries in situation H can be made regular by treating the outcome with the greatest probability as the anchor.

X27.26 **We also define a positive lottery, L: $x_1 > x_2 > 0$, and $p_1 + p_2 = 1$. Assume that we edit a positive lottery by extracting the certain gain, x_2. Write down an expression for the value of the lottery, $V(L)$, in terms of the certain gain and the perceived value of the possible further gain.**
$V(L) = v(x_2) + V(L_2)$, where lottery, $L_2 = (x_1 – x_2, 0, p_1, 1 – p_1)$.

X27.27 **Define a negative lottery, using the definition in X27.26 as a guide, and then write down an expression for its value.**
In a negative lottery, M: $-x_1 < -x_2 < 0$, and $p_1 + p_2 = 1$*, we can edit the lottery by removing the certain loss,* $-x_2$*. We can then write its value,* $V(M) = v(-x_2) + V(M_2)$*, where lottery,*
$L_2 = (-x_1 + x_2, 0, p_1, 1 - p_1)$.

X27.28 **Suppose that you are offered the choice of participation in the lotteries in situations K and M:**

L_{K1}: (900, 0; 0.25, 0.75); L_{K2}: (600, 300, 0; 0.25, 0.25, 0.5);
L_{M1}: (–900, 0; 0.25, 0.75); L_{M2}: (–600, –300, 0; 0.25, 0.25, 0.5).

Making your choice, given the constraints in Expressions 27.2 and 27.3, in which lotteries would you prefer to participate?
Note that in this question you should make a rational choice in the sense that you adopt the payoff rules set out in the text. In lottery L_{K2}*, there is a larger probability of making smaller gains than in lottery* L_{K1}*, but the expected value of participation is the same. Remembering that there is a common component of a gain of 0 with probability 0.5, the choice between lotteries might be reduced to* L_{K3} = (900, 0; 0.5, 0.5) *and* L_{K3} = (600, 300; 0.5, 0.5)*. The reverse holds in situation M: so we are likely to prefer the smaller probability of a larger loss, preferring to participate in lottery* L_{M1}.

X27.29 **Suppose that you are offered the choice between participating in the lotteries in situation N:**

L_{N1}: (200, –200; 0.5, 0.5); L_{N2}: (100, –100; 0.5, 0.5)

Which would you choose, assuming that your valuations abide by these constraints?
In both lotteries there is an equal probability of gain and loss, and gains are equal and opposite to losses. Applying the assumption of loss aversion, in such a situation, we prefer to face smaller gambles and choose L_{N2}.

X27.30 **Explain how non-linearities in the weighting function might help to explain the differences in preferences over the lotteries in situations E and F in X27.24.**
Given the non-linearities in the weighting function, we tend to overestimate the probability of events that have very low probability (and underestimate all other weights). This means that in situation E we tend to place more weight on the difference in probabilities in lottery L_{E1}*, where the probabilities are much larger than in lottery* L_2*. This suggests that people will place a weight on a payoff occurring with a probability of 0.9 that is around twice as large as when that payoff occurs with probability 0.45; and allowing for convexity in the face of gains, people will prefer the lottery offering the larger gain with the smaller probability.*

X27.31 **Show that it is possible to explain the version of Allais' paradox used in X27.22 by using the weighting rule $\pi(0.34) + \pi(0.66) < 1$. [*Hint:* You should write down expressions showing what we might infer about the valuation of the lotteries when there is a seeming preference reversal.]**
Assume that we observe the usual preferences: L_{A2} *over* L_{A1} *and* L_{B1} *over* L_{B2}*. If the weighting rule has the characteristic above, then people will tend to place a value on lottery* L_{B2}*,*
$v(L_{B2}) < 0.34v(L_{B1})$*, and this would be sufficient to explain the seemingly inconsistent rankings.*