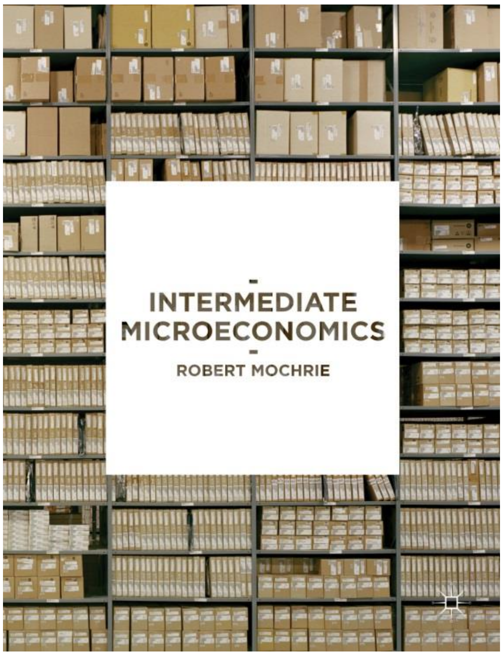
Solutions Manual: Part VII

Applying game theory

Summary answers to the 'By yourself' questions



Chapter 28

X28.1 **Confirm the following:**

a) There are two Nash equilibria in pure strategies: both choose to go the ball; or both choose to go to the family dinner.

Battle of the Sexes (1)		*Miss Bennet*	
		Ball	*Dinner*
Mr Darcy	*Ball*	*(6, 12)*	*(2, -2)*
	Dinner	*(4, 2)*	*(8, 4)*

For Mr Darcy, expecting Miss Bennet to go to the Ball, going to the Ball is the best reply; but expecting Miss Bennet to go to Dinner, going to Dinner is the best reply. Similarly, we see that for Miss Bennet, coordination of her action with Mr Darcy's is the best reply. So there are two pairs of consistent conjectures: (Ball, Ball) and (Dinner, Dinner).

b) Neither player has a dominant strategy.
Given that best replies depend on action chosen by the other player, they do not have a dominant strategy. We note that Mr Darcy prefers the outcome where they both go to Dinner, but Miss Bennet prefers the outcome where they go to the Ball.

c) Miss Bennet's best reply, β^*, is a function of her beliefs, δ^e, about Mr Darcy's probability distribution over the actions, and may be written:

$$\beta^*\left(\delta^e\right)=\begin{cases}0, \delta^e<\frac{1}{8}\\ [0,1], \delta^e=\frac{1}{8}\\ 1, \delta^e>\frac{1}{8}\end{cases}$$

For Miss Bennet, the expected value of going to the Ball, $E[V_B(Ball)] = 12\delta + 2(1 - \delta) = 2 + 10\delta$. The expected value going to Dinner, $E[V_B(Dinner)] = -2\delta + 4(1 - \delta) = 4 – 6\delta$.
The difference in expected values, $E[V_B(Ball)] - E[V_B(Dinner)] = -2 + 16\delta$. So, if $16\delta > 2$, or if the probability of Mr Bennet going to the Ball is large enough, Miss Bennet chooses to go to the Ball with certainty. Similarly, if $16\delta < 2$, the probability of Mr Darcy going to the Ball is sufficiently low that Miss Bennet prefers the strategy of going to dinner with certainty, so that $\beta^* = 0$. Lastly when $16\delta = 2$, Miss Bennet is indifferent between all mixed strategies.

d) Mr Darcy's best reply, δ^*, is a function of his beliefs, β^e, about Miss Bennet's probability distribution over the actions, and may be written:

$$\delta^*\left(\beta^e\right)=\begin{cases}0, \beta^e<\frac{3}{4}\\ [0,1], \beta^e=\frac{3}{4}\\ 1, \beta^e>\frac{3}{4}\end{cases}$$

The argument is broadly similar to that of part d). We see that for Mr Darcy, the expected value of going to the Ball, $E[V_D(Ball)] = 6\beta + 2(1 - \beta) = 2 + 4\beta$. The expected value going to Dinner, $E[V_D(Dinner)] = 4\beta + 8(1 - \beta) = 8 – 4\beta$. The difference in expected values, $E[V_D(Ball)] - E[V_D(Dinner)] = -6 + 8\beta$. We omit the argument, but we see here that if β is large enough, then Mr Darcy is certain to go the Ball.

e) There is a Nash equilibrium in mixed strategies in which Mr Darcy chooses to go the ball with probability, $\delta=\frac{1}{8}$, and Miss Bennet chooses to go to the ball with probability, $\beta=\frac{3}{4}$.
This follows from parts c) and d), where we see that only $(\beta^*, \delta^*) = \left(\frac{3}{4}, \frac{1}{8}\right)$

f) In this Nash equilibrium, Mr Darcy has an expected payoff of 5, and Miss Bennet an expected payoff of 3.25.

Mr Darcy's expected payoff to Ball is the same as his expected payoff to Dinner in this Nash equilibrium: $E[V_D(Ball)] = 2 + 4\beta^* = 5$. Similarly, Miss Bennet's expected payoff to each action is the same: $E[V_B(Ball)] = 2 + 10\delta^* = 3.25$.

X28.2 **Confirm the following:**

Battle of the Sexes (2)		Miss Bennet	
		Ball	Dinner
Mr Darcy	Ball	(6, 2)	(2, 6)
	Dinner	(4, 8)	(8, -2)

Table 28.2 A preference for avoiding Mr Darcy

a) **There is no Nash equilibrium in pure strategies.**
Suppose that Mr Darcy believes that Miss Bennet intends to go to the Ball; then he will choose to go to the Ball; but believing that Mr Darcy will go to the Ball, Miss Bennet chooses to go to Dinner. Yet, if Mr Darcy believes that Miss Bennet will go to Dinner, he too will choose to go to Dinner; and Miss Bennet would respond by choosing to go to the Ball. There is no pair of consistent conjectures, and so no Nash equilibrium in pure strategies.

b) **Neither player has a dominant strategy.**
Given that we are unable to find a Nash equilibrium in pure strategies, this is certainly true.

c) **There is a Nash equilibrium in mixed strategies in which Mr Darcy chooses to go to the ball with probability, $\frac{5}{7}$, and Miss Bennet chooses to go to the ball with probability, $\frac{3}{4}$.**
We simply confirm that Mr Darcy is indifferent between the strategies Ball and Dinner. This requires the expected payoffs $E[V_D(Ball)] = 6\beta + 2(1 - \beta) = 4\beta + 8(1 - \beta) = E[V_D(Dinner)]$. As before, this requires $\beta^* = ¾$.
We also confirm that Miss Bennet is indifferent between the strategies; which requires the expected payoffs $E[V_B(Ball)] = 2\delta + 8(1 - \delta) = 6\delta - 2(1 - \delta) = E[V_B(Dinner)]$. This condition is satisfied when $4\delta = 10(1 - \delta)$, or when $\delta^* = \frac{5}{7}$.

X28.3 **Illustrate the best replies of this game in a diagram similar to Figure 28.1. Explain why the Nash equilibrium in mixed strategies is unique.**
In a diagram with the probability of Miss Bennet going to the Ball, β, measured on the horizontal axis and the probability of Mr Darcy going to the Ball, δ, measured on the vertical axis, we see that Miss Bennet will wish to go to the Ball when she believes Mr Darcy will go to Dinner. So Miss Bennet's best reply curve starts from (0, 1), and is a vertical extending as far as $\left(0, \frac{5}{7}\right)$. At $\delta = \frac{5}{7}$, the best reply curve is horizontal, becoming vertical at $\left(1, \frac{5}{7}\right)$. The last segment of the curve is again a vertical line segment, running to the horizontal axis.
In the same way, Mr Darcy's best reply curve starts from the origin, and follows the horizontal axis to (0, ¾). There is then a vertical line segment running to (1, ¾); and lastly a horizontal line segment running to (1, 1). The crooked cross shape means that there can only be one point of intersection, the Nash equilibrium in mixed strategies $(\beta^*, \delta^*) = \left(\frac{3}{4}, \frac{5}{7}\right)$

X28.4 **Confirm the following, when $p = 0.5$:**

Mr Darcy's expected payoffs		*Miss Bennet's strategy*			
		Ball, Ball	*Ball, Dinner*	*Dinner, Ball*	*Dinner, Dinner*
Mr Darcy's action	*Ball*	*6*	*6p + 2(1 – p)*	*2p + 6(1 – p)*	*2*
	Dinner	*4*	*4p + 8(1 – p)*	*8p + 4(1 – p)*	*8*

Table 28.4 Mr Darcy's expected payoffs

a) **In pure strategies, Mr Darcy's best reply, $\delta^*(\sigma_B)$, to Miss Bennet's possible (pure) strategies are: (i) for σ_B = (*Ball*, *Ball*), $\delta^*(\sigma_B)$ = *Ball*; and (ii) $\delta^*(\sigma_B)$ = *Dinner*, otherwise.**
It is easy to verify that Mr Darcy chooses Ball if he believes that Miss Bennet is certain to choose Ball; and likewise Dinner, if he believes Miss Bennet is certain to choose Dinner. Considering the strategy σ_B = (Ball, Dinner) = BD, Mr Darcy's expected payoff to Ball, $E[V_B(Ball|BD)] = 6p + 2(1 – p) = 4$ (when $p = 0.5$); while his expected payoff to Dinner, $E[V_B(Dinner|BD)] = 4p + 8(1 – p) = 6$. So Dinner is the best reply to (Ball, Dinner).

b) **When playing his best reply, Mr Darcy's expected payoff to these strategies, E[$v(\delta^*)$]: (i) for σ_B = (*Dinner*, *Dinner*), E[$v(\delta^*)$] = 8; and (ii) otherwise, E[$v(\delta^*)$] = 6.**
These calculations are included in the previous part of the question.

c) **There is a Nash equilibrium in pure strategies, (σ_B, σ_D) = ((*Dinner*, *Ball*); *Dinner*).**
For Miss Bennet, in version (1) of the game, Dinner is the best reply to Dinner; and in version (2), Ball is the best reply to Dinner. So ((Dinner, Ball); Dinner) is a set of consistent conjectures, and therefore a Nash equilibrium.

X28.5 **Confirm that the Nash equilibrium in X28.4c is the only one in pure strategies, given that $0 < p < 0.75$.**
For there to be a mixed strategy, Mr Darcy must receive the same payoff to all actions. Taking the difference of expected payoffs that Mr Darcy receives facing Miss Bennet's strategy BD, $4p + 8(1 – p) – 6p – 2(1 – p) = 6(1 – p) – 2p = 6 – 8p > 0$ if $p < ¾$. So long as the probability of Miss Bennet being of type 1 is not too high, Mr Darcy will prefer to go to Dinner, rather than to the Ball.

X28.6 **Confirm that when $p > 0.75$, there is a Nash equilibrium in pure strategies (σ_B, σ_D) = ((*Ball*, *Dinner*); *Ball*).**
From the argument of X28.5, given that $p > 0.75$, for Mr Darcy, the expected payoff from choosing Ball is greater than the expected payoff from choosing Dinner.

X28.7 **Suppose that Brinda buys a car of exactly the same type at the same time as Anya. She identifies it as a peach, but after a month she is bored with it, and would simply like to replace it. For Brinda, WTA_B = 18,000; that is, she is willing to accept a loss of nearly £2,000.**

a) **Confirm that if the prior $\lambda = \pi = 0.01$, so that there is no learning, then Brinda will be able to find a willing buyer.**
Potential buyers consider the probability that Brinda is selling a lemon to be the same as the probability of her selling a lemon across the whole population of cars. Then, all buyers should be willing to pay up to 19,950 for the car, and since $WTP > WTA_B$, Brinda should be able to find willing buyers at a price she would accept.

b) Confirm that if the prior $\lambda = 0.9$, so that potential purchasers consider the fact that the car is offered for sale to be highly informative, then Brinda will not be able to find a willing buyer.

*If potential buyers believe that $\lambda = 0.9$, then WTP = 0.1*20,000 + 0.9*15,000 = 15,500 < 18,000 = WTA_B. Since WTP < WTA_B, Brinda cannot find a willing buyer at a price that she will accept.*

c) Find the range of values of λ for which Brinda will find a willing buyer.

Brinda needs WTP = $15{,}000\lambda + 20{,}000(1 - \lambda) \geq 18{,}000$; or $5\lambda \leq 2$; or that $\lambda \leq 0.4$.

X28.8 **Explain why for Anya, *WTA* = 0; and that irrespective of the value of λ, she will be able to find a willing buyer.**

Anya is a forced seller; the car has no usefulness to her, and so, effectively as a seller in the very short run, any revenue is advantageous.

X28.9 **Suppose that among the 10,000 people who buy their cars at the same time as Anya and Brinda, there are 25 forced sellers, like Anya; 25 willing sellers of standard quality cars, like Brinda, with *WTA* = 18,000; and 100 willing sellers of low quality cars, with *WTA* = 15,000.**

a) Calculate the average value to a potential buyer of a car that will be brought to market, given that: (i) potential buyers believe that only low-quality cars and forced sellers will bring cars to market; and (ii) potential buyers believe that all three groups of cars will be brought to market.

(i) If potential buyers believe that only forced sellers and owners of low quality cars bring their cars to the market, then WTP = $\frac{4}{5}*15{,}000+\frac{1}{5}*18{,}000$ *= 15,600.*

(ii) If potential buyers believe that only forced sellers, voluntary sellers and owners of low quality cars bring their cars to the market, then WTP = $\frac{2}{3}*15{,}000+\frac{1}{3}*18{,}000=16{,}000$.

b) Explain why the revision of the prior in (i) is consistent with the market reaching equilibrium, but the revision in (ii) is not.

For all participants in the market in case (i), WTP > WTA; so that there can be an equilibrium. In case (ii), for voluntary sellers, WTA > WTP (for potential buyers). We therefore expect voluntary sellers to take their cars from the market.

X28.10 **Suppose that there are no forced sales. Assuming that there are *n* potential sellers of standard quality cars, what is the highest *WTA* that Brinda could set if she is to succeed in selling her car?**

For Brinda to sell her car, we require WTA $\leq$ WTP =16,000.

X28.11 **Consider the situation facing the potential buyer, *B*. Confirm that with perfect information, *B*:**

a) facing a high price, would pay a_h only for a peach; and

Being able to distinguish between a peach and lemon, B, the price a_h is equal to B's WTP for a peach and greater that the WTP for a lemon.

b) facing a low price, would pay a_l for any car.

This price, a_l, equals WTP_B for a lemon and is less that WTP_B for a peach; so B will buy any car.

X28.12 **Assume now that there is asymmetric information. The seller assumes that the car is a lemon with probability λ_h, when the seller announces the high price, a_h; and with probability λ_l, when facing the low price, a_l.**

a) **Confirm that: (i) the seller of a peach will set a price of at least £19,000; and (ii) the seller of a lemon will set a price of at least £14,000.**
Since WTA_A(Peach) = 19,000 and WTA_A(Lemon) = 14,000, these are the minimum prices that a seller will accept.

b) **Suppose that all sellers of peaches set a price of £19,000; and all other sellers of lemons set a price of £14,000. What might *B* infer about *A*'s car if *A* sets a price of £19,000, even though the car is a lemon?**
Since all other cars offered for sale at a price of £19,000 are peaches, it would be reasonable for B to assume that A's car is a peach.

c) **Explain why you think that the situation outlined in part (b) is unlikely to arise. [*Hint:* If *A* believes that there is an advantage to setting a high price when selling a low-quality car, how will other sellers of lemons behave?] Discuss why it is unlikely that *B* will consider the price set by *A* to be informative about the quality of the car, so that $\lambda_h = \lambda_l = \lambda$.**
Were this to be true for A, it would also be true for all other potential sellers; so buyers would no longer be certain that a car offered at a high price is of high quality. Then we expect to see all cars being sold at a single price, and the probability of a car being of low quality does not depend on the price set.

d) **Show that if $\lambda < 0.2$, then the expected value of a car to a buyer, $E[v_B] > 19{,}000$, and peaches will be offered for sale.**
Expected value of a car offered for sale at price, a, given that a small proportion, λ, of the cars offered for sale are lemons will be: $E_V[WTP_B] = 15{,}000\lambda + 20{,}000(1 - \lambda) = 5{,}000(4 - \lambda)$. For this outcome, we require $E_V[WTP_B] \geq WTA_A(Peach)$, so that $5{,}000(4 - \lambda) \geq 19{,}000$; and $\lambda \leq 0.2$.

e) **Show that if $\lambda > 0.2$, then the expected value of a car to a buyer, $E[v_B] < 19{,}000$, so that only lemons will be offered for sale.**
If $\lambda > 0.2$, then potential buyers will consider the expected value of a car offered for sale to be less than 19,000. Sellers of peaches will withdraw them from the market, and so the only consistent conjecture is $\lambda = 1$.

f) **Confirm that there is a Bayes-Nash equilibrium in pure strategies, $(\sigma_A, \sigma_B, \lambda)$, with σ_A = (*Keep*, *Low*); σ_B = (*Reject*, *Accept*); and $\lambda = 1$. [*That is: A* keeps a peach, and offers a lemon for sale; B believes that all cars offered for sale are lemons, and so will only buy cars offered at a low price.]**
We see that from parts d) and e) that B's actions are consistent with the belief, $\lambda = 1$; that all cars offered for sale are lemons. High prices, which we would see if the proportion of lemons in the market was small, are rejected, and low prices are accepted. Expecting B to behave in this way, A keeps a car that is a peach, but offers a lemon for sale at low price.

X28.13 **Confirm that when half of the people who purchase lemons, and 2.5% of the people who buy peaches, seek to trade their cars in shortly after purchase, there is a second Bayes-Nash equilibrium in pure strategies in the game illustrated in Figure 28.3: $[\sigma_A^*, \sigma_B^*, \lambda^*]$ = [(*High*, *High*), (*Accept*, *Accept*), 0.17], with '*High*' price, p^*: £19,000 < p^* < £19,150.**
With half of the people buying lemons offering them for sale, this is 0.5% of all cars sold; while 2.5% of peaches are 2.475% of all cars. The proportion of lemons in the cars offered for sale, $\lambda \approx 0.17$. We know that with such a low proportion of lemons in the market, $E[WTP_B] \geq WTA_A(Peach)$, so that the belief is consistent with the decisions of the players; and furthermore that behaviour is consistent with beliefs.

X28.14 **Assume that there is perfect information, or that *q* is observable. Show that when a buyer and a seller meet, *WTP* > *WTA*, and the transaction will proceed.**

When a potential buyer, B meets seller, S, meet, with S offering a car of quality q, WTA_S = 10,000q < 15,000q = WTP_B, so that there will be a range of prices at which the transaction can go ahead.

X28.15 **Now assume that there is asymmetric information. Sellers know the value of *q*, while buyers only know the distribution of *q*.**

a) **Confirm that buyers are risk-neutral.**

Buyers' valuations are linear in quality, so $\frac{d^2V_B}{dq} = 0$; as required for risk neutrality.

b) **Explain why we should not expect buyers to offer to pay more than 15,000 for any car.**

Assume that all cars are offered for sale. Then $E[V_B]$ = ½(0 + 30,000) = 15,000, the average of the highest and lowest value.

c) **Suppose that a potential seller owns a unit of the good for which *q* > 1.5. Explain why this seller will not be able to complete a sale.**

If q > 1.5, then $V_S(q)$ > 15,000, so $WTA_S(q) > E[V_B]$; the seller will reject the highest price that the buyer is willing to pay, and so keeps the car.

d) **Suppose that all goods for which $q \leq 1.5$ are brought to the market. Show that *WTP* = 11,250, so that goods for which *q* > 1.125 remain unsold.**

We repeat the argument above. If only cars of quality $q \leq 1.5$ are brought to market, then $E[V_B | q \leq 1.5]$ = ½(0 + 22,500) = 11, 250. Then $E[V_B | q \leq 1.5] \geq V_S(q_S)$ if $q_S \geq 1.125$.

e) **Suppose that all goods for which $q \leq q_0$ are brought to market. Show that *WTP* = $0.75q_0$, so that goods for which $0.75q_0 < q \leq q_0$ remain unsold.**

This is a generalization of the previous two parts of the question. Assuming that all goods for which $q \leq q_0$ are brought to market, then $E[V_B | q \leq q_0] = 7{,}500q_0$; but then for all potential sellers with cars for which $q > 0.75q_0$, $E[V_B | q \leq q_0] < V_S(q)$, and so these cars are not brought to market.

f) **Show that the equilibrium condition for this market can be written, *WTP* = $E[q | q < q_0] = q_0$; and that this is satisfied when $q_0 = 0$.**

This condition emerges from the previous discussion. For the buyers' beliefs to be correct, it must be that all cars that they believe will be offered for sale will indeed be offered for sale. This requires the marginal seller to place on the car being brought to market the buyer's expected value (across all of the cars brought to market). The condition that we have written above formalizes this requirement.

X28.16 **Suppose that we defined the index of quality $q \sim [0, 1]$, with *WTA* = 1,000(14 + 5*q*) and *WTP* = 1,000(15 + 5*q*). By applying the equilibrium condition in Expression 28.9, explain the outcome in this market.**

For equilibrium, $E[WTP | q \leq q_0] = WTA(q_0)$; here we require $14 + 5q_0 = 15 + 2.5q_0$, so that $2.5q_0 = 1$, or $q_0 = 0.4$. There is an equilibrium in which cars of quality $q \leq 0.4$ are traded at price p = 16, 000.

X28.17 **Suppose that we define the index of quality $q \sim [0, 2]$, as in the Akerlof paper, but define $WTA = 1{,}000(a + q)$, and $WTP = 1{,}000(a + 1.5q)$. Confirm that the market will always collapse, irrespective of the value of a.**

For equilibrium, $E[WTP|q \leq q_0] = WTA(q_0)$; here we require $a + q_0 = a + 0.75q_0$, so that $0.25q_0 = 0$, and $q_0 = 0$. The market collapses.

X28.18 **Repeating X28.17, but with $WTP = 1{,}500(a + q)$.**

a) **Show that if $a = 0$, then the market collapses.**

For equilibrium, $E[WTP|q \leq q_0] = WTA(q_0)$; here we require $q_0 = 0.75q_0$, so that $0.25q_0 = 0$, and $q_0 = 0.4$. The market collapses.

b) **Show that if $0 \leq a \leq 1$, then $q^* = 2a$.**

For equilibrium, $E[WTP|q \leq q_0] = WTA(q_0)$; here we require $a + q_0 = 1.5(a + 0.5q_0)$, so that $0.25q_0 = 0.5a$, or $q_0 = 2a$. There is an equilibrium in which cars of quality $q \leq 2a$ are traded at price
$p = 3{,}000a$.

c) **Show that if $a > 1$, then $q^* = 2$, and the full market is served.**

The argument is as in part b). However, we note that when $a > 1$, then the expression, $E[WTP|q \leq q_0] = WTA(q_0)$, is satisfied for a value of $q_0 > 2$; but the maximum possible value, $q_1 = 2$. All cars are traded, at a price, $p = 3{,}000a$.

X28.19 **Explain why the willingness of a seller to provide a warranty to the buyer of a car, such as undertaking to pay the costs of repair of any mechanical faults arising in the 12 months following the purchase, might be interpreted as a signal that the car being offered for sale is a peach.**

We might think of the willingness to offer a warranty as a willingness to absorb the (uncertain) costs of future repairs. We assume that the seller has perfect information, which in this context means being able to predict the costs associated with the warranty. If the seller expects these to be high, the seller will be less likely to offer a full warranty. Since the costs of a warranty differ across types of cars, potential buyers can form inferences about the probability of a car being a lemon from the nature of the warranty that is offered.

X28.20 **For this outcome to be a Perfect Bayesian equilibrium, it must satisfy certain conditions. Derive expressions for which each would be satisfied.**

a) **Player A, given $Q = 1$, obtains a higher payoff from choosing signal $S = 1$, and receiving payment p_1, than from signal $S = 0$, and receiving payment p_0.**

We require the payment received, less the cost of production and the cost of signalling consistent with being in state 1 to be greater than signalling as if $S = 0$:
$p_1 - c_1 - s_1(q_1) \geq p_0 - c_1 - s_0(q_1)$, or that $p_1 - p_0 \geq s_1(q_1) - s_0(q_1)$.

b) **Player A, given $Q = 0$, obtains a higher payoff from choosing signal $S = 0$, and receiving payment p_0, than from signal $S = 1$, and receiving payment p_1.**

Player A also prefers to act in accordance with type, when $Q = 0$:
$p_1 - c_0 - s_0(q_1) \leq p_0 - c_0 - s_0(q_0)$, or that $p_1 - p_0 \leq s_1(q_0) - s_0(q_0)$.

c) **Player B, observing $S = 1$, will offer p_1; and observing $S = 0$, will offer p_0. [*Note:* Remember that this action has to be consistent with player A choosing $S = 1$ in order to lead player B into believing $Q = 0$.]**

We require player B to obtain surplus paying a higher price when $Q = 1$ than the lower price when $Q = 0$. $v_1 - p_1 \geq 0$; and $v_0 - p_0 \geq 0$.

d) Both players must be better off from taking part in the game than from not taking part. In particular, player A should not decide to keep the goods.
In addition to the conditions in a), b) and c), we require $p_1 - c_1 - s_1(q_1) \geq 0$, and that $p_0 - c_0 - s_0(q_0) \geq 0$

X28.21 **Confirm that there is a Perfect Bayesian equilibrium of this game, $[\sigma_A, \sigma_B, \beta]$ = [(0, 0.5), (1, 2), (0, 1)]. Confirm that here $e_2 = e^*$, so that workers of type Q_1 maximize their payoff (with education $e_1 = 0$); just preferring this to obtaining education, $e^* = 0.5$.**
Firstly, we note that player B believes that the signal e = 0 and e = 0.5 are informative; player B believes (correctly) that player A only emits signal e = 0.5 when Q = 2; and e = 0 when Q = 1. For player B, paying wage $m_1 = 2$ when e = 0.5, and wage $m_1 = 2$ when e = 0, the wage is then equal to the value of marginal product, and the participation condition is just met. When Q = 1, the cost of acquiring education e = 0.5, $c_1(0.5) = 1$; while for Q = 2, $c_2(0.5) = 0.25$. A worker of type 2 with education e = 0.5, obtains payoff $v_2(0.5) = 1.75 > 1 = v_2(0)$. A worker of type 1 with education 0 obtains payoff $v_1(0) = 1 = v_1(0.5)$. This is just enough education to allow separation to occur.

X28.22 **Explain why we expect any firm that sets wage $w = 1$ for workers who emit $E = e_2 \geq e^*$ to be unable to hire any. What would we conclude about the labour force of such a firm?**
Other firms will offer a higher wage to the high productivity workers. The labour force of such a firm will consist only of low productivity workers.

X28.23 **Explain why we would not expect any firm that offers a wage $w = 2$ to *all* workers to be able to trade.**
The average product of the workers hired $AP_L \leq 1 + f$, since all workers will find the contract attractive. Since $1 + f < 2$, the firm makes losses, and so will cease trading.

X28.24 **Find conditions that must be satisfied (i) for workers of type $Q = 2$ to choose education $E = e^*$; and (ii) for workers of type $Q = 1$ to choose education $E = 0$.**
To choose education $E = e^*$, this has to be the best outcome for workers of type Q = 2. If all firms offer different wages, with $w(e^*) = 2$ and $w(0) = 1$, then the incentive compatibility constraint is satisfied when $2 - 0.5e^* > 1$; or $e^* < 2$. We have already seen that when $e^* > 0.5$, then for workers of type Q = 1, $2(1 - e^*) < 1$.

X28.25 **Suppose that $f > 0.75$. Demonstrate that workers of type $Q = 2$ will then prefer to set $e = 0$, so that there is a pooling equilibrium, in which all workers obtain the same wage, $w_{ave} = 1 + f$, rather than investing in education to obtain the wage $w_2 = 2$.**
If no workers obtain education, firms offer wage $w_{ave} = 1 + f$, and make zero profits. This is preferable for workers of type Q = 2 if $1 + f > 2 - 0.5e^* = 1.75$, given that $e^* = 0.5$. If $f > 0.75$, then this condition will be satisfied.

X28.26 **Confirm that whether there is a pooling or a separating equilibrium, workers of quality $Q = 1$ obtain the same payoff as they would under conditions of perfect information.**
In a pooling equilibrium, all workers receive the same wage, which is equal to their average product. No workers signal, so the payoff is the same as when there is perfect information. In a separating equilibrium, since workers of type Q = 1 do not engage in signalling, there is no cost of signalling, and workers obtain the wage that they would under conditions of perfect information.

Chapter 29

X29.1 **Confirm that if $y = x(1 - x)$, then y is maximized when $x = 0.5$. Hence demonstrate that if $a(1 - a) > b(1 - b)$, firm A has greater market share.**

Differentiating, we obtain $\frac{dy}{dx} = 1 - 2x$ and $\frac{d^2y}{dx^2} = -2$. The first-order condition for a maximum, $\frac{dy}{dx} = 0$ is satisfied when x = ½. The second-order condition, $\frac{d^2y}{dx^2} = -2$ is satisfied for all values of x.

We note that for 0 < x < 1, y(x) = y(1 – x); and that for 0 < x < ½, y(x) is increasing. It follows that if a(1 – a) > b(1 – b), |a – 1/2| < |b – ½|, so that on the number line, a is closer to the mid-point, x = ½, than b.

X29.2 **Given that firm A believes that firm B will choose location b^e, where should firm A locate? How would you expect firm B to respond? Show that $(a^e, b^e) = (½, ½)$ is the only set of consistent conjectures, and the only possible Nash equilibrium in pure strategies.**

We can describe the location, but cannot write down an expression for it. Firm A should choose the location which maximizes its share of the market, choosing the location in the interval [½, b^e] that is closest to b^e, and so expecting to obtain market share max(b^e, 1 – b^e).

X29.3 **This model has frequently been related to competition between political parties in an election. Suppose that there are two political parties, K and L, and that it is possible to locate their electoral platforms, k and l, on a line between 0 (the most left-wing position) and 1 (the most right-wing position). We assume that voters' preferred positions, $x \sim U[0, 1]$, and that they will vote for the party closer to their position.**

a) **Confirm that if $k = 0.5$, then the best reply $l^*(0.5) = 0.5$; and that if $k < 0.5$, then $l^*(k)$: $k < l^* < 1 - k^*$.**

If k = 0.5, then 1 – k = k, so that there is no platform for which L can choose a policy that will command a majority of the population. If k < 0.5, then for any policy l: k < l < 1 – k, L obtains vote share 1 – ½(k + l), while K obtains vote share ½(k + l). Since l < 1 – k, (k + l) < 1, and L obtains a majority of the votes.

b) **Sketch a diagram showing these best replies.**

The diagram will be a line segment running from 0 to 1. We show point l as being closer to the centre of the line than point k.

c) **Confirm that $(k, l) = (0.5, 0.5)$ is the only Nash equilibrium in pure strategies.**

Suppose otherwise. Then there is some value of k: k $\neq$ 0.5 for which k will not wish to change position after l is known. But we have seen in part a) that if k $\neq$ 0.5, then L wins a majority for all values of l: k < l < 1 – k, and so we have a contradiction.*

X29.4 **Confirm that on simplifying Expression 29.7, we obtain:**

$$x^* = ½ - (p_A - p_B) \quad [29.8]$$

Writing $p_A - p_B = (x^*)^2 - \frac{3}{2}x^* + \frac{9}{16} - \left[(x^*)^2 - \frac{1}{2}x^* + \frac{1}{16}\right]$, *we obtain the result.*

X29.5 **Given our assumptions, firm A's objective is to maximize revenues.**

a) **Show that its revenue, $R_A(p_A; p_B) = p_A[½ - (p_A - p_B)]$.**

Revenue $R_A(p_A, p_B) = p_A.x^*$; and the result follows

b) **Both firms set their price at the same time. Denote firm A's belief about the price that firm B will set as p_B^e. Confirm that firm A's best-reply function, $p_A^*(p_B^e) = ½(p_B^e + ½)$.**

Differentiating the revenue function, $\frac{\partial R_A}{\partial p_A} = \frac{1}{2} - 2p_A + p_B{}^e$*, and setting this to zero, the result follows.*

c) **Show that firm *B*'s best reply may be written: $p_B{}^*(p_A{}^e) = ½(p_A{}^e + ½)$.**
This result follows immediately by symmetry.

d) **Confirm that there is a Nash equilibrium, $(p_A{}^*, p_B{}^*) = (½, ½)$; so that the firms achieve revenues $R_A(p_A{}^*, p_B{}^*) = R_B(p_A{}^*, p_B{}^*) = ¼$, and that the market will be covered if $v > \frac{9}{16}$.**
For consistent conjectures, $p_A{}^* = p_A{}^e$; and $p_B{}^* = p_B{}^e$. We require $2p_A{}^* - p_B{}^* = 2p_B{}^* - p_A{}^* = ½$; so that $p_A{}^* = p_B{}^*$ (from the first equality) and $p_A{}^* = p_B{}^* = ½$; substituting, we obtain $R_A{}^* = R_B{}^* = ¼$, and for market coverage $C_A(½) = C_B(½) = ½ + (½ - ¼)^2 = \frac{9}{16} \leq v$. So long as consumers value the good at more than $v = \frac{9}{16}$, then they will all buy the good.

X29.6 **Suppose that a social planner insists upon minimum differentiation, so that *a* = *b* = 0.5.**

a) **Confirm that if $v - ¼ > p_A > 0$, firm *B* can obtain the whole market by setting a price p_B: $p_A > p_B > 0$, but that if $p_A = p_B$, then the firms share the market entirely.**
If $v - p_A > ¼$, then the market will be covered; for consumers for whom $x = 0$, or $x = 1$, $C(x) = (½)^2 + p_A$. Since $p_A > 0$, then for any value of p_B: $0 < p_B < p_A$, firm B undercuts firm A and so obtain the whole market, while making profits.

b) **Hence confirm that the only Nash equilibrium in prices is $(p_A{}^*, p_B{}^*) = 0$.**
The argument is the same as for competition in prices generally. If firm B believes that firm A might undercut it (and make profits), then it is possible for firm B to reduce its price to less than it expects firm A to charge, acquire the whole market, and make profits. Only by setting the Nash equilibrium pair $(p_A{}^*, p_B{}^*) = (0, 0)$ are profits eliminated, so that there is no possibility of undercutting, and since $p_A{}^* = p_B{}^*$, the market is shared.

X29.7 **Suppose that the firms have chosen to engage in maximum differentiation, so that they select the locations at the end points of the line segment: *a* = *b* = 0.**

a) **Confirm that if the firms set prices p_A and p_B, then the location of the marginal consumer will be $x^* = ½[1 + p_B - p_A]$.**
We write the cost of purchase from firm A, for the consumer at x^* as $C(x^*, 0) = (x^*)^2 - p_A$; and the cost of purchase from firm B as $C(x^*, 1) = (1 - x^*)^2 - p_B$. For the consumer to be indifferent between these alternatives, $C(x^*, 0) = C(x^*, 1)$, so that $2x^* - 1 = p_B - p_A$, and the result follows.

b) **By obtaining the revenues of each firm, show that their best-reply functions may be written in implicit form as:**

$$p_A{}^*(p_B{}^e): 2p_A{}^* - p_B{}^e - 1 = 0; \text{ and}$$
$$p_B{}^*(p_A{}^e): 2p_B{}^* - p_A{}^e - 1 = 0;$$

where $p_B{}^e$ is firm *A*'s conjecture of price, p_B, and $p_A{}^e$ is firm *B*'s conjecture of price p_A.
Writing $R_A(p_A, p_B{}^e) = p_A x^* = ½p_A[1 + p_B{}^e - p_A]$. Differentiating with respect to p_A, $\frac{\partial R_A}{\partial p_A} = \frac{1}{2}\left(1 + p_B{}^e - 2p_A\right)$; and setting the partial derivative to zero, and rearranging the expression, we obtain the required expression. Noting that for firm B, market share is $1 - x^* = ½[1 + p_A - p_B]$, we can form a similar expression to R_A for R_B, and the required expression follows directly.

c) **Show in the Nash equilibrium (where there are consistent conjectures), $(p_A{}^*, p_B{}^*) = (1, 1)$.**

For consistent conjectures, the each firm's best reply is its conjecture of its output, so that $p_f^e = p_f^*$. The system in part b) can be written as $2p_A^* - p_B^* = 1$; and $4p_B^* - 2p_A^* = 2$. Then adding together the two equations, $3p_B^* = 3$, and the result follows immediately.

d) Confirm that for the market to be covered, $v > 5/4$.

For market coverage, the marginal consumer, at x = ½, must be willing to purchase the good. The cost of purchase, $C(½, 1) = C(½, 0) = (1 - 0.5)^2 + 1 = \frac{5}{4}$.

X29.8 **Given firm *B*'s conjectured revenue, $R_B^e(p_B) = \{1 - x[p_B^*(p_A^e)]\}.p_B^*(p_A^e)$, show that firm *B*'s best reply, p_B^* satisfies the condition:**

$$2p_B^*(p_A^e) - p_A^e = (1 - a)^2 - b^2 \quad [29.15]$$

We write conjectured revenue $R_B^e = (1 - x^*[p_B^*(p_A^e)])p_B^*(p_A^e)$. Now, $1 - x^* = 1 - \frac{p_B - p_A + (1 - 2b + b^2) - a^2}{2(1-a-b)}$. Writing this as a single fraction, we obtain $1 - x^* = \frac{p_A - p_B + 1 - 2a + a^2 - b^2}{2(1-a-b)} = \frac{p_A - p_B + (1-a)^2 - b^2}{2(1-a-b)}$. It follows that $R_B^e = \left(\frac{p_A^e - p_B + (1-a)^2 - b^2}{2(1-a-b)}\right)p_B$, and differentiating this expression with respect to p_B, we obtain $\frac{\partial R_B^e}{\partial p_B} = \left(\frac{p_A^e - 2p_B + (1-a)^2 - b^2}{2(1-a-b)}\right)$. We set this expression to zero, obtaining the condition in expression [29.15] upon rearrangement.

X29.9 **Demonstrate that in equilibrium, firm *A* sets price, p_A^*:**

$$p_A^* = \tfrac{1}{3}(3 - b + a)(1 - a - b) \quad [29.19]$$

We rewrite the reaction functions as $4p_A^* - 2p_B^* = 2(1 - b)^2 - 2a^2$; and $2p_B^* - p_A^* = (1 - a)^2 - b^2$. Add these expressions together, we obtain $3p_A^* = 1 - 2a - a^2 + 2 - 4b + b^2$. Expression [29.19] follows from factorization.

X29.10 **Given the Nash equilibrium prices, (p_A^*, p_B^*) in *Stage 2* of the game, confirm that the marginal consumer is located at position $x^* = \tfrac{1}{6}(3 - b + a)$.**

Given that $x^* = \frac{p_B - p_A + (1 - 2b + b^2) - a^2}{2(1-a-b)}$, we see that $p_B^* - p_A^* = \tfrac{1}{3}(1 - a - b)[(3 - a + b) - (3 - b + a)] = \tfrac{2}{3}(b - a)(1 - a - b)$, and also that $(1 - b)^2 - a^2 = (1 - b - a)(1 - b + a)$. Extracting a common factor of $(1 - a - b)$ from the expression, $x^* = \tfrac{1}{3}(b - a) + \tfrac{1}{2}(1 - b + a)$, and the result follows by simplifying the fractions.

X29.11 **Substituting from Expressions 29.11, 29.18 and 29.19:**

a) Confirm that firm *A*'s revenue in equilibrium, R_A^*, can be written as:

$$R_A^*(a, b) = \tfrac{1}{18}(1 - b - a)(3 - b + a)^2 \quad [29.20]$$

Writing $x^* = \tfrac{1}{6}(3 - b + a)$, and $p_A^* = \tfrac{1}{3}(3 - b + a)(1 - a - b)$, the result follows immediately since $R_A^* = x^*.p_A^*$.

b) By partially differentiating Expression 29.20 with respect to location, *a*, confirm that:

$$\frac{\partial R_A^*}{\partial a} = -\frac{1}{18}(3 - b + a)(1 + b + 3a) < 0 \quad [29.21]$$

Applying the product rule of differentiation, $\frac{\partial R_A}{\partial a} = -\tfrac{1}{18}(3 - b + a)^2 + \tfrac{2}{18}(1 - b - a)(3 - b + a)$. Extracting the common factor, $\tfrac{1}{18}(3 - b + a)$, the result follows immediately.

X29.12 **We use Expression 29.22 to obtain the number of firms, *n*, that enter the market.**

a) Confirm that $x^* = \frac{n}{2}\left(p^* - p_f + \frac{1}{n^2}\right)$.

From Expression 29.22, expanding the brackets, we obtain $p_f + (x_f*)^2 = p* + \left((x_f*)^2 - \frac{2x_f*}{n} + \frac{1}{n^2}\right)$. The required expression follows by algebraic rearrangement.

b) Explain why firm *f* makes total sales:

$$q(p_f, p*) = n\left(p* - p_f + \frac{1}{n^2}\right) \quad \textbf{[29.23]}$$

The answer to part a) indicates the sales made in the region between firm f and firm f + 1. We add the sales made in the region between firm f and firm f – 1 as well.

c) Write an expression for the firm's profit, Π_f. Show that the first-order condition, $\frac{\partial \Pi_f}{\partial p_f} = 0$, is satisfied when $p_f = \frac{1}{2}\left(p* + \frac{1}{n^2}\right)$, and that for a symmetric equilibrium, $p_f = p* = \frac{1}{n^2}$, so that each firm makes profits $\Pi_f* = n^{-3} - F$.

Profit is the difference between revenue and costs, so that $\Pi_f = np_f\left(p* - p_f + \frac{1}{n^2}\right) - F$. To find the profit maximizing price, p_f*, we partially differentiate the profit function with respect to p_f, obtaining $\frac{\partial \Pi_f}{\partial p_f} = n\left(p* - 2p_f + \frac{1}{n^2}\right)$. Setting this derivative to zero, we see that the first-order condition stated above is satisfied. Requiring the equilibrium to be symmetric, so that we treat firm f as a representative firm, we obtain the result that $p_f = p* = \frac{1}{n^2}$; and profit $\Pi_f = \frac{n}{n^2}\left(\frac{1}{n^2} - \frac{1}{n^2} + \frac{1}{n^2}\right) - F = \frac{1}{n^3} - F$.

d) Firms continue entering the market until all profits are eliminated. Confirm that $\Pi_f = 0$ if $n = F^{-\frac{1}{3}}$, (where *F* is each firm's fixed costs); and that firms then set the price, $p* = F^{\frac{2}{3}}$.

These results follow directly from the breakeven condition.

X29.13 **Define the marginal quality valuation, *v**, at which a potential customer will be indifferent between the products of quality θ_1 and θ_2, sold at prices p_1 and p_2. Show that $v* = \frac{p_2 - p_1}{\theta_2 - \theta_1}$. Calculate the market shares, q_1 and q_2, that each firms enjoys.**

Marginal quality valuation v: $v*\theta_1 - p_1 = v*\theta_2 - p_2$. Then $(\theta_2 - \theta_1)v* = p_2 - p_1$ and the result follows. Given market coverage, with valuation v: $v_0 \le v \le 1 + v_0$, firm 1 makes sales to consumers with valuation between v_0 and v*, while firm 2 makes sales to consumers with valuations between v* and $1 + v_0$. We obtain market shares $q_1 = \frac{p_2 - p_1}{\theta_2 - \theta_1} - v_0$ and $q_2 = 1 + v_0 - \frac{p_2 - p_1}{\theta_2 - \theta_1}$.*

X29.14 **Given our assumptions about the costs of production, the firms seek to maximize their revenues, R_f. Firms decide on their prices at the same time, so that each seeks to maximize its conjectured revenues, R_f^e, by forming a conjecture about the other firm's price, $p_{\sim f}^e$, and then choosing the best reply, $p_f*(p_{\sim f}^e)$.**

a) Write down each firm's conjectured revenues, R_f^e, as the product of its market share, q_f^e, and price, p_f.

For firm 1, $R_1^e = \left(\frac{p_2^e - p_1}{\theta_2 - \theta_1} - v_0\right)p_1$, while for firm 2, $R_2^e = \left(v_1 - \frac{p_2 - p_1^e}{\theta_2 - \theta_1}\right)p_2$.

b) By partially differentiating each firm's conjectured revenue with respect to its own price, $\frac{\partial R_f^e}{\partial p_f}$, confirm that the best-reply functions can be written:

$$p_1*(p_2^e) = ½ [p_2^e - v_0(\theta_2 - \theta_1)] \quad \textbf{[29.31a]}$$

$$p_2*(p_1^e) = ½ [p_1^e + v_1(\theta_2 - \theta_1)] \quad \textbf{[29.31b]}$$

Taking the partial derivatives of each firm's expected revenue with respect to its price, we obtain:
$\frac{\partial R_1^e}{\partial p_1} = \frac{p_2^e - 2p_1}{\theta_2 - \theta_1} - v_0$ *; and* $\frac{\partial R_2^e}{\partial p_2} = v_1 - \frac{2p_2 - p_1^e}{\theta_2 - \theta_1}$. *Setting these partial derivatives to zero and rearranging the expressions, we obtain Expressions 29.31a) – b).*

X29.15 **Confirm that in equilibrium, firm 2 sets price, p_2*:**

$$p_2{}^* = \tfrac{1}{3}(2v_1 - v_0)(\theta_2 - \theta_1) \qquad \textbf{[29.35]}$$

Substitution Substituting Expression [29.34] into Expression [29.31b], we obtain, $p_2{}^* = \frac{1}{2}\left[\frac{1}{3}(v_1 - 2v_0)(\theta_2 - \theta_1) + v_1(\theta_2 - \theta_1)\right]$, *and Expression [29.35] follows on simplifying.*

X29.16 **Show that:**

a) **The firms' equilibrium outputs are (q_1*, q_2*):**

$$q_1{}^* = \tfrac{1}{3}(v_1 - 2v_0); \text{ and } q_2{}^* = \tfrac{1}{3}(2v_1 - v_0)$$

[29.36]

We obtain the difference in prices, $p_2{}^* - p_1{}^* = \left[\frac{1}{3}(2v_1 - v_0)(\theta_2 - \theta_1)\right] - \left[\frac{1}{3}(v_1 - 2v_0)(\theta_2 - \theta_1)\right] = \frac{1}{3}(v_1 + v_0)(\theta_2 - \theta_1)$. *Then* $q_1{}^* = \frac{p_2{}^* - p_1{}^*}{\theta_2 - \theta_1} - v_0 = \frac{1}{3}(v_1 - 2v_0)$; *and* $q_2{}^* = v_1 - \frac{p_2{}^* - p_1{}^*}{\theta_2 - \theta_1} = \frac{1}{3}(2v_1 - v_0)$.

b) **The firms' equilibrium profits are (q_1*, q_2*):**

$$R_1{}^* = \tfrac{1}{9}(v_1 - 2v_0)^2(\theta_2 - \theta_1); \text{ and } R_2{}^* = \tfrac{1}{9}(2v_1 - v_0)^2(\theta_2 - \theta_1)$$

[29.37]

This result follows immediately, since revenues are the product of price and quantity sold.

X29.17 **Using the method outlined above, confirm that there will be maximum product differentiation in quality.**

For each firm, differentiating profits, R_f, in terms of the firm's own choice of quality, θ_f, we obtain* $\frac{\partial R_1{}^*}{\partial \theta_1} = -\tfrac{1}{9}(v_1 - 2v_0)^2 < 0$; *and* $\frac{\partial R_2{}^*}{\partial p_2} = \tfrac{1}{9}(2v_1 - v_0)^2 > 0$. *This indicates that firm 1 reduces its profits whenever it increases the quality of its output, while firm 2 increases its profits whenever it increases quality. So firms choose the greatest possible product differentiation.*

X29.18 **Repeat the argument above, but demonstrating that a decrease in quality, θ_1, also leads to decreased intensity of price competition, and thus allows higher prices and profits for both firms.**

Following a decrease in quality index θ_1, firm 1's reaction function moves to the right, so that its slope remains unchanged, but intersects the vertical axis lower down. Firm 2's reaction moves to the left, so that there is again no change to the slope, but the intersection with the vertical axis moves up. Sketching these changes in a diagram, we see that both prices increase, and since firm's profits depend on the difference in quality, profits increase. The reduction in quality competition is beneficial for the firms.

X29.19 **We consider the sub-games of length 1.**

a) **Confirm that facing price p_2^1 for a product of quality v = 1, the customer, C, will choose to buy the product if $p_2^1 \leq 1$.**

Facing a sub-game of length 1, the consumer is fully informed, and so WTP(1) = 1. The consumer will buy the good at that price, or any lower one.

b) **Confirm that facing price p_2^0 for a product of quality v = 0, the customer, C, will choose to buy the product if p_2^0 = 0.**

By the argument above, since WTP(0) = 0, the consumer will only buy the good at price p_2^0 = 0.

X29.20 **For the sub-games of length 2, beginning with firm *F*'s pricing choice in *Stage 2*:**

a) **Confirm that if $v = 1$, in the perfect equilibrium $(p_2^1)^* = 1$.**
The firm will choose the highest price at which it can make sales in order to maximize revenues.

b) **If $v = 0$, show that the firm is indifferent among all prices $p_2^0 \geq 0$. [*Hint:* Think what customer *C* will choose to do if $p_2^0 > 0$.]**
If the firm chooses price $p_2^0 = 0$, then it makes the sale, but there is no contribution to profit. If it sets $p_2^0 > 0$, then it fails to make sales, so revenues are again zero, and there is no contribution to profit.

X29.21 **Define the updating rule for the potential customer's beliefs as follows:**

$$\beta^e = \begin{cases} 1, \text{if } (p_1, A_1): p_1 \leq p_1^*, A_1 = A^* \\ 0, \text{otherwise} \end{cases} \quad [29.38]$$

Assume that this updating rule is known to the firm. Confirm that in *Stage 1*:

a) **If the potential customer observes:**

i. **action pair (p_1^*, A_1^*), then if $p_1^* \leq 1$, the customer will choose *Buy*;**
The potential customer observes enough advertising and a low enough price to believe that the company is investing in advertising to demonstrate high quality.

ii. **action pair (p_1, A_1^*), with $p_1 < p_1^* \leq 1$, the customer will choose *Buy*.**
The potential customer considers that the firm has offered the goods for sale at a deeper discount than is necessary to confirm that the good is of high quality, and so is willing to buy.

iii. **action pair (p_1^*, A_1), with $A_1 < A_1^*$, the customer will choose *Don't buy*.**
The potential customer does not observe any investment in demonstrating quality, and so believes that the good is of low quality; and would only be willing to pay price $p_1 = 0$.

b) **Anticipating the customer's decision, then if $v = 1$:**

i. **the firm chooses action pair (p_1^*, A_1^*) rather than (p_1, A_1^*);**
This follows, since setting $p_1 < p_1^*$ reduces prices without affecting sales, and therefore revenues and profits.

ii. **the firm chooses (p_1^*, A_1^*) rather than (p_1^*, A_1) if:**

$$1 + p_1^* - A_1^* - 2c_1 \geq 0 \quad [29.39]$$

The firm selling high quality goods experiences costs $A_1^* + 2c_1$. It can only generate revenues $1 + p_1^*$ by setting $A_1 = A_1^*$; so Expression [29.39] is the participation constraint for the firm; where it is satisfied, the firm can make profits from signalling.

c) **But if $v = 0$, then the firm chooses action pair (0, 0); unless $p_1^* \geq A_1^*$, in which case it would choose (p_1^*, A_1^*).**
When $v = 0$, for the firm to make any positive revenue, it must be able to sell its output at a price above the advertising cost. This requires $p_1^* \geq A_1^*$. But such behaviour is inconsistent with the structure of Bayesian learning.

X29.22 **Confirm that when the cost $c_1 = 0.25$, there is a Perfect Bayesian Equilibrium of this game, in which $(A_1^*, p_1^*) = (0.75, 0.5)$.**
Beginning with the customer, we see that the customer will believe that a firm is of high quality only if the advertising value exceeds the price in period 1; satisfying the requirement

that this cannot be a profitable choice for a firm selling low quality goods. We also note that the participation constraint stated above will be satisfied; the firm will generate revenue $R = 1 + p_1*$, and costs $C = 0.75 + 0.5$; so that its profit $\Pi = 0.25$.

Chapter 30

X30.1 **eBay uses a proxy bid system. Describe how this works, and what a bidder has to do to win the auction.**

Bidders submit an amount (up to their true WTP) as the proxy bid. Assuming that the proxy bid is higher than the current highest bid, the bid that is published (in the bidding system) is the smallest possible increment above the highest bid; then if another bidder has entered a proxy bid higher than your own proxy bid, a higher bid will be entered. Note that a bidder making a proxy bid, which is less than the true WTP, may revise the proxy during the auction.

X30.2 **Form the payoff matrix for this game in normal form, and confirm the best replies to bids of 0 and v_I. Show that for Erica, bidding v^E = 18,000 is a dominant strategy, whereas Felicity is then indifferent between actions. Hence confirm that there are two Nash equilibria, (b_E*, b_F*) = (18,000, 0), and (b_E*, b_F*) = (18,000, 15,000). Why does Giselle prefer the outcome in which Felicity bids her valuation? How could she make certain that this will happen?**

Car auction		Felicity	
		Bid $b_F = 0$	Bid $b_F = 15{,}000$
Erica	Bid $b_E = 0$	(0, 0)	(0, 15,000)
	Bid $b_E = 18{,}000$	(18,000, 0)	(3,000, 0)

From the table, we see that the bid of 18,000 is Erica's best reply both when Felicity bids b_F= 0 and when she bids b_F = 15,000. We also note that while bid b_F = 15,000 is the best reply to bid b_E = 0, Felicity is indifferent between the bids b_F = 0 and b_F = 15,000 when Erica bids b_E = 18,000. There are two pairs of consistent best replies; (b_E, b_F*) = (18,000, 0), and (b_E*, b_F*) = (18,000, 15,000). These are the Nash equilibria of the game. Giselle prefers the equilibrium in which Felicity bids her valuation because this maximizes her revenue. She can ensure that this will occur by setting a minimum bid of her own valuation.*

X30.3 **What would happen in this auction if Erica agreed that she would pay Felicity €6,000 so long as b_F = 0?**

In this case, Erica would bid €18,000, but with Felicity bidding 0, we consider that the bid secures the car but that Giselle cannot ask Erica to pay any money for it; so she obtains it for the €6,000 payment to Felicity. An auction can be manipulated by the bidders.

X30.4 **Assume that Giselle sets a reserve price, $\underline{b}$ = 12,000, and announces an additional rule: that in the event of the two bids being equal, she will sell the car to Erica.**

a) **Obtain expressions for Erica's best replies for conjectured bids, b_F^e: (i) b_F^e < 12,000; (ii) 12,000 ≤ b_F^e < 18,000; (iii) b_F^e = 18,000; and (iv) b_F^e > 18,000.**

(i) Expecting Felicity to bid b_F^e < 12,000, Erica wins the auction with a bid of 12,000, or higher. (ii) b_F^e: 12,000 ≤ b_F^e ≤ 18,000, Erica wins the auction with a bid b_E: $b_E \geq b_F^e$. (iv) For b_F^e > 18,000, Erica does not wish to win the auction, and so the best reply $b_E^*(b_F^e) < b_F^e$.

b) **Obtain expressions for Felicity's best replies for conjectured bids, b_E^e: (i) b_E^e < 12,000; (ii) 12,000 ≤ b_E^e < 15,000; (iii) b_E^e = 15,000; and (iv) b_E^e > 15,000.**

(i) Expecting Erica to bid b_E^e < 12,000, Felicity wins the auction with a bid of 12,000, or higher. (ii) b_E^e: 12,000 ≤ b_E^e < 15,000, Felicity wins the auction with a bid b_F: $b_F > b_E^e$. (iv) For b_E^e ≥ 15,000, Felicity does not wish to win the auction, and so the best reply $b_F^*(b_E^e) < b_E^e$.

c) **Illustrate, on separate diagrams: (i) Erica's best replies; (ii) Felicity's best replies; and (iii) the Nash equilbria, where conjectures are consistent.**

These are illustrated in Figure 30.1 in the textbook, and are described fully in the text.

d) Confirm that there are Nash equilibria in which both Erica and Felicity bid in excess of their valuations, but that there is no Nash equilibrium in which they both bid in excess of Erica's valuation.

If one bidder believes that the other bidder will enter a bid that is less than her valuation, she wins the auction with any bid that is greater than the other bid, including bids that are greater than her own valuation. If both enter bids that are greater than Erica's valuation, then the winner regrets entering such a high bid, and would have done better to offer less than the other bid; so that the winning bid is not a best reply to the other bid, and the outcome is not a Nash equilibrium of the game.

X30.5 **Show that should Erica bid less than her *WTP*, then there are (non-equilibrium) action profiles in which she would do worse than by bidding her *WTP*. Show that there are other (non-equilibrium) action profiles in which she would do worse by bidding more than her *WTP* than by bidding her *WTP*. What do you conclude about the strategy $b_E = v_E$ compared with all other strategies?**

If Erica bids b_E: $b_E < WTP_E$, then it is possible that Felicity will make a bid, b_F: $b_E < b_F < WTP_E$ and Erica loses the surplus that would come from winning, but paying less than her WTP. Similarly, if Erica bids b_E: $b_E > WTP_E$, then it is possible that Felicity will make a bid, b_F: $WTP_E < b_F < b_E$ and Erica then wins the auction, but paying more than her WTP, she makes a loss.

X30.6 **As before, a seller proposes to sell a car by second-price, sealed-bid auction. There are $n > 2$ potential bidders, indexed $i = 1, 2, \dots, n$ with valuations $v_1 > v_2 \dots > v_n > 0$, each choosing an action, b_i. In the event of the winning bids being equal, the seller will accept the winning bid made by the player with the lowest index.**

a) What is the payoff received by player 1: (i) when bid b_1 is the highest bid, and bid b_2 the second-highest bid; and (ii) when bid $b_1 < b_2$?

Player 1 receives (i) payoff $\pi_1 = v_1 - b_0$, when making the winning bid; and (ii) payoff $\pi_1 = 0$, when outbid by player 2.

b) Under what circumstances is bid b_i a winning bid? Under what circumstances would bid $b_i > 0$ be a best reply?

Bid b_i is the winning bid when it is the highest of the n bids. Bid $b_i > 0$ is a best reply for valuation v_i: $v_i > \max\{b_j\}$, where $\{b_j\}$ is the set of bids made by other players, so that $\max\{b_j\}$ is the payment made. We require bidder i to make the highest bid, and the second highest bid to be less than the bidder i's valuation.

c) Confirm that there are Nash equilibria in which: (i) $b_i = v_i$ for all n players; (ii) $b_1 = v_2$ and $b_i = 0$ for all of the other players; (iii) $b_1 > v_1$ and $b_i = v_1$ for all of the other players; (iv) $b_2 > v_1$ and $b_i = 0$ for all of the other players.

(i) Bidder 1 wins, and pays v_1. All players receive a payoff of zero, and none can increase their payoff by bidding higher (or lower) and breaking the tie. Given other bids, each bid is a best reply, and this is a Nash equilibrium.

(ii) Bidder 1 wins, and $b_2 = 0$. So bidder 1 pays $b_2 = 0$ and obtains payoff $\pi_1 = v_1$; all other players receive a payoff of zero. No player can increase payoff by bidding higher, player 1 does not increase payoff by bidding lower (unilaterally). So all bids are best replies to the other bids, and this is a Nash equilibrium.

(iii) Bidder 1 wins, and pays v_1. All players receive a payoff of zero, and none can increase their payoff by bidding higher (or lower) and breaking the tie. As before, given other bids, each bid is a best reply, and this is a Nash equilibrium.

(iv) Bidder 2 wins, and pays 0. Bidder 2 receives a payoff $\pi_2 = v_2$, while all other players receive a payoff of zero. No player can increase their payoff by bidding higher (or lower) and breaking the tie. Once again, given other bids, each bid is a best reply, and this is a Nash equilibrium.

d) **Show that the strategy $b_i > v_i$ is weakly dominated by the strategy $b_i = v_i$.**
Bidding $b_i > v_i$, there are occasions where the bid will win, when bidding $b_i = v_i$ would be a losing bid. But on those occasions, it must be that there is some bid b_j: $b_i > b_j > v_i$, and the winner obtains payoff $\pi_i = v_i - b_j < 0$. In all other situations (where $b_j > b_i$, or where $v_i > b_j$), either the bid b_i is a losing bid, and so the payoff to bidding either v_i or b_i will be zero; or else the bid v_i is a winning bid, and so the payoff to bidding either v_i or b_i will be $\pi_i = v_i - b_j$. We can identify action profiles in which bidding v_i yields a higher payoff than bidding $b_i > v_i$, but none in which bidding $b_i > v_i$ yields a higher payoff than bidding v_i. The strategy of bidding $b_i = v_i$ is therefore weakly dominant.

e) **Show that the strategy $b_i < v_i$ is weakly dominated by the strategy $b_i = v_i$.**
Bidding $b_i < v_i$, there are occasions where the bid will lose, when bidding $b_i = v_i$ would be a winning bid. But on those occasions, it must be that there is some bid b_j: $v_i > b_j > b_i$, so that the payoff to bidding the true valuation, $\pi_i(v_i) = v_i - b_j > 0$, while the payoff to bidding b_i (and losing), $\pi_i(b_i) = 0$. In all other situations (where $b_j > v_i$, or where $b_i > b_j$), either the bid v_i is a losing bid, and so the payoff to bidding either v_i or b_i will be zero; or else the bid b_i is a winning bid, and so the payoff to bidding either v_i or b_i will be $\pi_i = v_i - b_j$. Again, we identify action profiles in which bidding v_i yields a higher payoff than bidding $b_i < v_i$, but none in which bidding $b_i < v_i$ yields a higher payoff than bidding v_i. The strategy of bidding $b_i = v_i$ is therefore weakly dominant.

f) **Confirm that for there to be a Nash equilibrium in which player *i* wins, $b_i^* > v_1$ and $b_{\sim i}^* \leq v_i$; but that the strategies in this Nash equilibrium are weakly dominated by the strategies in the Nash equilibrium, $b_i^0 = v_i$.**
We assume that the losing players obtain payoff $\pi_{\sim i} = 0$. We see here that for player 1 to win, she must abandon the weakly dominant strategy of bidding her valuation and enter bid $b_1 > b_i^*$. But then, she will pay more than her valuation, and be worse off. Similarly, bidder i obtains payoff $\pi_i = v_i - \max\{b_{\sim i}\} > 0$. He cannot increase his payoff, given other bids, but can reduce it by bidding less than $\max\{b_{\sim i}\}$. All actions in this action profile are best replies given the other players' actions, and this is a Nash equilibrium.
Suppose that bidder 1 always follows the strategy $b_1 = v_1$. We know that this guarantees player 1 a non-negative payoff, and that bidder 1 is never worse off adopting this strategy than in the alternative discussed above. Then player i, who follows the aggressive strategy of bidding more than his valuation will win the auction, but receive payoff $\pi_j = v_j - v_1 < 0$. This follows directly from our argument that the strategy $b_i^0 = v_i$ is weakly dominant for all players, and so is the basis of a weakly dominant equilibrium of the game.

X30.7 **Suppose that an object is brought to an ascending (English) auction. There are only two bidders, *A* and *B*, willing to pay $v_A = 3$ and $v_B = 2$. The auctioneer sets the initial bid $b_0 = 1$, and will require bid increases of 1 unit. After every point at which they have to make a decision, *A* has the choice of making another bid or of quitting the auction.**

a) **Assume that *A* makes the initial bid. Draw a decision tree showing the game in extensive form and confirm that the sub-game perfect equilibrium action profile, A^* = {*Bid*, *Bid*, *Bid*}; and that player *A* obtains the object by paying price $p = 3$, so that neither player makes a surplus.**

The decision tree will have three nodes. At each one, the bidder making a choice chooses between the actions Bid and Stop. Choosing Bid, the game continues to the next node (and the amount that must be paid for the object increases by one unit). Choosing Stop, the player obtains payoff $\pi_i = 0$, while the other player (who wins the auction) obtains payoff $\pi_j = v_j - b_j$, where b_j is the last bid made.

At the first node, A chooses between the actions Bid $b_1 = 1$ and Stop. Choosing Stop, the game ends, with payoffs $(\pi_A, \pi_B) = (0, 3)$. Choosing Bid, the game continues to the next node, at which B chooses between the actions Bid $b_2 = 2$ and Stop. Much as at the preceding node, choosing Stop, the game ends, with payoffs $(\pi_A, \pi_B) = (2, 0)$. Choosing Bid, the game continues to the third (and last) node. Here A chooses between the actions Bid $b_3 = 3$ and Stop. Choosing Stop, the game ends, with payoffs $(\pi_A, \pi_B) = (0, 1)$. Choosing Bid, the game ends with payoffs $(\pi_A, \pi_B) = (0, 1)$.

Applying the principle of backward induction, we see that in the subgame of length 1, player A is indifferent between Bid and Stop. It is therefore consistent with the subgame equilibrium that A chooses Bid. In the same way, in the subgame of length 2, player B anticipates receiving a payoff of zero from Stop, but assigning probability p_1 to player 1 choosing Stop in the subsequent subgame, expects to obtain payoff p_1 from choosing Bid. Bid is therefore part of the subgame perfect action profile. We apply a very similar argument to player A making the choice between Bid and Stop at the initial node (in the subgame of length 3).

b) Repeat the exercise, assuming that *B* makes the initial bid. Confirm that the sub-game perfect equilibrium action profile, $A^* = \{Bid, Bid, Stop\}$, and that player *A* obtains the object by paying price $p = 2$, making a surplus $v_A - p = 1$.

The game has the same structure, except that the player function assigns player B to the first and last nodes, and player A to the second node. Player A can now be certain in the subgame of length 2 that player B will choose Stop, so that payoffs $(\pi_A, \pi_B) = (1, 0)$. Player B is therefore indifferent in the subgame of length 3 between choosing Bid and Stop, since both lead to a payoff of zero. This confirms that there is a subgame perfect action profile as stated above.

c) Repeat the exercise, assuming that there is a Japanese auction. Show that *B* drops out when the required bid $p = 3$, so that the outcome is the same as in part (a).

This follows directly, so long as the clock hand moves at finite intervals, stopping in between them. Neither A nor B need drop out when the clock shows $p = 2$ (although we should perhaps argue that it is consistent with the description of the game to assign B a mixed strategy defined as a probability distribution over the actions Stop and Continue). But B must drop out when $p = 3$, ensuring that A wins the auction, and pays price $p = 3$.

X30.8 **As in X30.6, a seller proposes to sell a car, but now by an English ascending auction. With $n > 2$ potential bidders, indexed $i = 1, 2, \ldots, n$ with valuations $v_1 > v_2 \ldots > v_n > 0$, and perfectly informed bidders, show that there are sub-game perfect equilibria: (a) in which player 1 makes an initial bid, $b_1 = v_2$, and every other bidder immediately stops bidding; and (b) in which some player, i, immediately bids $b_i = v_1 - 2$, bidder 1 announces $b_1 = v_1 - 1$, and bidding then stops.**

a) *At the bid $b_1 = v_2$, for another bidder to win the auction, they must make a bid $b_j > v_2$; such a bidder then risks winning the auction, and obtaining a negative payoff, so that for all other bidders, Stop is a best reply to the initial bid.*

b) *With player 1 choosing the action, Bid $v_1 - 1$, there is no other player for whom choosing the action Bid v_1, generates a positive expected payoff, since the payoff to bidding and winning, $\pi_j(v_i) = v_j - v_i < 0$; whereas the payoff to Stop, $\pi_j(0) = 0$. So player 1 is certain of generating* *surplus when bidding $v_1 - 1$. (We must assume that bidder i, when bidding b_i, hopes to win and avoid a loss, so that $b_i \leq v_i - 2$.)*

X30.9 **Suppose that Giselle decides to sell her car by a Dutch auction.**

a) **Explain why Erica should be certain that she will win the auction if she follows the strategy b_E: $b_E \geq 15{,}000$.**
Felicity will not bid more than her valuation of 15,000, since she would then make a loss.

b) **Confirm that Erica cannot make a surplus with the bid $b_E = 18{,}000$.**
If $b_E \geq 18{,}000$, Erica is certain to win; but her surplus $\pi_E = v_E - b_E \leq 0$.

c) **Confirm that if Erica follows the strategy $b_E \geq 15{,}000$, then Felicity will be indifferent between all bids b_F: $b_F \leq b_E$.**
With $b_E > v_F$, Felicity would prefer not to win the auction, and so chooses $b_F \leq b_E$.

X30.10 **Generalizing the previous case, suppose that a seller proposes to sell a car by first-price, sealed-bid auction. There are $n > 2$ potential bidders, indexed $I = 1, 2, \ldots, n$ with valuations $v_1 > v_2 \ldots > v_n > 0$, each choosing an action, b_i. In the event of the winning bids being equal, the seller will accept the winning bid made by the player with the lowest index.**

a) **Suppose that player n conjectures that $b_1^e = \max\{b_i^e\} > v_n$. Write down player n's best reply.**
We define player n's best reply $b_n^*(b_1^e)$: $b_n \leq b_1^e$.

b) **Suppose that player n conjectures that $b_1^e = \max\{b_i^e\} < v_n$. Write down the set of bids for which player n obtains a positive payoff upon winning.**
For player n to win, $b_n^*(b_1^e)$: $b_n > b_1^e$ and for player n to make a surplus, $v_n \geq b_n^*$.

c) **Confirm that there is no equilibrium in which bid $b_n > v_n$ and bidder n wins the auction.**
Since $b_n > v_n$, bidder n can reduce the bid and either lose the auction, or win it and generate a surplus. In either case, b_n is not a best reply to the actions of the other bidders.

d) **Confirm that there is no equilibrium in which $b_n < v_n$ and bidder n wins the auction.**
If $b_n < v_n$, and bidder n wins the auction, then all other players could choose the action $b_i = v_n$, win the auction, and achieve surplus $\pi_i = v_i - v_n > 0$; so that their actions are not a best reply to bidder n's choice.

e) **Hence or otherwise confirm that in equilibrium, $v_1 \geq b_1^* \geq v_2$; and $b_1^* = \max\{b_i^*\}$.**
It is sufficient here to consider the two player case. Bidder 1 can be certain of winning the auction by choosing this strategy; and no other bidder will want to match this bid. The auction is efficient in the sense of allocating the good to the person with the highest valuation.

X30.11 **Confirm that there are Nash equilibria in which:**

a) $b_1 = b_2 = v_2$; $b_3 = b_4 = \ldots = b_n = 0$;

b) $b_1 = b_2 = \ldots = b_n = v_2$;

c) $b_1 = b_2 = ½(v_1 + v_2)$; $b_3 = b_4 = \ldots = b_n = 0$;

d) $b_i = v_{i+1}$ for $i < n$; $b_n = v_n$.
All that is required for this question is to confirm that the conditions in X30.10e) are satisfied. We see that in all four cases, $v_1 \geq b_1^* \geq v_2$; and $b_1^* = \max\{b_i^*\}$.

X30.12 **Suppose that an art school regularly runs auctions of students' work at the end of their graduation show. Do you consider this to be a private-value auction or a common-value auction?**
At the point of graduation, most students' work will be of interest to bidders because it appeals to their personal taste, rather than there being a market for that particular students' work, in which the future (financial) value of the art will determine its value. This is a private value auction.

X30.13 **Consider instead auctions of high-value, rare works of art. Remembering that these are often purchased by individuals or organizations as financial investments, discuss whether they should be treated as private-value or common-value auctions.**
We might well consider these to be common-value auctions. The purchase is made as a financial investment; it depends upon expectations of future prices.

X30.14 **Suppose that following the death of its previous owner, a painting by Raphael (a 16th Italian artist) comes to the market. A potential bidder obtains a report from an art historian indicating that the painting was probably completed by an assistant, rather than by the master. Why might this signal affect the bidder's *WTP*?**
Since it is 500 years since Raphael's death, the supply of new works is very limited (relying effectively on new attributions, or the discovery of lost works). We might think of Raphael as the creative director in a large workshop, delegating much of the routine work to skilled assistants, but managing the process of design directly and executing the highest quality work in the workshop. Work undertaken by a skilled assistant therefore lacks the direct input of the master, and so is substantially less rare. Given the argument about ownership being a financial investment, the greater the rarity of the object, and belief that rarer objects are more exquisitely finished, the more valuable it will be.

X30.15 **Confirm that any strategy $b_i > v_i$ is weakly dominated by the strategy $b_i = v_i$. [*Hint:* Think of the payoff to winning when $b_i > \max\{v_j\} > v_i$.]**
We suppose that all other bidders follow a strategy $b_j = v_j$. Then if $b_i > \max\{v_j\} > v_i$, bidder i wins the auction, and pays $v_j > v_i$, which is worse than bidding v_i, losing the auction, and obtaining payoff $\pi_i = 0$.

X30.16 **Confirm that any strategy $b_i < v_i$ is weakly dominated by the strategy $b_i = v_i$. [*Hint:* Think of the payoff to losing when $v_i > \max\{v_j\} > b_i$.]**
We suppose that all other bidders follow a strategy $b_j = v_j$. Then if $v_i > \max\{v_j\} > b_i$, bidder i loses the auction obtaining payoff $\pi_i = 0$, whereas by following strategy $v_i = b_i$, bidder i wins and pays $v_j < v_i$, which is better.

X30.17 **Confirm that if $v_i > \max\{v_j\}$, then any bid b_i^*: $b_i^*(b_j^*) \geq \max\{v_j\}$ is a best reply; and that if $v_i < \max\{v_j\}$, then any bid b_i^*: $b_i^*(b_j^*) \leq \max\{v_j\}$ is a best reply. Hence demonstrate that there is a Nash equilibrium in which every bid $b_i^* = v_i$.**
We assume that all other bidders have adopted the strategy of bidding their valuation. So, if player i bids at least the highest valuation of the other bidders, bidder i is certain to win the auction, paying a price $\max\{v_j\}$. Where bidder i has the highest valuation, this means that the bidder makes a surplus from the transaction. Where bidder i does not have the highest valuation, it may lead to winning, but making a loss. In this latter case, bidding no more than the highest of other bidders' valuations leads to a zero payoff, while bidding more will lead to a loss.

If every bidder follows the strategy of bidding their own valuation, then the winner cannot increase surplus by changing her bid; but no other bidder can increase surplus either by increasing their bid by enough to win the auction, or by varying it and still losing.

X30.18 **Confirm that Expression 30.11 simplifies to give:**

$$b_E^*(v_E) = ½(10{,}000 + v_E). \quad [30.12]$$

Sketch a graph of b_E^*. Confirm that b_E^* is the expected value of v_F given that $v_F \le v_E$.

Integrating, b_E^* =

$$v_E - \frac{\left.0.5x^2 - 10{,}000x\right]_{10{,}000}^{v_E}}{v_E - 10{,}000} = v_E - \frac{0.5{v_E}^2 - 0.5(10{,}000)^2 - 10{,}000v_E + 10{,}000^2}{v_E - 10{,}000} = v_E - \frac{\frac{1}{2}(v_E - 10{,}000)^2}{v_E - 10{,}000}.$$

The result follows immediately on dividing through by v_E – 10,000.

In a diagram with v_E on the horizontal axis, and b_E^* on the vertical, the graph of b_E^* is an upward sloping line, intersecting the vertical axis at b_E^* = 5,000, with gradient, m = 0.5.

X30.19 **Suppose that there are several bidders in an auction. They all know that the others will wish to shade their bids. How would you expect the degree of shading of bids to increase as the number of bidders increases?**

We expect bidders to assume that they value the object being auctioned the most. That is, among the n potential bidders, their valuation is the highest. We have argued that in this type of auction, it is therefore sensible to shade the bid so that we do not expect the bidder with the second highest valuation to be able to deviate from the equilibrium strategy, bid their true valuation and win the auction. With a uniform distribution, it is possible to show that bidders should then expect the other values to be spaced equally in the interval between the lowest possible value and their own value. The degree of shading is then the fraction $\frac{n-1}{n}$ of the interval

v_i – v_0.

X30.20 **Suppose that there are two bidders, whose valuations are drawn from a uniform distribution, *U*[0, 1].**

a) **What is the probability of bidder 1, whose valuation is v_1, winning the auction? Write down an expression for this bidder's expected payoff, and hence obtain the bid function associated with a symmetric equilibrium.**

Bidder 1 wins the auction with $Pr(v_2 < v_1) = v_1$. Using the argument above, we can write the rate of change of bidder 1's expected payoff with respect to the valuation, $\frac{dE[\pi_1]}{dv_1} = v_1$, which is the probability of winning. So bidder 1's expected payoff can be written as

$$E[\pi_1] = \int_0^{v_1} x dx = \left[\frac{x^2}{2}\right]_0^{v_1} = \frac{{v_1}^2}{2}.$$

We also know that $E[\pi_1] = v_1(v_1 - b_1)$ so that we require b_1 = $0.5v_1$; the bidding rule (for both bidder 1 and bidder 2) is to bid half of their valuation.

b) **Repeat part (a), but assume that there are *N* bidders.**

Bidder 1 wins the auction with $Pr(v_1 = \max\{v_i\}) = {v_1}^{n-1}$. Using the argument above, we can write the rate of change of bidder 1's expected payoff with respect to the valuation, $\frac{dE[\pi_1]}{dv_1} = {v_1}^{n-1}$, which is the probability of winning. So bidder 1's expected payoff can be written as

$$E[\pi_1] = \int_0^{v_1} \left(x^{n-1}\right) dx = \left[\frac{x^n}{n}\right]_0^{v_1} = \frac{{v_1}^n}{n}.$$

We also know that $E[\pi_1] = {v_1}^{n-1}(v_1 - b_1)$ so that $(n-1){v_1}^n = n{v_1}^{n-1}b_1$, or $b_1 = \frac{n-1}{n}v_1$; the bidding rule (for both bidder 1 and bidder 2) is to shade their bid by a fraction $\frac{1}{n}$ of their valuation.

c) How would you explain the effect of increasing the number of bidders?
The larger the number of bidders, the closer any bidder would expect the next highest valuation to their own to be, and so the optimal degree of shading will be smaller.

X30.21 **Rewrite Expression 30.16 in terms of value v_i, the difference in probabilities of the two bidders receiving the good and the difference in the expected payments. What interpretation would you place on this result?**
We can rewrite Expression 30.16 as $E[\pi_{ij}] = E[\pi_i] + (v_j - v_i)\theta_i$, which means that for player j, following the equilibrium strategy for player i, the expected payoff from participation in the auction is the expected payoff to player i, adjusted by the difference in the players' values times the probability of player i winning.

X30.22 **How can the seller's expected revenues be the same in first-price and second-price auctions when the mechanisms used are different?**
The seller's revenue expected revenue can be the same in different forms of auctions because the different bidding rules reflect the interaction between the probability of winning, and the surplus given that the player wins.

X30.23 **Suppose that bidder *j* considers impersonating bidder *i*. Write down an expression equivalent to Expression 30.14. Using this expression to substitute for π_j in Expression 30.15, confirm that the probability of being given the good increases with bidder *j*'s valuation.**
The equivalent to expression [30.14] is $E[\pi_{ij}] = v_j\theta_i - P_i$; and this compares with the situation facing player i, for whom $E[\pi_i] = v_i\theta_i - P_i$. It follows that $E[\pi_{ij}] = E[\pi_i] + (v_j - v_i)\theta_i$. That is, we find that for player j, seeking to imitate player i, the difference from the expected payoff received by player i is also the difference in values multiplied by the probability of player i winning.
Now, for there to be an equilibrium bid, we can demonstrate that player j's probability of winning (from first principles) satisfies the condition $\frac{dE[\pi_j]}{dv_j} = \theta_j$, which is to say that the rate of change of the expected payoff with the underlying value is the probability of winning. Since $\theta_j > 0$, the expected value is increasing.
We also note that the probability of being given the good is the probability of the bidder having the highest valuation, and this certainly increases with the valuation.

X30.24 **Consider the derivation of Expression 30.8. Give an intuitive explanation of its generalization in Expression 30.20.**
In Expression [30.8], we define how the expected payoff from participation in the auction changes for one of two players in the auction, making the assumption that the game is symmetric, so that both players follow the same bidding rule. We find that this expected payoff increases with the underlying valuation, and that the rate of increase of the expected payoff as the underlying value increases is the probability of the other bidder having a lower value, so that the rate of change is the probability density function.
Expression [30.20] is a generalization of this claim to a model in which there are n bidders, and the probability distribution of the bidders' values is not known. It states that the rate of change of the expected payoff for any bidder as the bidder's value increases is the probability of winning the auction, which is effectively the probability of that bidder's value being the highest of the n bidders' values.

X30.25 **Obtain expressions for the following:**

a) **The distribution function, G, for $n-1$ independently distributed values, v_i, with distribution function, $F = F(v)$, which are all less than the value v^*.**
We write down the distribution function G: $G(v) = [F(v)]^{n-1}$

b) **The probability density function, $g(v)$: $g = \frac{dG}{dv}$.**
Differentiating the density function, we obtain g: $g(v) = \frac{dG}{dv} = (n-1)[F(v)]^{n-2}f(v)$.

c) **The expected payment P^*, given that the winner's *WTP* is v^*.**
From expression [30.20], g(v) is the rate of change of the expected payoff to participation in the auction with respect to bidder valuation. So the expected payoff $E[\pi]$ = $\int_{\underline{v}}^{v}(n-1)[F(x)]^{n-2}f(x)dx$, where $\underline{v}$ is the minimum possible value.

X30.26 **Assume that the distribution function, F: $F(v) = \frac{v-10{,}000}{10{,}000}$, so that valuations are drawn from a uniform distribution, with $10{,}000 \le v \le 20{,}000$, and that there are only two bidders. Show that:**

a) **the distribution function, defined in X30.25, is G: $G(v) = F(v)$;**
$G(v) = F(v)^{n-1}$. With n = 2, $G(v) = F(v) = \frac{v-10{,}000}{10{,}000}$. This is the probability of one bidder having a lower value than v.

b) **the probability density function g: $g(v) = \frac{1}{10{,}000}$; and**
Differentiating the distribution function, $g(v) = \frac{dG}{dv} = \frac{1}{10{,}000}$; since the distribution function is linear.

c) **the expected payment is P^*: $P^*(v^*) = \dfrac{\int_{10{,}000}^{v^*}\frac{x}{10{,}000}dx}{\int_{10{,}000}^{v^*}\frac{dx}{10{,}000}} = \frac{1}{2}(v^*+10{,}000)$**
We recognize the ratio of integrals as the bid that the winner will make divided by the probability that the winner has valuation of no more than v, and so interpret the expression P*(v*) as the expected payment received by the seller given that the winner of the auction has value v*.*

$$P^*(v^*) = \frac{\int_{10{,}000}^{v^*}\frac{x}{10{,}000}dx}{\int_{10{,}000}^{v^*}\frac{dx}{10{,}000}} = \frac{\left[\frac{x^2}{20{,}000}\right]_{10{,}000}^{v^*}}{\left[\frac{x}{10{,}000}\right]_{10{,}000}^{v^*}} = \frac{\frac{(v^*)^2-10{,}000^2}{2}}{v^*-10{,}000} = \frac{1}{2}\frac{(v^*+10{,}000)(v^*-10{,}000)}{(v^*-10{,}000)} = \frac{1}{2}(v^*+10{,}000)$$

X30.27 **Assume that the distribution function F: $F(v) = v$, so that all valuations are drawn from a uniform distribution with $0 \le n \le 1$and that there are n bidders. Show that:**

a) **the distribution function, defined in X30.25, is G: $G(v) = [F(v)]^{n-1} = v^{n-1}$;**
With distribution function F(v), then as in X30.25, the distribution function of n – 1 values, all of which are less than v is G: $G(v) = [F(v)]^{n-1} = v^{n-1}$.

b) **the probability density function g: $g(v) = (n-1)v^{n-2}$; and**
Again, this follows from X30.25.

c) **the expected payment is P^*: $P^*(v^*) = \dfrac{\int_0^{v^*}(n-1)x^{n-1}dx}{\int_0^{v^*}(n-1)x^{n-2}dx} = \left(\frac{n-1}{n}\right)v^*$.**
The expected payment will be the bid made when the higher bidder has a value v divided by the probability of the highest bidder having such a value.*

$$P^*(v^*) = \frac{\int_0^{v^*}(n-1)x^{n-1}dx}{\int_0^{v^*}(n-1)x^{n-2}dx} = \frac{\left[\frac{(n-1)}{n}x^n\right]_0^{v^*}}{\left[\frac{(n-1)}{(n-1)}x^{(n-1)}\right]_0^{v^*}} = \frac{(n-1)}{n}\frac{(v^*)^n}{(v^*)^{n-1}} = \left(\frac{n-1}{n}\right)v^*$$

X30.28 **Consider the special case where bidders' valuations are distributed according to a uniform distribution with supports [$\underline{v}$, $\bar{v}$]. The distribution function is then $F(v) = \frac{v-\underline{v}}{\bar{v}-\underline{v}}$ and the density function $f(v) = \frac{1}{\bar{v}-\underline{v}}$. The expected value of the 2nd order statistic (the second highest of the n values drawn from the distribution) is then $v_{(2)} = \frac{2\underline{v}+(n-1)\bar{v}}{n+1}$.**

a) **What is the seller's expected revenue in a second-price auction, where the highest *WTP* is v^*?**

The seller expects to receive revenue $P^*(v^*) = \frac{2\underline{v}+(n-1)v^*}{n+1}$

b) **Using revenue equivalence, how much will bidders be willing to offer in a first-price auction?**

The expected payment made in the second price auction is the optimal bid in the first price auction, so that for a bidder with value v, the bid function is b: $b(v) = \frac{2\underline{v}+(n-1)v}{n+1}$

c) **In an all-pay auction, bidders pay the amount that they bid, but only the highest bidder receives the object. Explain why revenue equivalence holds in this case, and use the result of part (b) to calculate optimal bids.**

In an all-pay auction, the allocation rule is the same as for the first and second price auctions – the player making the highest bid wins. It follows that in equilibrium, bids will increase in underlying value according to expression [30.20].