

Contents

<i>Preface</i>	xv
1 Introduction	1
1.1 The need for mathematics in economics	3
1.2 Economic theory, economic models and mathematics	6
1.3 Summary	9
Exercises	9
SECTION A THE BUILDING BLOCKS OF ECONOMIC ANALYSIS	
A1 Tools of the trade: the basics of algebra	13
Learning Objectives	13
A1.1 Algebraic notation	14
A1.2 Arithmetic in algebra	14
A1.3 Brackets in algebra	17
A1.4 Inequalities	21
A1.5 Fractions	23
A1.6 Transposing an expression	27
A1.7 Summary	30
Learning Check	30
Exercises	30
Appendix A1 Powers and exponents	31
A2 Linear relationships in economic analysis	34
Learning Objectives	34
A2.1 Economic relationships	35
A2.2 Using graphs to show economic relationships	35
A2.3 Functions	40
A2.4 Functional notation	44
A2.5 Linear functions	44
A2.6 Summary	47
Learning Check	47
Worked Example	47
Exercises	49
Appendix A2 Graphs in Excel	51

A3 Non-linear relationships in economic analysis	55
Learning Objectives	55
A3.1 Polynomial functions	56
A3.2 Graphs of non-linear functions	57
A3.3 Other non-linear functions	58
A3.4 Logarithms and logarithmic functions	60
A3.5 Exponential functions	62
A3.6 Functions with more than one independent variable	66
A3.7 Inverse functions	69
A3.8 Summary	70
Learning Check	70
Worked Example	70
Exercises	73
Appendix A3 Non-linear graphs in Excel	75
SECTION B LINEAR MODELS IN ECONOMIC ANALYSIS	
B1 The principles of linear models	83
Learning Objectives	83
B1.1 Linear functions	83
B1.2 A simple breakeven model	87
B1.3 Simultaneous equations	89
B1.4 Obtaining linear equations from graphs	91
B1.5 Summary	92
Learning Check	92
Worked Example	92
Exercises	94
B2 Market supply and demand models	96
Learning Objectives	96
B2.1 Market demand and supply	96
B2.2 A partial equilibrium model	98
B2.3 An excise tax in a competitive market	101
B2.4 Elasticity	104
B2.5 Summary	107
Learning Check	107
Worked Example	107
Exercises	109
B3 National income models	111
Learning Objectives	111
B3.1 A national income model	111
B3.2 The national income model in diagram form	118
B3.3 A national income model including a government sector	120
B3.4 A national income model with a government sector and foreign trade	123
B3.5 A national income model with a monetary sector	124
B3.6 Summary	126
Learning Check	127
Worked Example	127
Exercises	129

B4 Matrix algebra: the basics	130
Learning Objectives	130
B4.1 The vocabulary of matrix algebra	131
B4.2 Special matrices	133
B4.3 Matrix algebra	133
B4.4 Matrix addition	134
B4.5 Matrix subtraction	135
B4.6 Multiplication by a scalar	135
B4.7 Matrix multiplication	135
B4.8 Using matrix algebra to represent economic models	138
B4.9 Summary	140
Learning Check	140
Worked Example	140
Exercises	142
Appendix B4 Matrix algebra with Excel	143
B5 Matrix algebra: inversion	149
Learning Objectives	149
B5.1 The matrix inverse	149
B5.2 Using a matrix inverse	150
B5.3 Calculating the matrix inverse	152
B5.4 Determinants	154
B5.5 Calculating the matrix inverse using determinants	158
B5.6 The determinant and non-singularity	159
B5.7 Cramer's rule	161
B5.8 Summary	163
Learning Check	163
Worked Example	163
Exercises	165
Appendix B5.1 Matrix inverse with Excel	166
Appendix B5.2 Confirmation of the determinant method	168
B6 Economic analysis with matrix algebra	171
Learning Objectives	171
B6.1 A partial equilibrium market model	171
B6.2 The effect of an excise tax on market equilibrium	175
B6.3 A basic national income model	177
B6.4 A national income model with government activity	179
B6.5 Summary	181
Learning Check	181
Worked Example	182
Exercises	184
B7 Input–output analysis	186
Learning Objectives	186
B7.1 Input–output tables	186
B7.2 Input–output coefficients	188
B7.3 Input–output analysis	190
B7.4 Summary	195
Learning Check	195

Worked Example	195
Exercises	198
SECTION C OPTIMIZATION IN ECONOMIC ANALYSIS	
C1 Quadratic functions in economic analysis	201
Learning Objectives	201
C1.1 Quadratic functions	201
C1.2 Characteristics of quadratic functions	202
C1.3 Breakeven analysis	205
C1.4 Market equilibrium	209
C1.5 Quadratic functions with no real roots	210
C1.6 Summary	211
Learning Check	211
Worked Example	211
Exercises	212
Appendix C1 Derivation of the roots formula	213
C2 The derivative and the rules of differentiation	215
Learning Objectives	215
C2.1 The slope of linear and non-linear functions	215
C2.2 The derivative	219
C2.3 Rules of differentiation	223
C2.4 Non-differentiable functions	228
C2.5 Summary	229
Learning Check	230
Worked Example	230
Exercises	232
C3 Derivatives and economic analysis	235
Learning Objectives	235
C3.1 Curve sketching	235
C3.2 The derivative and the concept of marginality	237
C3.3 Analysing elasticity	238
C3.4 Analysing other types of elasticity	241
C3.5 Analysing revenue	242
C3.6 Analysing production	244
C3.7 Analysing costs	246
C3.8 The consumption function	247
C3.9 National income	248
C3.10 Summary	249
Learning Check	249
Worked Example	249
Exercises	251
C4 The principles of optimization	253
Learning Objectives	253
C4.1 An example of optimization	253
C4.2 Optimization in general	257
C4.3 Local and global maxima and minima	259

C4.4	Points of inflection	260
C4.5	Summary	262
	Learning Check	263
	Worked Example	263
	Exercises	264
C5	Optimization in economic analysis	266
	Learning Objectives	266
C5.1	Profit maximization	266
C5.2	Profit maximization: perfect competition	267
C5.3	Profit maximization: monopoly	270
C5.4	The effect of tax on profit maximization	272
C5.5	The imposition of a lump-sum tax	272
C5.6	The imposition of a profit tax	274
C5.7	The imposition of an excise tax	275
C5.8	Maximizing taxation revenue	276
C5.9	Summary	279
	Learning Check	279
	Worked Example	279
	Exercises	281
C6	Optimization in production theory	283
	Learning Objectives	283
C6.1	The theory of production	283
C6.2	The theory of costs	287
C6.3	Relationship between the cost functions	289
C6.4	Summary	290
	Learning Check	290
	Worked Example	290
	Exercises	292
 SECTION D OPTIMIZATION WITH MULTIPLE VARIABLES		
D1	Functions of more than two variables	295
	Learning Objectives	295
D1.1	Partial differentiation	295
D1.2	Second-order partial derivatives	299
D1.3	Generalization to n -variable functions	302
D1.4	Differentials	304
D1.5	The total differential	305
D1.6	The total derivative	307
D1.7	Implicit functions	309
D1.8	Summary	310
	Learning Check	310
	Worked Example	311
	Exercises	311
	Appendix D1 Rules for partial differentiation	312
D2	Analysis of multivariable economic models	314
	Learning Objectives	314
D2.1	Partial market equilibrium	314

D2.2	A national income model	317
D2.3	Elasticity of demand	319
D2.4	Production functions	320
D2.5	Utility functions	323
D2.6	Summary	327
	Learning Check	327
	Worked Example	327
	Exercises	328
D3	Unconstrained optimization	330
	Learning Objectives	330
D3.1	General principles of unconstrained optimization	330
D3.2	Profit maximization	335
D3.3	Price discrimination	336
D3.4	Profit maximization revisited	338
D3.5	Summary	340
	Learning Check	340
	Worked Example	341
	Exercises	342
	Appendix D3 The Hessian matrix	343
D4	Constrained optimization	345
	Learning Objectives	345
D4.1	The principles of constrained optimization	345
D4.2	Lagrange multipliers	348
D4.3	Interpretation of the Lagrange multiplier	349
D4.4	Output maximization subject to a cost constraint	352
D4.5	Cost minimization subject to an output constraint	354
D4.6	Maximizing consumer utility subject to a budget constraint	355
D4.7	Summary	357
	Learning Check	357
	Worked Example	357
	Exercises	358
	Appendix D4 The bordered Hessian matrix	359
 SECTION E FURTHER TOPICS IN ECONOMIC ANALYSIS		
E1	Integration and economic analysis	363
	Learning Objectives	363
E1.1	Notation and terminology	363
E1.2	Rules of integration	364
E1.3	Definite integrals	366
E1.4	Definite integrals and areas under curves	367
E1.5	Consumer's surplus	369
E1.6	Producer's surplus	371
E1.7	Capital stock formation	371
E1.8	Summary	372
	Learning Check	373

Worked Example	373
Exercises	374
E2 Financial analysis I: interest and present value	376
Learning Objectives	376
E2.1 Financial mathematics	376
E2.2 Time preference	377
E2.3 Arithmetic and geometric series	377
E2.4 Simple and compound interest	379
E2.5 Nominal and effective interest rates	381
E2.6 Depreciation	382
E2.7 Present value	384
E2.8 Basic investment appraisal	385
E2.9 Internal rate of return	386
E2.10 Interest rates and the price of government bonds	388
E2.11 Summary	390
Learning Check	390
Worked Example	391
Exercises	392
Appendix E2 Financial calculations in Excel	393
E3 Financial analysis II: annuities, sinking funds and growth models	396
Learning Objectives	396
E3.1 Annuities	396
E3.2 The value of an annuity	397
E3.3 NPV of an annuity	399
E3.4 Repayment annuity	400
E3.5 Sinking funds	400
E3.6 The mathematical constant e and rates of growth	401
E3.7 Calculus and e	406
E3.8 Rates of growth	408
E3.9 Summary	410
Learning Check	410
Worked Example	410
Exercises	411
E4 An introduction to dynamics	413
Learning Objectives	413
E4.1 Difference equations	413
E4.2 The equilibrium position	415
E4.3 The solution to a difference equation	416
E4.4 Stability of the model	418
E4.5 A macro model with a government sector	423
E4.6 Harrod–Domar growth model	424
E4.7 Market equilibrium	424
E4.8 Summary	427
Learning Check	427
Worked Example	427
Exercises	428

E5 Probability in economic analysis	430
Learning Objectives	431
E5.1 Uncertainty and probability	431
E5.2 Understanding probability	432
E5.3 Basic rules of probability	433
E5.4 Bayes' theorem	437
E5.5 Probability distributions	439
E5.6 Decision-making under uncertainty	444
E5.7 Summary	448
Learning Check	448
Worked Example	449
Exercises	451
Appendix 1: The Greek alphabet in mathematics	454
Appendix 2: Solutions to Knowledge Check activities	455
Appendix 3: Solutions to Progress Check activities	457
Appendix 4: Outline solutions to end-of-module exercises	478
<i>Index</i>	510

1

Introduction

You may be wondering what mathematics has to do with economic analysis. Like many students you have a serious interest in studying economics and understanding how economics and economic analysis contribute to both microeconomic and macroeconomic activities. As we shall see throughout this text, serious students of modern economic analysis need a number of essential mathematical skills and techniques. Such skills and techniques are necessary to allow you to properly understand economic theory, economic behaviour and modern economic analysis. Let's consider the following scenarios.

Scenario 1

Following the banking crisis that began in 2008, particularly in the US and the UK, a number of governments had to put considerable emergency funding into their banking system to support banks and other financial institutions that were close to collapse. As a result, government expenditure and therefore borrowing increased dramatically. The Finance Minister has now decided that the government deficit (the difference between what the government collects in taxes and what it spends) needs to be cut back significantly to bring the government budget more into balance. However, the Minister is concerned about the impact that reducing government spending will have on particular sectors of the economy. Reducing government expenditure has consequences for the firms who supply the government with goods and services, for their employees, for their shareholders and often for the wider community as well. One sector in particular is under serious scrutiny: the government is thinking of cancelling a couple of major naval shipbuilding contracts. These defence cuts would initially save the government a good deal of money. However, the shipbuilding companies would be badly affected and would have to reduce their workforce considerably. In turn this would lead to a loss in tax revenue for the government and increased welfare payments for those who lost their jobs. The Minister has asked for your economic analysis of the overall impact of such a decision.

Scenario 2

You have been approached by both easyJet and Ryanair, two highly successful budget airlines operating in Europe. They're concerned that the European Union is considering imposing additional taxes on passengers who book short-haul flights by adding the tax to the ticket price charged by the airline. The declared purpose behind such taxes is to encourage passengers to switch to more environmentally friendly transport (such as electric trains) by making short-haul air travel more expensive, and thereby

to reduce the carbon footprint of travel and contribute to a reduction in global warming. Once again, you've been asked to undertake an economic analysis to assess the impact that such a tax would have on their businesses. How will such a tax affect demand for airline travel? How will it affect airline revenue? How will it affect their profitability?

Scenario 3

The government is looking for ways of increasing both its tax revenue and its popularity with the public – not a combination that's easy to achieve. One option is to capitalize on the current unpopularity of senior executives in the banking sector among the public. There was considerable criticism that senior executives were awarded very large performance bonuses at the time when banks were struggling financially and had to be financially supported by the government. The government is now thinking of introducing a special tax on bankers' bonuses. It's looking for your economic advice as to what level of tax it should introduce in order to maximize the amount of tax revenue it collects in this way.

Scenario 4

As part of its economic growth strategy, the South African government is looking for ways to help small businesses start up and expand. It's thinking of encouraging the central bank (the South African Reserve Bank) to increase the money supply in the hope that this will reduce the interest rate that businesses borrowing money have to pay. Generally, an increase in the money supply in an economy makes it cheaper and easier to borrow money. The proposed measure is expected to increase the demand from firms for investment funding and so stimulate economic growth as firms borrow more money to help them expand their economic activities. What economic advice can you give on the impact such a policy would have on the economy?

These scenarios are all realistic – and real. They illustrate the situations that economists and economic analysis are frequently involved in. Some of the scenarios involve analysing and assessing what will happen at a *microeconomic* level – at the level of individual markets, organizations or people. Some of the scenarios involve analysing and assessing at the *macroeconomic* level – at the level of the whole economy or some part of it. At both the micro- and macro-levels it's likely that you will need to do a number of things. First, you will need to analyse each scenario in order to establish the general economic impact that we would expect to happen – that is, using economic theory to predict in general what economic changes are likely to occur. For example, in the easyJet/Ryanair scenario you want to be able to explain in general the effects that introducing a tax would have regardless of the precise value of the tax. Second, you want to quantify the exact effect at the micro- or macroeconomic level – in other words, be able to accurately predict for easyJet/Ryanair the impact of a specific tax level on their business. This combination of understanding the *general* effects of an economic change as well as being able to quantify the *exact* effects of a specific change is important in economics.

Through our economic analysis we want to be able to understand the general principles at work in a given scenario as well as the specific details. For example, you wouldn't want to be in a situation where you analysed the impact of, say, a €10 tax on each flight and then have to repeat the analysis if the tax changed to €15 or then do it all again

because the tax was now going to be €20. What you do need to be able to do is to assess the impact no matter what the exact tax might be. And this is where mathematics comes into economic analysis.

1.1 The need for mathematics in economics

We begin with a bold statement: in order to develop a comprehensive understanding of both economic theory and economic analysis you need a detailed understanding of key mathematical principles and of the role that mathematics can play in the study of economics. You may have opened this text on mathematics and economics with a degree of uncertainty, being unsure what to expect, and possibly even some concern. As part of your studies of economics you may well have been surprised to realize that it is necessary to undertake a formal course in mathematical economic analysis (often under the name of applied economics, quantitative economics or similar) and that the use of mathematics in economics is more widespread than you realized. It may also be the case that the prospect of having to recollect and use key mathematical principles and skills acquired at school is not one that fills you with much enthusiasm. Mathematics in general has a poor reputation with many students, who simply cannot see its relevance in the real world.

However, it is an inescapable feature of the serious study of economics that you need to be familiar and comfortable with key mathematical methods and you need to develop the skills necessary to apply such methods to the economic models that you will gradually build and explore. But it's important for us to stress from the very beginning that this text is *not* a text on mathematical economics as such but rather on the *use* of mathematics in economic analysis. You may be forgiven at this stage in your studies for wondering what the difference is and whether it really matters. In this textbook our main focus is on:

- Seeing how mathematics is used and the value it adds to economic analysis
- Helping you develop your own skills in using such mathematics to improve your own economic analysis
- Increasing the level of your own mathematical confidence
- Developing your awareness of the widespread and typical uses of mathematics in economics.

As you work through the material in this book you will gradually recognize that mathematics need not be viewed as a discipline separate from economics but, rather, one that can be used in an integrated way to help develop economic models and economic theory. We stress again, however, that the purpose of this text is not to turn you into a mathematician but to allow you to develop the mathematical skills and knowledge that you will require as an economist.

There are a number of reasons why the use of mathematics in economics has steadily increased over the years. One important reason is that mathematics is a useful tool in the study of economics. While it is possible to undertake some limited economic analysis by relying on verbal analysis and logic without much use of mathematics, an appropriate use of mathematical notation and solution methods can make life much, much easier for the economist.

Let's return to the easyJet scenario. As a first step in trying to assess the impact that an EU tax might have on passenger numbers, we might consider thinking about the

factors that would affect the number of people flying on one of easyJet's routes, say from Edinburgh to London. At this stage we do not need to know the precise effects, only to grasp what might influence the number of passengers choosing to fly this route. In standard economic terms, we'd say we want to identify the key factors affecting the demand for seats on this route.

Progress Check 1.1

Before reading on, take a few minutes to list the main factors you think could affect the demand for seats on this route.

Applying basic common sense, and possibly some personal experience, you may have said that the demand for airline seats on this route will depend on a variety of factors: the price charged by easyJet for the seat, the price charged by its competitors, such as British Airways, who also fly this route, the cost of alternative travel such as the rail fare from Edinburgh to London, or people's income levels. You might have suggested other factors as well. In other words, we can build up a *verbal* picture of the economic situation. But, as we all know, words can sometimes get in the way of understanding, with different people reading different things into a particular phrase, and it can sometimes take a lot of words to describe a relatively straightforward situation. A much more concise and unambiguous way of summarizing such an economic situation is provided through simple mathematical notation, such as:

$$Q_d = b_1P + b_2P_C + b_3P_A + b_4Y \quad (1.1)$$

where we use letters to stand for some of the factors we've thought of. Here we use:

Q_d = the total number of easyJet seats people want to buy on this route (often referred to as the *quantity demanded* and pronounced 'queue dee')

P = the price of easyJet's airline seats on this route

P_C = the price of competitor airlines' seats on this route

P_A = the price of alternative travel such as railways

Y = level of consumer income.

In economics we refer to these factors as *variables* – since they vary or change according to the economic situation under consideration. We use other letters (b_1, b_2, b_3, b_4) to represent what are called the *parameters* of the relationship (frequently these are the specific numerical values that are appropriate to the particular economic relationship). Notice also that we show Eq. 1.1 using *italics*. There is no particular reason for this other than the fact that it makes them stand out in the text and so helps you to realize that they are referring to parts of a mathematical expression.

Such a mathematical presentation offers a number of advantages to the economic analyst, once you get used to them. First, to those who understand the mathematical symbols used, the use of mathematical notation to describe such economic relationships provides a definitive and unambiguous statement of the relationship. A purely verbal description of an economic relationship is more prone to misinterpretation and confusion than a mathematical one. It is for this reason that relationships such as the example above are shown in mathematical terms. However, not only can we use mathematics to describe such a relationship, but we can also apply mathematical reasoning and logic. Mathematics is a particularly powerful tool in enabling us to make logical

deductions about economic behaviour patterns. In the above example an economist with the appropriate mathematical understanding can work out the effect of, say, a change in consumer income on the quantity demanded of the product under the critical assumption that the other factors in the equation remain unchanged. This is a very common assumption used in economic analysis and one that we'll use often. If we wanted to work out the general effect on Q_d (demand for easyJet seats on this route) if Y (consumer income) changed, then the only way we can do this is to make the assumption that all the other factors stay exactly as they are. If we did not make this assumption but allowed other factors to change at the same time, it would be impossible to work out what was causing Q_d to change. Economists refer to this assumption using the phrase 'other things being equal' or with the Latin expression '*ceteris paribus*' (pronounced 'ketter-iss parry-bus'), which literally means 'with other things the same' or 'other things being equal'.

Progress Check 1.2

Suppose British Airways lowers its prices on this route. Other things being equal, what effect would you expect this to have on demand for easyJet seats?

Given that the expression b_2P_C in (1.1) is used to show the effect of competition on Q_d , what numerical value would you expect b_2 to take: negative, positive or zero?

Other things being equal, it seems reasonable to assume that if BA increase their prices, easyJet's prices will appear cheaper and so more attractive to the customer. In other words, we would expect an increase in demand for easyJet seats as a result of an increase in BA prices. This suggests that the numerical value for b_2 would be positive – an increase in a competitor's price, P_C , would lead to an increase in Q_d . This is our first mathematical economic analysis. You may also have worked out that we'd expect b_1 to be negative – if easyJet themselves charge a higher price then we'd expect this to have a negative effect on demand; we'd expect b_3 to be positive since, again, an increase in the prices of alternative forms of travel is likely to boost demand for air travel; b_4 is slightly less clear but we'd probably conclude that, if people have more income to spend, they'd probably travel more so we might think that b_4 would be positive also.

These examples illustrate how mathematical economic analysis can help us in the scenario outlined at the beginning of the module where the EU was considering imposing a tax on the price charged for short-haul flights. Although we've only just started looking at mathematical economic analysis we can use expression (1.1) to work out that such a tax would affect P and P_C but not P_A and Y . In other words, the effect of the tax on passenger demand, Q_d , would come through the impact of a higher price that easyJet would have to charge, P , and through the impact of the higher price its competitors would also have to charge, P_C – assuming of course that all airlines were charged the same tax. But we also see that the two effects might counterbalance each other to some extent. The effect on Q of a higher value for P will be negative (through b_1), so easyJet will lose passengers thanks to the higher price they have to charge because of the tax imposed. On the other hand, airline competitors will also have to increase their prices and, as we already know through b_2 , this would have a positive effect on easyJet demand – increasing passenger numbers, other things being equal. So, on the one hand easyJet would lose passengers and on the other would gain passengers. What would the net effect be? In part this would depend on the exact numerical values taken by b_1 and

b_2 (which of course we don't know). Although we don't have exact numerical values for the two parameters it's clear that there are three possibilities if we ignore the positive and negative signs:

- b_1 is bigger than b_2
- b_1 is the same as b_2
- b_1 is less than b_2 .

Progress Check 1.3

Look at each of the three possibilities in turn. Overall, would easyJet lose passengers or gain passengers for each of these possibilities?

The three possibilities effectively show how competitive easyJet's prices are relative to those of its airline competitors. In the first possibility it would lose more customers than it gains. In the second possibility the gains and losses would leave it as it is. In the third possibility it would win more than it lost. You have now completed your second mathematical economic analysis.

An important point to note at this stage is that the relationship we've been looking in Eq. 1 is expressed mathematically but contains no actual numbers. This is a common misunderstanding of the role of mathematics in economics. Of course, there are frequent occasions when we wish to use specific numerical values in such an equation. A business organization – easyJet for example – would wish to obtain precise and accurate forecasts of quantity demanded given specific values for the other variables in the equation. From the viewpoint of studying economic principles and theory, however, such number values are frequently irrelevant. Economists are often concerned with establishing key principles of economic behaviour – independently of whatever specific numbers happen to be appropriate. They might wish to work out, for example, the general principles of individual consumer behaviour if income changes. They might want to understand how firms would react if their labour costs increased. They might want to work out how both consumers and firms would react if interest rates increased. Accordingly, in this book we shall frequently be using general mathematical notation to establish general conclusions about economic behaviour. Naturally, we shall also be illustrating such important deductions with specific numerical values, although these are generally used primarily as an aid to understanding. In the real world considerable effort and attention is paid to obtaining and using such numerical values. This is the area known as *econometrics* or *econometric analysis* – another important and related area of economics.

1.2 Economic theory, economic models and mathematics

This leads us to another important area: the link between economic theory, economic models and mathematics. In economics we typically begin by observing something that's happening in a certain section of the economy. How much are consumers paying for the latest Apple smartphone? How are firms responding to the changes in the currency exchange rate? How is the energy industry responding to the latest government

incentives to invest in green energy production? Using the example we have been using so far, we might observe that a particular level of demand for easyJet flights occurs. The economist will ask why this product was purchased by consumers and why this particular quantity of the product at this price. Typically, we will then try to develop a theoretical explanation of this observed economic behaviour (which is what we provided in Eq. 1.1). Such a theoretical explanation will generally involve the construction of an economic *model* (again, as we have provided in Eq. (1.1)). Other professions use models in their work. An architect may create a scale model of a new building so that people can see what it will look like. An engineer may use a model to help with the design of a new aeroplane so that he can see how changes in design may affect the aeroplane's performance.

In economics, we use models to help us understand various aspects of economic behaviour. There is no particular reason why an economic model has to be mathematical or why the underpinning theory needs to be expressed in mathematical terms. Indeed, much early economic thinking did not make use of mathematics as such. However, as we have seen, there are factors that may strongly encourage us to make use of mathematics in the model-building process. In addition, if the model is mathematical it will involve an equation (or equations) linking certain economic variables together. Typically, we will then wish to examine the model in a mathematical manner. This will involve:

- Setting out the key assumptions on which the model is built
- Using these assumptions to examine the logical deductions to be obtained from the model
- Reaching conclusions about predicted economic behaviour
- Comparing our conclusions with actual economic behaviour.

Naturally, such a process is not usually as simple as it first appears. The whole process, in fact, will be iterative: we specify key assumptions, make logical deductions, reach conclusions and then we may find that the conclusions derived from the model are inconsistent with observed economic behaviour. We then have to return to the model for further development and refinement until we are satisfied that the model provides a reasonable explanation of the observed economic phenomenon (or until we abandon this theory because of its repeated failure to provide such an explanation). Mathematics in economics, therefore, is primarily concerned with the application of mathematical principles and logic to the theoretical aspects of economic analysis. Frequently, the next stage is a rigorous empirical investigation of the theory that has been developed thus far.

At this stage, econometrics comes into play. Econometrics is primarily concerned with the measurement of economic data and economic relationships. Using both mathematics and the principles of statistical inference, econometrics seeks to empirically evaluate a theoretical economic model. In this book we are not concerned with econometrics or indeed with empirical evaluation of economic models as such, although we do need to be aware of its critical role in the process of economic analysis. Figure I.1 illustrates the process.

We must remember, however, that any economic model – whether mathematical or not – is a simplified representation of a far more complex real-world situation. The purpose of models in this context is to reduce these real-world complexities to a level that can be understood and analysed. By definition, a model restricts its attention to

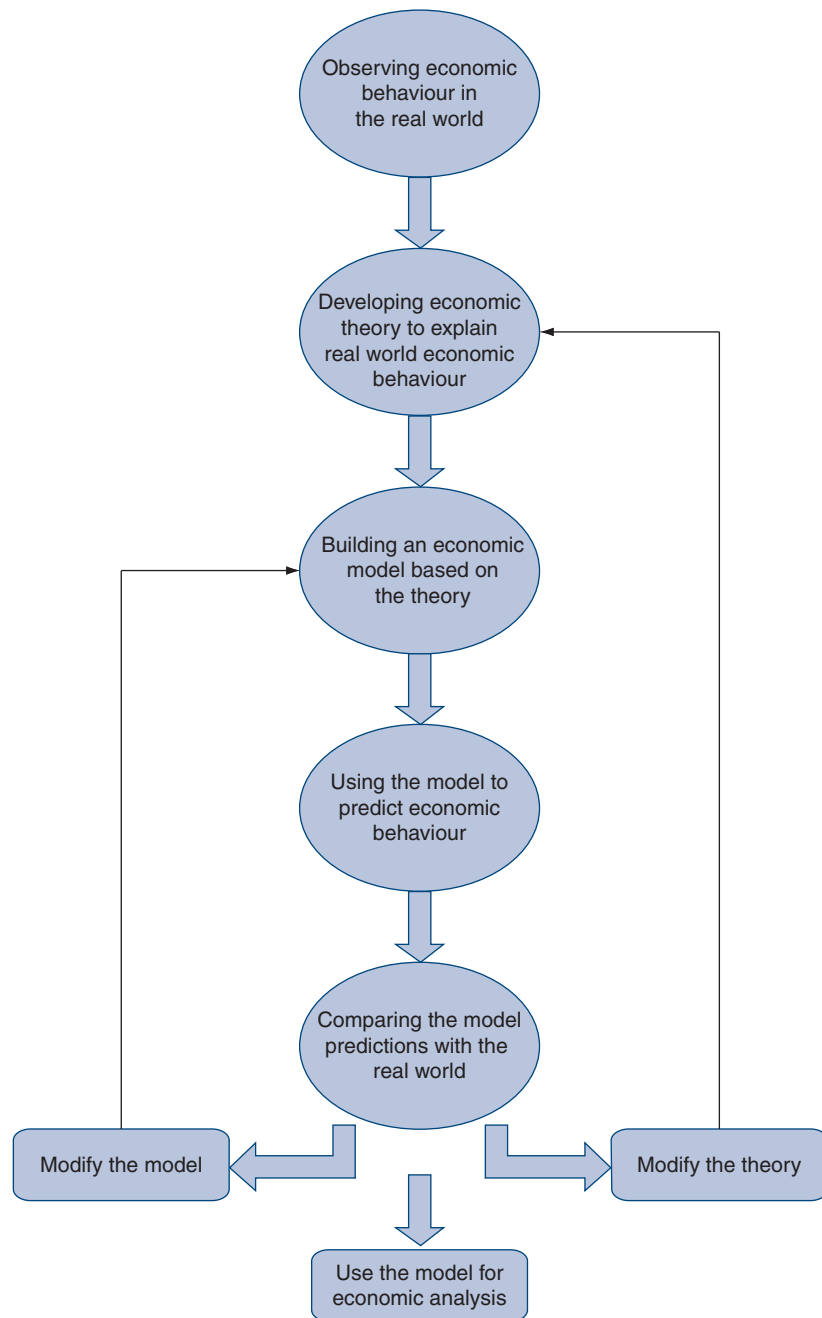


Figure I.1 Theory and models in economic analysis

what are seen to be the key features of the situation under investigation. So, in the context of our earlier example, there will be numerous factors influencing the quantity of a good that is purchased. An economic model will, however, focus on only a few of these factors – naturally, the ones thought to be most important in the context of the analysis. We did this earlier with Eq. 1.1 for easyJet.

1.3 Summary

We are now in a position to begin our investigation into the uses of mathematics in the study of economic analysis. Mathematics plays a critical role in providing economists with the logic and analytical tools needed to develop and investigate economic theories which are at the heart of economics and the study of economic behaviour. Without an adequate understanding of mathematics and its role in economics your career in this subject area will be severely curtailed. By the time you reach the end of this text we're convinced that your knowledge and appreciation of the usefulness of mathematics to the economist will have undergone a fundamental change.

Exercises

- 1.1** Earlier in this module we looked at the idea of a demand situation – considering the factors that will affect demand for a product. Consider the other side of the picture: the quantity of the good supplied by the individual firm. What variables do you think we would wish to link with the quantity supplied? What numerical values do you think each of these would take? Try using easyJet as an example again.
- 1.2** How do you think such numerical values could be obtained in practice?
- 1.3** Consider a variable, C , which represents the annual expenditure (consumption) of a particular individual. What variables do you think would influence consumption? Assess whether you would expect each variable to have a positive or a negative influence.
- 1.4** Consider an individual's consumption of a particular good – coffee. Identify a set of variables that you feel would influence such consumption and develop a simple model for determining such consumption. Consider the assumptions you are making – explicitly and implicitly – in your model. Practically, how could you test how good your model was?
- 1.5** In the context of Exercise 1.4 consider the annual national import of coffee. What variables do you think would influence imports of this good? How does your list of variables compare with that of Exercise 1.4 and how do you explain the differences?

Section A

The building blocks of economic analysis

A

In the introductory module we hope we convinced you that being able to use mathematics in economic analysis is essential if you want to study economics seriously. In this section we provide the necessary building blocks to enable you to understand and to start using mathematics and economic analysis effectively together. The section is divided into three modules.

Module A1 Tools of the trade: the basics of algebra

This module provides a refresher and reminder of the key principles of mathematics, in particular algebra and algebraic calculations, which are essential for the topics that follow.

Module A2 Linear relationships in economic analysis

This module looks at how we can express economic relationships using the simplest form of mathematics, which involves linear, or straight-line, equations. We'll look at the idea of functions and show how to use graphs to illustrate linear economic relationships.

Module A3 Non-linear relationships in economic analysis

Finally, in this section we look at non-linear relationships in economics. While linear equations are useful and easy to use, they're often restrictive in terms of building an accurate economic model. Frequently, economic relationships need to be modelled in a non-linear way. In this module we look at non-linear equations and equations involving several variables.

You may already be familiar and comfortable with the material in some of these modules. To help you check whether or not it's worth your while reading a particular module, we've included a Knowledge Check activity towards the start of each module. This is a short activity that will help you work out if you already know about the material of that module. If you answer the Knowledge Check correctly we suggest you don't need to read that module but move straight on to the exercises at the end of the module as they'll give you extra practice at seeing the connection between maths and economics. If you find any of the exercises especially challenging you can always go back and read the relevant part of that module.

Module A1

Tools of the trade: the basics of algebra

A

This module reviews a number of the basic principles in algebra. As we work through the text, you will see that economic analysis makes a lot of use of algebra to support and develop economic theory and to reach conclusions about economic behaviour: how firms will respond to a tax change; how consumers will respond to a change in interest rates; how government will respond to a change in exchange rates and so on. As mentioned in the Introduction, one of the main benefits of using mathematics in economic analysis is to help us to deduce general economic conclusions without having to resort to specific numerical values. Algebra allows us to do this, and although some algebra procedures may at first seem more like black magic than reasoned economic logic, you will find that, with practice, such manipulations begin to make sense. It may be some time since you last had to use algebra, so this module is intended to refresh your memory. The material that follows will allow you to gradually develop your own skills with algebra. However, if you find that you are unable to follow some of the algebraic manipulations that take place later in the text, you can return to the appropriate part of this module to help you.

Learning Objectives

By the end of this module you should be able to:

- Use algebraic notation to show economic relationships
- Work with brackets
- Work with inequalities
- Work with fractions
- Transpose an algebraic expression.

Knowledge Check A1

To check how comfortable you are with algebra already, try solving these:

(i) $y = \frac{3x + 3}{2x + 5}$

Find an expression for x

Knowledge Check A1 (Continued)(ii) Simplify the expression $7x/4x^2 - 8x/2x^3$

Check your answers in Appendix 2. If you got the correct answers try out some of the exercises at the end of the module for extra practice and then move to module A2.

And, if you've no idea what to do, read on. You will be able to do these by the time you complete the end of this module.

If you want to find out more about where algebra came from try <http://en.wikipedia.org/wiki/Algebra>

A1.1 Algebraic notation

We start by looking at how algebra can be used to show a simple economic situation. For example, individual consumers would normally distinguish between their *gross* income and their *disposable* income. Gross income would mean all the income that they had: what they earned in wages or salaries; what they received in interest on their savings; or dividend payments from shares they had bought in companies. In most economies, though, your gross income is not what you actually have to spend. Typically, there are compulsory deductions from your gross income, such as government tax on your salary/interest/dividends, compulsory payments into a health insurance scheme in case you fall ill and compulsory payments into a pension scheme for when you retire. Disposable income is that income the consumer has left to spend after any deductions (such as tax) have been taken from their gross income. We've already seen in the Introduction how we can use mathematical notation to help illustrate simple economic models, and we will do the same here. We'll use Y_g to refer to gross income and D to refer to all deductions and we'll use Y_d for disposable income. Using algebraic notation we would then write:

$$Y_d = Y_g - D \quad (\text{A1.1})$$

That is, disposable income, Y_d , is simply gross income, Y_g , less deductions, D .

As a slight digression, it is worth knowing that certain economic variables, like income, conventionally tend to be shown algebraically using specific letters. For example:

- Y is used for income
- P for price
- C for consumption
- G for government spending

and so on. We'll do the same throughout this book. And in case you're wondering why we use Y for income and not I , it is because I is used to refer to investment. It is also common practice to use subscripts with a variable when there may be different versions of that variable. That's why we have Y_g and Y_d .

A1.2 Arithmetic in algebra

Even with such a simple expression as Eq. A1.1 it is clear that we can obtain two related expressions:

$$D = Y_g - Y_d \quad (\text{A1.2})$$

$$Y_g = Y_d + D \quad (\text{A1.3})$$

Eq. A1.2 indicates that deductions, D , are simply the difference between gross income, Y_g , and disposable income, Y_d , and Eq. A1.3 indicates that gross income is equal to disposable income plus deductions. While Eqs A1.2 and A1.3 are easily obtained using some simple logic it will also be worth exploring the algebraic arithmetic. These principles will be useful when we look at more complex expressions.

To find an expression from Eq. A1.1 where D equals some combination of the other two variables, we can rearrange Eq. A1.1 (or any other algebraic expression) by understanding that if *one side* of an algebraic expression is altered we keep the algebraic relationship exactly the same *as long as* we alter the *other side* of the expression in exactly the same way. This is an important rule in algebra and one that we will use a lot. From Eq. A1.1 we have:

$$Y_d = Y_g - D$$

The rule says that the algebraic expression remains unchanged in terms of the underlying relationship if we alter both sides of the expression in the same way. If we add D to each side we have:

$$D + Y_d = Y_g - D + D$$

and by simple inspection we see that the two D s on the right-hand side will cancel each other out to give:

$$D + Y_d = Y_g$$

If we now subtract Y_d from both sides (which again leaves the relationship unchanged as both sides of the equation are treated in the same way) this gives:

$$D + Y_d - Y_d = Y_g - Y_d$$

where, again, the two Y_d terms on the left-hand side cancel each other out to give

$$D = Y_g - Y_d$$

It is important to realize that this equation and Eq. A1.1 are identical.

We've deliberately taken a detailed, step-by-step approach, but you won't always need to be as methodical because it soon becomes obvious how to use this type of arithmetic with algebraic expressions.

Progress Check A1.1

Using the algebra we've just shown, try the following examples yourself and then carry on reading the text.

Rearrange each of the following expressions so that you have an expression in the form $Y =$

(i) $5Y + 3X - 10 = 25$

(ii) $A - C = Y + 10 - B$

(iii) $6A = 4Y - 5C$

(iv) $0.2X - 0.75Z = 0.3Y + 1512$

We'll work through each of these in the next section, but try them yourself first.

Taking each in turn, we have:

(i) $5Y + 3X - 10 = 25$

Using the rule from earlier, we can add 10 to both sides to give:

$$5Y + 3X - 10 + 10 = 25 + 10$$

with the two 10s on the left cancelling each other out to give

$$5Y + 3X = 35$$

Next we can subtract $3X$ from both sides to get:

$$5Y + 3X - 3X = 35 - 3X$$

Again, the two $3X$ s on the left cancel each other out, giving:

$$5Y = 35 - 3X$$

Finally, we can divide both sides by 5:

$$\frac{5Y}{5} = \frac{35}{5} - \frac{3X}{5}$$

Which, if we do the maths, gives:

$$Y = 7 - \frac{3X}{5}$$

Once again, it is important to remember that, although this equation and the one we started with look very different, they are in fact identical.

(ii) $A - C = Y + 10 - B$

We want to rearrange this to get Y onto the left-hand side of the equation and everything else on the right-hand side. We can do this in several different ways, but let's first add B to both sides to get:

$$A - C + B = Y + 10 - B + B$$

And, with the two B s on the right-hand side cancelling each other out, we get:

$$A - C + B = Y + 10$$

Now we can subtract 10 from both sides, giving:

$$A - C + B - 10 = Y + 10 - 10$$

or

$$A - C + B - 10 = Y$$

If we now simply swap over the left-hand and right-hand sides we get:

$$Y = A - C + B - 10$$

You might have done this in a different order, but should still have reached the same result.

(iii) $6A = 4Y - 5C$

Add $5C$ to both sides:

$$6A + 5C = 4Y - 5C + 5C$$

$$6A + 5C = 4Y$$

We now divide both sides by 4:

$$\frac{6A}{4} + \frac{5C}{4} = \frac{4Y}{4}$$

Simplifying and switching both sides gives:

$$Y = 1.5A + 1.25C$$

(iv) Finally, we had $0.2X - 0.75Z = 0.3Y + 1512$

Subtracting 1512 we obtain:

$$0.2X - 0.75Z - 1512 = 0.3Y$$

Dividing through by 0.3 we obtain:

$$\frac{0.2X}{0.3} - \frac{0.75Z}{0.3} - \frac{1512}{0.3} = Y$$

Rearranging and simplifying gives:

$$Y = 0.67X - 2.5Z - 5040$$

The last example did not show all the detailed steps and calculations, but you should be able to follow what's happening.

A1.3 Brackets in algebra

The use of brackets in algebra is quite common, and we need to be familiar with how to use them. Let's go back to Eq. A1.1 where we had:

$$Y = S - D$$

Let's now define D , deductions, as:

$$D = f + tY_g \tag{A1.4}$$

where f is a fixed amount deducted from each person's gross income while t is a proportionate tax (expressed as a decimal) deducted from gross income Y_g . For example, suppose the government taxes everyone €100 and also sets income tax at 25% of gross income; f would be 100 and t would be 0.25, implying that deductions would be a fixed sum of €100 regardless of actual income plus 25% of gross income earned. We can now substitute Eq. A1.4 into Eq. A1.1:

$$Y_d = Y_g - D$$

$$Y_d = Y_g - (f + tY_g) \tag{A1.5}$$

Eq. A1.5 could be simplified by removing the brackets and rearranging the expression. However, we must remember that we cannot simply remove the brackets from the expression to give:

$$Y_d = Y_g - f + tY_g$$

You should be able to see what is wrong with this expression. We should subtract *both* f and Y_g and not just f . This gives a simple rule that, if we wish to remove brackets from an expression, then *all* the terms within the brackets must have the same arithmetical operation performed on them. In Eq. A1.5, for example, we must multiply each term within the brackets by a negative sign (since this is the mathematical operator immediately before the bracket expression). This then gives:

$$Y_d = Y_g - f - tY_g$$

We then collect all the Y_g terms together (collecting the common terms together in this way is something we will frequently want to do in economics):

$$Y_d = Y_g - tY_g - f$$

We now have two Y_g terms. We can now rewrite this equation as:

$$Y_d = 1Y_g - tY_g - f$$

or rearrange it as

$$Y_d = (1 - t)Y_g - f \quad (\text{A1.6})$$

If you look carefully at Eq. A1.6 you will note that it is the same as the previous equation. It may seem odd that we want to remove brackets first and then reintroduce them, but what we have been able to do with Eq. A1.5 is to derive an expression where similar terms appear together to help interpretation and evaluation of the expression. We can generalize the approach by saying that:

$$ab + ac = a(b + c)$$

where a is a term common to both parts. To see how this works let's go back to where we had:

$$Y_g - tY_g$$

or, as we wrote it

$$1Y_g - tY_g$$

The common term here is Y_g , so we have:

$$ab + ac = a(b + c)$$

In other words:

$$a = Y_g$$

$$b = 1$$

$$c = -t \text{ (remember the minus sign)}$$

so

$$a(b + c) = Y_g(1 - t)$$

We could just as well have had three, four or more terms inside the brackets and the same approach would be appropriate. Similarly, we could have had more than one term, a , before the bracket. For example:

$$(a + b)(c + d)$$

would give

$$ac + bc + ad + bd$$

and you can see that each term within the first set of brackets has, in turn, been multiplied by each term within the second set of brackets. Notice, though, that the order in which we multiply does not matter. This principle is readily extended to more than two sets of brackets or to brackets containing more than two expressions.

In Eq. A1.6 we showed this as $(1 - t)Y_g$, which is the same

Progress Check A1.2

For each of the following expressions multiply out the brackets and, where relevant, simplify the expressions.

(i) $10x(3a - c)$

(ii) $(5x - 3y)(2x + 4y)$

(iii) $3(x + y - z) - (4y + 2)x$

Try these first before reading on.

Taking each in turn we have:

$$(i) \quad 10x(3a - c)$$

And we can multiply the two terms inside the brackets by $10x$ to give:

$$30ax - 10cx$$

Notice that it doesn't matter whether we write $30ax$ or $30xa$.

$$(ii) \quad (5x - 3y)(2x + 4y)$$

We multiply each of the two terms in the second bracket first by $5x$ and then by $-3y$ (remember the minus sign):

$$\text{Multiply by } 5x: \quad 5x2x + 5x4y$$

$$\text{Multiply by } -3y: \quad -3y2x - 3y4y$$

When we multiply a variable like x by itself we get x^2 (x squared). (If you don't remember how to work with powers like x^2 you might want to read through the short appendix to this module on p. 31.) This would give us:

$$5x2x + 5x4y = 10x^2 + 20xy$$

and

$$-3y2x - 3y4y = -6xy - 12y^2$$

and combining these two expressions together we get

$$10x^2 + 20xy - 6xy - 12y^2$$

or

$$10x^2 + 14xy - 12y^2$$

$$(iii) \quad 3(x + y - z) - (4y + 2)x$$

If you think this appears complicated, remember to break it into parts and then add the parts together at the end. Let's take the first part of this and multiply the brackets through by 3:

$$3(x + y - z) = 3x + 3y - 3z$$

and now the second part (remembering the minus sign)

$$-(4y + 2)x = -4xy - 2x$$

and if we now collect both parts together we have

$$3x + 3y - 3z - 4xy - 2x$$

and collecting common terms together we have

$$x + 3y - 3z - 4xy$$

Multiple brackets

We have seen how we can multiply out brackets in an expression. There are times when we have multiple sets of brackets. For example:

$$3x(4 - y(15 - x))$$

Again, this looks complicated but if we do it part by part it's straightforward. We multiply this out in much the same way, but making sure that we start with the *inside*

set of brackets first – those around $(15 - x)$ – and then gradually work outwards. So, multiplying out the inside set first we have:

$$-y(15 - x) = -15y + xy$$

and then

$$3x(4 - 15y + xy)$$

and then

$$12x - 45xy + 3x^2y$$

Expressions involving multiple sets of brackets can be simplified using this approach: find the innermost set of brackets, work out that expression, find the next innermost set of brackets, work that out – and so on.

Progress Check A1.3

Simplify each of the following expressions:

- (i) $15x(3x - 2y(y - x))$
- (ii) $(4x - 3y(4x + 3y)(5x))$
- (iii) $(2x - 3y + 4z(2x + 3(15y)))$

For (i) we have:

$$15x(3x - 2y(y - x)) = 15x(3x - 2y^2 + 2xy)$$

(multiplying the $(y - x)$ term by $2y$ and remembering the change in sign when we have two negatives multiplied). This then becomes:

$$15x(3x - 2y^2 + 2xy) = 45x^2 - 30xy^2 + 30x^2y$$

Note that we cannot simplify further: the last two terms are not identical.

(ii) $(4x - 3y(4x + 3y)(5x))$

Multiplying together the two bracket terms inside the outside bracket, $(4x + 3y)$ and $(5x)$, we have:

$$(4x - 3y(20x^2 + 15xy))$$

Multiply through by $-3y$:

$$4x - 60x^2y + 45xy^2$$

(iii) $(2x - 3y + 4z(2x + 3(15y)))$

Multiply through the two terms on the right of the expression $3(15y)$:

$$(2x - 3y + 4z(2x + 45y))$$

Multiply through by $4z$:

$$2x - 3y + 8xz + 180yz$$

A1.4 Inequalities

So far we have explored algebraic expressions in the form of *equations*, where an expression on the left-hand side is set exactly equal to another expression on the right-hand side. Occasionally we will wish to explore relationships that are expressed in the form of an *inequality*. For example, we may have:

$$x > y$$

which is read as ‘ x is greater than y ’ and where the symbol $>$ indicates that x must take values greater than y at all times. Similarly, we may have:

$$x < y \quad x \text{ always takes a value less than } y$$

$$x \geq y \quad x \text{ always takes a value which is greater than or equal to } y \text{ (i.e. } x \text{ values cannot be less than } y \text{ but they could be the same as } y \text{ or greater than } y)$$

$$x \leq y \quad x \text{ always takes a value which is less than or equal to } y \text{ (i.e. } x \text{ values cannot be greater than } y).$$

Let’s go back to Eq. A1.4 where we had:

$$D = f + tY_g$$

where t is a tax imposed on gross income, Y_g . If we express the tax as a decimal (e.g. a tax that took 25% of income would be shown as 0.25) then we would have:

$$t \geq 0 \quad \text{i.e. the tax rate could not be negative}$$

$$t < 1 \quad \text{the tax rate must be less than 1 (or less than 100%).}$$

The first inequality could be rewritten instead as:

$$0 \leq t$$

so we could merge the two inequalities together to give

$$0 \leq t < 1$$

That is, t must lie within a range between 0 but less than 1.

It will also be worth exploring how inequalities are affected if we manipulate them using the algebraic principles developed earlier. We have already seen that we can manipulate equations in any way we wish as long as we alter both sides of the equation in the same way. Let us see if the same principle applies to inequalities. Consider:

$$x < y$$

where $x = 2$ and $y = 10$. Then:

$$2 < 10$$

which is clearly correct. Suppose we add 4 to both sides:

$$2 + 4 < 10 + 4$$

$$6 < 14$$

which is still correct. Suppose we now subtract 20 from both sides:

$$6 - 20 < 14 - 20$$

$$-14 < -6$$

which is still correct (although you may have to think about this one: -14 is lower (less) on the negative scale than -6 so the inequality holds true).

So addition and subtraction do not affect the inequality. What about multiplication and division? We had:

$$2 < 10$$

If we multiply both sides by 5:

$$2 \times 5 < 10 \times 5$$

$$10 < 50$$

which is correct. Suppose we now multiply by -2 :

$$10 \times -2 < 50 \times -2$$

$$-20 < -100$$

which is clearly *incorrect* since -100 is a larger negative number and is less than -20 . This leads us to a simple manipulation rule when dealing with inequalities: *if both sides of an inequality are multiplied/divided by a negative number, the direction of the inequality is reversed.*

So, if we had:

$$x < y$$

and multiplied through by $-n$, we would have

$$-nx > -ny$$

Progress Check A1.4

Simplify the following expressions by collecting all variable terms on one side and all numerical values on the other:

(i) $4x + 7 < 3x - 5$

(ii) $4x - 3 > 6x + 2$

(iii) $-4x + 5 \geq 6 - 3x$

For (i) we have:

$$4x + 7 < 3x - 5$$

Subtracting 7 gives:

$$4x + 7 - 7 < 3x - 5 - 7$$

$$4x < 3x - 12$$

Subtracting $3x$:

$$4x - 3x < 3x - 12 - 3x$$

$$x < -12$$

That is, x must always take values that are less than -12 .

(ii) $4x - 3 > 6x + 2$

Add 3 to give:

$$4x > 6x + 5$$

Subtract $6x$:

$$4x - 6x > 5$$

$$-2x > 5$$

Divide through by -2 :

$$x < -2.5$$

remembering that as we divide through by a negative value we must reverse the inequality sign. That is, x is less than or equal to -2.5 .

(iii) $-4x + 5 \geq 6 - 3x$

Add $3x$:

$$-4x + 3x + 5 \geq 6$$

$$-x + 5 \geq 6$$

Subtract 5:

$$-x \geq 6 - 5$$

$$-x \geq 1$$

Divide through by -1 :

$$x \leq -1$$

again remembering to reverse the direction of the inequality.

A1.5 Fractions

We now look at the use of fractions in algebra. You will already be familiar with numerical fractions such as:

$$\frac{2}{3} \text{ or } \frac{1}{10} \text{ or } \frac{72}{100}$$

You may also remember that the number on the top of the fraction expression is referred to as the *numerator* and the one on the bottom as the *denominator*. In algebra we may have fractions such as:

$$\frac{a}{b} \text{ or } \frac{a^2 - 3b}{2a - b^2} \text{ or } \frac{15 - b}{3a^2 - 2ab}$$

The rules for manipulation of algebraic fractions are virtually the same as those for numerical fractions.

Multiplication

To multiply two or more fractions we multiply the numerator terms together and then multiply the denominator terms together. For example:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} = \frac{ac}{bd}$$

Division

To divide one fraction by another, we invert (turn upside down) the fraction we are dividing by and then multiply the two fractions together:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Addition/subtraction

To add or subtract two fractions, we put them over a *common denominator* and add/subtract the numerators. We will illustrate this with a numerical example first. Suppose we want:

$$\frac{3}{4} + \frac{1}{2}$$

A common denominator is a number of which 4 and 2 (the two original denominators) are exact multiples. In this case one common denominator would be 4 since the denominator 4 goes into this exactly once and the other denominator 2 goes into this exactly twice. We then use these multiples (1 and 2) to multiply the respective numerators. That is:

$$\frac{3}{4} + \frac{1}{2} = \frac{(1 \times 3) + (2 \times 1)}{4} = \frac{3 + 2}{4} = \frac{5}{4}$$

Note that we have multiplied the first numerator, 3, by 1 since its denominator (4) goes into the common denominator exactly once. We have multiplied the second numerator (1) by 2 since its denominator goes into the common denominator exactly twice.

Choosing a common denominator

When deciding which common denominator to use there's frequently an obvious number that will be exactly divisible by each of the two fraction denominators. There are times, however, when such a number is not immediately obvious. In such a case an easy approach is simply to use a common denominator that is the result of multiplying the two fraction denominators together. For example:

$$\frac{3}{7} + \frac{2}{3}$$

As there is no obvious common denominator that springs to mind, we choose 21 (7×3). The arithmetic would then be:

$$\frac{(3 \times 3) + (2 \times 7)}{21} = \frac{9 + 14}{21} = \frac{23}{21}$$

Progress Check A1.5

Simplify each of the following expressions:

- (i) $3/5 + 8/4$
- (ii) $2/6 + 3/7$
- (iii) $4/5 + 2/3 + 6/8$

For (i) we use a common denominator of 20 (5×4) to give:

$$\frac{(3 \times 4) + (8 \times 5)}{20} = \frac{12 + 40}{20} = \frac{52}{20}$$

Notice that while we can leave the result as $52/20$ we can simplify further since both the numerator and denominator can be divided through by a common factor.

For example, divide both through by 2 (and since we are applying the same arithmetic to top and bottom it will leave the expression unchanged):

$$\frac{52}{20} = \frac{26}{10}$$

26/10 can be further simplified, again dividing by 2 to give:

$$\frac{26}{10} = \frac{13}{5}$$

Of course, we could have divided the top and bottom of 52/20 through by 4 straightaway. This type of simplification is quite common in economics, and so it is worthwhile familiarizing yourself with it, particularly with algebraic rather than arithmetic examples.

(ii) $2/6 + 3/7$

A common denominator of 42 (6×7) gives:

$$\frac{(2 \times 7) + (3 \times 6)}{42} = \frac{14 + 18}{42} = \frac{32}{42}$$

This can be simplified again by dividing through by 2:

$$\frac{32}{42} = \frac{16}{21}$$

(iii) $4/5 + 2/3 + 6/8$

Although we have not explicitly looked at three fractions being added together, we can simply add the first two and then add this product to the third (although with practice we might be able to perform the arithmetic in one step rather than two). We have:

$$\frac{(4 \times 3) + (2 \times 5)}{15} = \frac{22}{15}$$

We use 5×3 as a common denominator for the first two fractions

and then

$$\frac{22}{15} + \frac{6}{8} = \frac{(22 \times 8) + (6 \times 15)}{120} = \frac{176 + 90}{120} = \frac{266}{120}$$

We use 15×8 as a common denominator for the next two fractions

Simplifying gives:

$$\frac{266}{120} = \frac{133}{60} \text{ (dividing through by 2).}$$

We could have performed the arithmetic in one step as:

$$\frac{4}{5} + \frac{2}{3} + \frac{6}{8}$$

$$\frac{4(3 \times 8) + 2(5 \times 8) + 6(5 \times 3)}{(5 \times 3 \times 8)} = \frac{96 + 80 + 90}{120} = \frac{266}{120}$$

Subtraction

So far we have looked only at the addition of fractions; but exactly the same approach applies to subtraction. For example:

$$\frac{3}{5} - \frac{2}{3} = \frac{(3 \times 3) - (2 \times 5)}{15} = \frac{9 - 10}{15} = \frac{-1}{15}$$

Progress Check A1.6

Simplify each of the following expressions:

- (i) $3/4 - 2/3$
 (ii) $2/5 - 1/9$
 (iii) $2/5 - 4/7 + 1/8$

For (i) we have:

$$\frac{3}{4} - \frac{2}{3} = \frac{(3 \times 3) - (2 \times 4)}{4 \times 3} = \frac{9 - 8}{12} = \frac{1}{12}$$

For (ii):

$$\frac{2}{5} - \frac{1}{9} = \frac{(2 \times 9) - (1 \times 5)}{45} = \frac{18 - 5}{45} = \frac{13}{45}$$

For (iii):

$$\frac{2}{5} - \frac{4}{7} + \frac{1}{8} = \frac{2(7 \times 8) - 4(5 \times 8) + 1(5 \times 7)}{(5 \times 7 \times 8)} = \frac{112 - 160 + 35}{280} = \frac{-13}{280}$$

Fractions with algebraic expressions

The same principles apply to algebraic expressions. For example:

$$\frac{x}{x+2} \times \frac{3x}{2x^2}$$

Multiplying the two denominators gives:

$$(x+2)(2x^2) = 2x^3 + 4x^2$$

and then multiplying the two numerators gives

$$\frac{3x^2}{2x^3 + 4x^2}$$

However, if we divide both the numerator and denominator by x^2 we have:

$$\frac{3}{2x+4}$$

Note that when we are cancelling out terms in an algebraic expression we must be careful to ensure that the term being used appears in all parts of the expression, as in this case.

Progress Check A1.7

Simplify each of the following expressions:

- (i) $(3x - 6)/x^2$ divided by $5x/3x^2$
 (ii) $5x/4x^2 + 3x^3/5x$
 (iii) $7x/4x^2 - 8x/2x^3$

Taking each in turn, for (i) we have:

$$\frac{(3x-6)}{x^2} \div \frac{5x}{3x^2}$$

Recollecting that if we invert the second term and then multiply we have:

$$\frac{(3x-6)}{x^2} \times \frac{3x^2}{5x}$$

Notice that we can cancel the x^2 term on both top and bottom to give:

$$\frac{(3x-6)}{1} \times \frac{3}{5x} = \frac{9x-18}{5x}$$

We cannot cancel the x terms since they do not appear in each part of the final expression (the -18 term does not have an x attached to it).

For (ii):

$$\frac{5x}{4x^2} + \frac{3x^3}{5x}$$

we have a common denominator of $(4x^2)(5x)$ which is $20x^3$:

$$\frac{5x(5x) + 3x^3(4x^2)}{20x^3} = \frac{25x^2 + 12x^5}{20x^3}$$

If we wished we could simplify further as:

$$\frac{25 + 12x^3}{20x}$$

We can divide through by the common term x^2

One useful way of checking whether we can simplify by cancelling a common term is to break the fraction into its component parts:

$$\frac{25x^2 + 12x^5}{20x^3} = \frac{25x^2}{20x^3} + \frac{12x^5}{20x^3}$$

It will then be apparent that both parts of the expression have a common term which can be cancelled (x^2 in this case).

For (iii):

$$\frac{7x}{4x^2} - \frac{8x}{2x^3}$$

we have a common denominator of $(4x^2)(2x^3)$ or $(8x^5)$ giving:

$$\frac{7x(2x^3) - 8x(4x^2)}{8x^5} = \frac{14x^4 - 32x^3}{8x^5}$$

Cancelling through by $2x^3$ we have:

$$\frac{7x - 16}{4x^2}$$

A1.6 Transposing an expression

The last aspect of algebra that we shall examine relates to the *transposition* of an expression (basically, rearranging it into another form). For example, consider the expression:

$$-ax = bx - cy + d \quad (\text{A1.7})$$

We wish to rearrange this into an expression such that:

$x =$ an expression involving all other terms.

The first step is to collect x terms together. From Eq. A1.7 we can subtract bx from both sides to give:

$$-ax - bx = -cy + d$$

The two terms on the left-hand side have an x term in common, so we have:

$$x(-a - b) = cy + d$$

Dividing both sides through by $(-a - b)$ gives:

$$x = \frac{cy + d}{-a - b}$$

Progress Check A1.8

From Eq. A1.7 derive an expression for y .

We have:

$$-ax = bx - cy + d$$

Subtracting bx gives:

$$-ax - bx = -cy + d$$

Subtracting d :

$$-ax - bx - d = -cy$$

Multiplying through by -1 :

$$ax + bx + d = cy$$

Dividing through by c :

$$y = \frac{ax + bx + d}{c}$$

We may also apply these principles to a more complex expression. Suppose we wish to derive an expression for x from:

$$y = \frac{x + 2}{x - 4}$$

Multiplying through by $(x - 4)$:

$$y(x - 4) = x + 2$$

Multiplying out the left-hand side:

$$yx - 4y = x + 2$$

Adding $4y$ to both sides:

$$yx = x + 2 + 4y$$

Subtracting x :

$$yx - x = 2 + 4y$$

The left-hand side terms have x in common, so:

$$x(y - 1) = 2 + 4y$$

and dividing through by $(y - 1)$:

$$x = \frac{2 + 4y}{y - 1}$$

Although this type of manipulation looks complicated it is simply a matter of practice and applying a few basic rules. Use the algebraic principles we have developed to:

- Remove any fractions by cross-multiplication
- Multiply out any brackets
- Collect x terms on one side
- Find any factors/multiples of x
- Divide through by the x coefficient.

If you're in any doubt as to whether you've applied these principles correctly, choose a couple of numerical values for x and solve for y using the original expression. Then use these y values in your transposed result and see whether you get the same x values (which, of course, you will if you've not made a mistake anywhere).

Progress Check A1.9

Find an expression for x from:

(i) $y = \frac{x - 5}{x + 3}$

(ii) $y = \frac{3x + 3}{2x - 5}$

For (i), using the steps above:

$$y(x + 3) = x - 5$$

$$yx + 3y = x - 5$$

$$yx - x = -3y - 5$$

$$x(y - 1) = -3y - 5$$

$$x = \frac{-3y - 5}{y - 1}$$

For (ii):

$$y = \frac{3x + 3}{2x - 5}$$

$$y(2x - 5) = 3x + 3$$

$$2yx - 5y = 3x + 3$$

$$2yx - 3x = 5y + 3$$

$$x(2y - 3) = 5y + 3$$

$$x = \frac{5y + 3}{2y - 3}$$

A1.7 Summary

This brings us to the end of this module on basic algebra and, although at times it might have looked complicated, algebra follows a set of basic rules. As long as you know what the rules are and have a steady, methodical approach to working with algebraic expressions, you will soon see how it works. If you've been able to follow what we've been doing in this module then you're ready to move on to where we can really start seeing how mathematics can be used in economic analysis. If, at any stage in the text, you have difficulty following the algebraic manipulations, return to the relevant part of this module and re-read that section.

Learning Check

Having read this module you should have learned that:

- A basic rule in algebra is that if both sides of an expression are changed in the same way the expression remains unchanged
- When you're working with multiple brackets, start with the ones on the inside and work outwards
- If both sides of an expression are multiplied or divided by a negative value, the direction of an inequality is reversed
- To divide by a fraction, turn it upside down and multiply
- To add/subtract fractions, put them over a common denominator and add/subtract the numerators.

Exercises

A1.1 For each of the following equations find the simplest form:

i) $y = \frac{3x + 3}{2x - 5}$

ii) $7x/4x^2 - 8x/2x^3$

(These were in the Knowledge Check at the start of this module.)

A1.2 For each of the following expressions find the simplest form:

i) $5(x - y) + 2(y - 3x)$

ii) $4x(3x - 2) + 0.5(x - 4y)$

iii) $(x - 2y)(3y - 5x)$

iv) $z(2x - y) - z(5x - 2)$

v) $3x(5 - 2x(y - x(3x - 6)))$

vi) $0.4y(3x(2 - 4y) + 2y(5 - 3x(x - 10)))$

A1.3 For each of the following expressions find the simplest form:

i) $x/(x - 1) \times 2/x(x - 4)$

ii) $7x/2x^3 + 5x/2x$

iii) $15x^3/3x^2 - 0.5x^2/3x$

iv) $6x/3x(5x - 10) + 2x^3/4(3 - 10x)$

Exercises (Continued)

A1.4 A company selling a particular product knows that the quantity of the product demanded by customers is given by the expression:

$$Q_d = 100 - 5P$$

where Q_d is the quantity of the product demanded and P is the price charged. Similarly, the quantity that the company is willing to supply is given by:

$$Q_s = -100 + 20P$$

where Q_s is the quantity supplied and P is the price charged. Equilibrium is defined as the price charged so that $Q_d = Q_s$. Find the price that will give equilibrium. What quantity will be demanded/supplied at this price?

A1.5 For the firm in A1.4, we now have:

$$Q_d = a - bP$$

$$Q_s = c + dP$$

Find an algebraic expression that will allow you to determine equilibrium price. Check this using the parameters in A1.4.

Appendix A1 Powers and exponents

When using mathematics in economic analysis we frequently come across terms such as x^2 or x^5 or $x^{-0.5}$. You will need to be able to use expressions like these. This is relatively straightforward once you understand that such notation is in fact a form of mathematical shorthand. Suppose we want to show some simple arithmetic:

$$10 \times 10 = 100$$

$$10 \times 10 \times 10 = 1000$$

$$10 \times 10 \times 10 \times 10 = 10,000$$

and so on. There's nothing wrong with showing such arithmetic in this way. However, it can be more convenient at times to use mathematical shorthand:

$$10 \times 10 = 10^2$$

where we say that the result of multiplying 10 by itself is 10^2 where the term 2 is known as the *power* or *exponent*. The power/exponent simply shows how many times we multiply a number/variable by itself. So, from above, we have:

$$10 \times 10 = 100 = 10^2$$

$$10 \times 10 \times 10 = 1000 = 10^3$$

$$10 \times 10 \times 10 \times 10 = 10,000 = 10^4$$

If we were using variables rather than numbers we'd have:

$$a^2 = a \times a$$

$$a^3 = a \times a \times a$$

$$a^4 = a \times a \times a \times a$$

Sometimes we'll come across exponents that appear a little odd, for example a^{-1} . This looks like a multiplied by itself -1 times. Earlier we saw that:

$$10^4 = 10,000$$

$$10^3 = 1000$$

$$10^2 = 100$$

Clearly there is a pattern here. As the exponent drops from 4 to 3 to 2 a zero is 'lost' from the actual number on the right. So, if we continue this pattern, we obtain:

$$10^4 = 10,000$$

$$10^3 = 1000$$

$$10^2 = 100$$

$$10^1 = 10$$

$$10^0 = 1$$

$$10^{-1} = 0.1$$

$$10^{-2} = 0.01 \text{ and so on.}$$

Now let's consider the items we have added. Normally we wouldn't bother writing 10 as 10^1 but write just 10 instead. $10^0 = 1$ might seem odd at first but it follows from the logic of the sequence. In fact, we'll state without proof that any number/variable to the power 0 equals 1: this is worth remembering. The negative exponents are just as logical. Negative exponents show decimals in exponent form. Sometimes these are written in a different way. Recollect that:

$$0.1 = \frac{1}{10}$$

and

$$10 = 10^1$$

so

$$0.1 = \frac{1}{10} = \frac{1}{10^1}$$

So:

$$10^{-1} = 0.1 = 1/10^1$$

Similarly:

$$10^{-2} = 1/10^2$$

and a^{-3} would be $1/a^3$.

Just as we can carry out algebraic arithmetic on ordinary numbers or variables, so we can do much the same when dealing with exponents. There are four simple rules for doing algebraic arithmetic with exponents.

Rule 1

$$a^n \times a^m = a^{n+m}$$

For example:

$$10^2 \times 10^4 = 10^6$$

$$a^5 \times a^3 = a^8$$

Here $n = 2$ and
 $m = 4$
Try it out using the
actual numbers if
you're not sure

Rule 2

$$a^n / a^m = a^{n-m}$$

For example:

$$10^6 / 10^3 = 10^3$$

$$a^5 / a^4 = a^1 = a$$

Rule 3

$$(a^n)^m = a^{nm}$$

For example:

$$(10^3)^2 = 10^6$$

$$(a^2)^4 = a^8$$

We're squaring the number 10^3 so this is actually Rule 1: $10^3 \times 10^3$

Rule 4

$$(ab)^n = a^n b^n$$

For example:

$$(3 \times 10)^3 = 3^3 10^3$$

This may be seen more clearly if we write:

$$(3 \times 10)^3 = (3 \times 10) \times (3 \times 10) \times (3 \times 10)$$

The order in which we multiply is unimportant, so we can rearrange this as:

$$(3 \times 10)^3 = (3 \times 3 \times 3) \times (10 \times 10 \times 10) = 3^3 10^3$$

An example involving variables rather than numbers is:

$$(ab)^4 = a^4 b^4$$

Powers and exponents are easy to work with if you remember the rules.

Index

- a priori* probability 432
- actual consumption and investment 118–19
- addition
 - fractions 24–5, 30
 - inequalities 21, 22
 - matrices 134, 143–5, 146
- Addition Law 436–7, 449
- adjoint matrix 158, 172–3, 177–8
- air travel short-haul tax 1–2, 3–6
- algebra 11, 13–33
 - arithmetic in 14–17
 - brackets in 17–20, 30
 - fractions 23–7, 30
 - inequalities 21–3, 30
 - notation 14
 - powers and exponents 31–3
 - transposing an expression 27–9
- alien cofactors 157–8, 169
- amortization annuities 400
- annual percentage rate/annual equivalent rate (APR/AER) 381–2, 391
- annuities 396–400, 410, 410–11
 - NPV 399, 411
 - repayment annuity 400, 401
 - value of an annuity 397–8
- antilog 60
- area under a curve 367–9, 373
- argument of a function 41
- arithmetic
 - in algebra 14–17
 - fractions 23–6
 - and functions 44
 - matrix algebra 133–8, 143–5
- arithmetic mean 440–1, 443
- arithmetic series 377–8, 391
- assumptions
 - ceteris paribus* 5
 - economic models 7
- augmented matrix 152–4
- autonomous consumption 112
- average cost 59, 287
 - relationship to marginal cost 289–90, 290
- average fixed cost 263
- average product
 - and marginal product 283–7, 290, 290–2
 - maximization 250–1
- average revenue 242, 249, 270, 271
- average total cost 59
- average variable cost 264
- axiomatic probabilities 432
- balance of trade 124, 127–9, 195–8
- Bayes' theorem 437–9, 449, 449–51
 - controversy over 439
- bond prices 388–90, 391–2
- bordered Hessian matrix 359–60
- brackets 17–20
 - multiple 19–20, 30
- breakeven analysis 47–9, 71–2, 73
 - quadratic functions 205–9
 - simple model 87–8, 92, 92–4
- budget constraint 325
 - maximizing consumer utility subject to a 355–6
- budget surplus 127–9
 - and balancing the budget 327–8
- capital
 - production theory 320–3
 - productivity of 424
 - profit maximization 338–40
- capital stock formation 371–2
- cash flows 385–7
- causality 43
- certain annuities 397, 397–8, 410
- ceteris paribus* assumption 5
- chain rule 226–7, 229
- Chart wizard 51–4, 75–7
- circular flow of income 111–13
- closed economy 111–20
- Cobb–Douglas production function 61–2, 410
 - partial differentiation 320–3, 327
- cobweb models 426
- cofactor matrix 158, 172–3, 177–8
- cofactors 156
 - alien 157–8, 169
- column vectors 132
- common denominator 24–5, 30
- common logarithms 60
- comparative static analysis 101–4, 107, 114–17, 130
- competitive market
 - excise tax in 101–4, 107, 175–6
 - perfect competition *see* perfect competition
 - tax revenue maximization 277–8
- complementary solution 416, 418, 427
- compound interest 379–81, 391, 403–5
- conditional equality 87
- conditional events 433, 434–5
- conditional probability 435, 449, 449–51
 - Bayes' theorem 437–9
- constant returns to scale 323
- constants
 - derivative of a constant 229
 - differentiation of a constant times a power term 224, 229
 - e see e* (mathematical constant)
 - integral of a constant times a function 365
 - intercept 84, 85, 92, 112, 203, 205
- constrained optimization 294, 323, 345–60
 - bordered Hessian matrix 359–60
 - consumer utility maximization subject to a budget constraint 355–6

- cost minimization subject to an output constraint 354–5
- Lagrange multipliers 348–52, 353, 355, 356, 357, 358
- output maximization subject to a cost constraint 352–3
- principles of 345–8
- consumer income
 - gross and disposable 14–15
 - and quantity demanded 4, 5
- consumer utility *see* utility
- consumer's surplus 369–70, 371, 372
- consumption 14, 111–14, 116–17, 121, 127, 178–9
 - multiplier 318
 - over time 413–14
- consumption function 112–14, 430
 - analysis using derivatives 247–8
- contingent annuities 397, 410
- continuous interest rate 404–5
- continuous random variables 441–3
- convergence to equilibrium 418, 419, 423, 425–6, 427
- coordinates 36–7, 45–6, 47, 57–8
- cost
 - analysing using derivatives 246–7
 - average *see* average cost; average fixed cost; average variable cost
 - constraint and output maximization 352–3
 - fixed costs 45–6, 92, 93, 94
 - increases and pricing 163–5
 - marginal *see* marginal cost
 - minimization subject to an output constraint 354–5
 - relationship between the cost functions 289–90
 - theory of 287–9
 - total *see* total cost
 - variable costs 45–6, 92, 93–4
- Cramer's rule 161–2, 163
- cross-partial derivatives 300–2, 303, 303–4, 310
- cross-price elasticity of demand 241, 320, 327
- cubic functions 56, 57, 70
- curve sketching 235–7
- damped oscillations 419–20, 421, 427
- debt repayment 400–1, 450–1
- decision theory 431
 - decision-making under uncertainty 444–8
- decreasing returns to scale 323
- deductions 14–15, 17
- definite integrals 366–7, 373
 - and the area under a curve 367–9, 373
- demand
 - for bonds 390
 - elasticity of 104–7, 107, 238–40, 243–4, 319–20, 327, 337–8
 - factors affecting 4–6, 43
 - final 186–8, 189, 191, 192–4, 196–7
 - market models 96–8
 - for money 125
 - multivariable functions 67–8
 - sketching a demand function 235–6
- denominator 23
 - common 24–5, 30
- dependent variable 35, 36, 37, 42–3, 47
- depreciation 382–3
- depreciation rate 383
- derivatives 199–200, 218, 219–23, 230, 230–2
 - and the concept of marginality 237
 - cross-partial 300–2, 303, 303–4, 310
 - curve sketching 235–7
 - partial *see* partial derivatives
 - rules of differentiation 223–8, 229, 230
 - total derivative 307–9, 310
 - use in economic analysis 200, 235–52
 - analysing costs 246–7
 - analysing elasticity 238–41
 - analysing production 244–6
 - analysing revenue 242–4
 - consumption function 247–8
 - national income 248–9
- destinations of output 186, 187
- determinants 152, 154–60, 163
 - calculating the matrix inverse 158–9, 168–70
 - Cramer's rule 161–2, 163
 - in Excel 166, 167
 - and non-singularity 159–60
 - properties 157–8
- difference equations 413–29
 - macroeconomic model 413–24, 427–8
 - equilibrium position 415–16, 418, 419, 427–8
 - Harrod–Domar growth model 424
 - model with government sector 423
 - solutions to a difference equation 416–18, 427
 - stability of the model 418–22, 423, 427
 - market equilibrium 424–6
- difference quotients 85–6, 216, 217–18, 219
- differences/sums
 - derivatives of 225, 229
 - integrals of 365
- differentials 304–6
 - exponential functions 406–7
 - total differential 305–6, 307, 310, 311, 351
- differentiation 215–34
 - derivatives *see* derivatives
 - non-differentiable functions 228–9, 230
 - partial *see* partial differentiation
 - rules of 199, 223–8, 229, 230
 - slope of linear and non-linear functions 215–20
- dimensions of a matrix 132
- diminishing marginal utility, law of 324, 327
- diminishing returns, law of (or diminishing marginal product) 232, 245
- direct price elasticity of demand 319, 327
- discontinuities 228
- discount factor 385, 386, 387
- discount rate 385, 386–8, 391
- discrete random variables 439–41
- disposable income 14–15
- divergence from equilibrium 418, 420, 427
- division
 - dealing with exponents 33
 - fractions 23, 30
 - inequalities 22
- domain of an independent variable 45

- due annuities 397, 397–8, 399, 410, 411
 repayment annuities 400, 401
 dynamics 361, 413–29
 see also difference equations
- e* (mathematical constant) 60, 64–5
 calculus and 406–8
 and growth rates 401–6, 408–10
 econometrics 6, 7
 economic behaviour 7, 8
 economic dependency 42–3
 economic growth 449–51
 economic models
 dynamics 361, 413–29
 economic theory, mathematics and 6–8
 market models *see* market models
 multivariable *see* multivariable economic models
 national income models *see* national income models
 representation with matrix algebra 138–9, 140
 economic theory 2
 economic models, mathematics and 6–8
 effective interest rate 381–2, 391
 elastic demand 106, 107, 239, 240, 244, 249
 elasticity 238–41, 249
 of demand 104–7, 107, 238–40, 243–4, 319–20, 327, 337–8
 elasticities and growth rates 409–10
 of production 322–3
 elements of a matrix 132
 empirical (frequentist) probability 432–3, 439, 448
 EMV *see* expected monetary value
 endogenous variables 112–14
 equations 7, 34, 35
 linear 45–6, 54, 91–2
 equilibrium
 market equilibrium *see* market equilibrium
 national income models 113–14, 118–20, 123, 125–6, 127, 127–9, 318
 difference equation model 415–16, 418, 419, 427–8
 matrix algebra 178, 180–1, 182, 182–4
 with a government sector 121
 events 432, 449
 conditional 433, 434–5
 independent 433, 434
 mutually exclusive 433
 Excel
 determinants 166, 167
 financial calculations 393–5
 graphs in
 linear 51–4
 non-linear 75–80
 inbuilt financial formulae 393, 394
 matrix algebra 143–8
 matrix inverse 166–7
 excise tax 107
 in a competitive market 101–4, 107, 175–6
 effect on market equilibrium 103, 175–6
 maximizing tax revenue 107–9, 276–8
 and profit maximization 275, 279
 on short-haul flights in Europe 1–2, 3–6
 exogenous variables 112, 114–18, 127
 expansion path 355
 expected monetary value (EMV) 444–6
 expected utility 447–8, 449
 expected value 440–1, 443
 experiments, random 432
 explosive macroeconomic model 418, 420, 427
 explosive oscillations 420–1, 422, 426, 427
 exponential functions 62–6, 70
 differentiation 406–7
 growth models 401–10, 410
 integration 407–8
 rates of growth 408–10
 exponents (powers) 31–3, 62–4
 extrapolation 38, 47
 final demand 186–8, 189, 191, 192–4, 196–7
 financial analysis 361, 376–412
 annuities 396–400, 410, 410–11
 arithmetic series 377–8, 391
 compound interest 379–81, 391, 403–5
 depreciation 382–3
 Excel and 393–5
 geometric series 378–9, 380, 383, 391, 396, 398, 399, 400, 401
 growth models 401–10, 410
 interest rates and the price of government bonds 388–90, 391–2
 investment appraisal 384, 385–8
 nominal and effective interest rates 381–2, 391
 present value *see* net present value; present value
 simple interest 379–80, 391
 sinking funds 400–1, 410
 time preference 377, 384, 390
 financial markets 377
 financial (monetary) sector 124–6, 182–4
 firms 111–13
 first derivative 221–3, 236
 stationary points at zero value 254–5, 256, 263
 first-order conditions 256–7, 259, 268, 286, 330
 profit maximization 335, 336–7, 341
 unconstrained optimization 334, 336–7, 339, 340, 343
 first-order difference equations 414
 first-order partial derivatives 298–9, 300, 303, 310, 311
 fixed costs 45–6
 average fixed cost 263
 breakeven analysis 92, 93, 94
 foreign trade 123–4, 127, 127–9, 195–8
 fractions 23–7, 30
 with algebraic expressions 26–7
 frequentist probability 432–3, 439, 448
 function of a function, derivative of 226–7, 229
 functional independence 90
 functions 35, 40–6, 47
 implicit 309–10, 348
 linear *see* linear functions
 multivariable *see* multivariable functions

- non-differentiable 228–9, 230
- non-linear *see* non-linear functions
- notation 44
- Gauss–Jordan elimination method 152–4, 159, 163
- general equilibrium 98
- generalized power rule for
 - differentiation 224, 229
 - partial differentiation 312
- geometric series 378–9, 380, 383, 391, 396, 398, 399, 400, 401
- global maxima and minima 259–60, 263
- government bond prices 388–90, 391–2
- government expenditure 1, 14, 120–2, 127–9
 - input–output analysis 191, 192–4, 195–8
 - multiplier 122, 180–1, 318, 328
 - spending the budget surplus 327–8
- government sector 120–4, 127
 - budget surplus 127–9
 - and balancing the budget 327–8
 - dynamic model 423
 - effect of changes in the tax rate on the multiplier 248–9
 - matrix algebra 179–81
 - model including foreign trade 123–4
 - tax revenue maximization 107–9, 211–12, 276–8
- gradient (slope)
 - linear functions 84–6, 92, 112–13, 215–16
 - non-linear functions 216–21
 - quadratic functions 205–6
- graphs 34, 35
 - in Excel 51–4, 75–80
 - functions with more than one independent variable 67–9
- linear *see* linear graphs
- non-linear functions 57–8, 59, 70, 75–80
- obtaining linear equations from 91–2
- national income model 118–20
- quadrants 39–40
- three-dimensional 67–9, 295–8
- Greek alphabet 454
- gross income 14–15
- growth, proxy measures for 449–51
- growth models 401–10, 410
 - growth rates 408–10
 - Harrod–Domar macroeconomic model 424
- Harrod–Domar growth model 424
- Hessian matrix 342–4
 - bordered 359–60
- households 111–13
- identity matrix 133, 140, 150, 151, 152–4
- implicit functions 309–10, 348
- imports 123, 124
- income
 - circular flow of 111–13
 - consumer income 4, 5, 14–15
 - national income models *see* national income models
- income elasticity of demand 241, 320, 327
- increasing returns to scale 323
- indefinite integrals 364–6
- independent events 433, 434
- independent variable 35, 36, 37, 42–3, 47, 51
 - functions with more than one 66–9, 70
- indifference curves 325–6, 356
- indifference map 325–6
- inelastic demand 106, 107, 239, 240, 244, 249, 319, 320
- inequalities 21–3, 30, 352
- inferior goods 320
- inflection, points of 260–2
- input–output analysis 82, 186–98
 - input–output coefficients 188–90, 195, 196
 - input–output inverse 191–2, 193–4, 195
 - input–output tables 186–8, 195
- integration 361, 363–75
 - capital stock formation 371–2
 - consumer’s surplus 369–70, 371, 372
 - definite integrals 366–9, 373
 - exponential functions 407–8
 - notation and terminology 363–4
 - probability density function 442
- producer’s surplus 371, 372
 - rules of 364–6
- intercept (constant) 84, 85, 92, 112, 203, 205
- interest 376–95
 - compound 379–81, 391, 403–5
 - depreciation 382–3
 - exponential growth models and 403–6
 - investment appraisal 385–8
 - simple 379–80, 391
 - time preference 377
 - interest rate 124–5, 182–4, 377, 380–1
 - effective 381–2, 391
 - nominal 381–2
 - and the price of government bonds 388–90, 391–2
- internal rate of return (IRR) 386–8, 391, 394, 395
- interpolation 38–9, 47
- inverse function rule for differentiation 227–8
- inverse functions 69–70, 70, 71
- inverse matrix *see* matrix inverse
- investment 111–12, 113, 118, 124–5
 - appraisal 384, 385–8
 - capital stock formation 372
 - change in and national income 114–17, 119, 178–9
 - Harrod–Domar growth model 424
 - multiplier 115–17, 178–9, 318
- IRR *see* internal rate of return
- IS schedule (investment and savings schedule) 125–6
- iso lines
 - indifference curves 356
 - iso-costs and iso-quants 353, 355
 - profit 346–8
- Keynesian multiplier 115–16
- labour
 - average product (AP) of labour 250–51, 283–4,
 - marginal product of labour (MPL) 232, 237–8, 245–7, 250, 284, 321, 339
 - production maximization 230–2, 249–51
 - production theory 320–3
 - profit maximization 338–40

- labour market 195–8
- Lagrange functions 348, 352, 354, 356, 358, 359
- Lagrange multipliers 348–52, 353, 355, 356, 357, 358
interpretation 349–52
- Laplace expansion 154–5, 156–7, 161, 163
- law of diminishing marginal utility 324, 327
- law of diminishing returns (or diminishing marginal product) 232, 245
- Law of Total Probability 437, 449
- limit (limiting value of a function) 217–18, 220
- linear equations 45–6, 54, 91–2
- linear functions 44–6, 83–6, 92, 215
gradient (slope) 84–6, 92, 112–13, 215–16
integration and area under the line 367, 368, 369
intercept 84, 85, 92, 112, 203, 205
non-linear functions derived from 58–9
transformation of non-linear functions to using logarithms 61–2
- linear graphs 35–40, 47
obtaining linear equations from 91–2
plotting 45–6
using Excel 51–4
- linear models 81, 83–95
simple breakeven model 87–8
simultaneous equations 89–90
- linear relationships 11, 34–54
see also linear functions; linear graphs
- LM schedule (liquidity–money schedule) 125–6
- loan repayments 400–1, 450–1
- local maxima and minima 259–60, 262, 263
- logarithmic functions 60–2, 70
derivatives of 227, 229
log rule for integration 365
- logarithms 60–2
- logical deductions 4–5, 7
- lump-sum tax 272–4, 279
- macroeconomic growth model 424
- marginal cost 237, 238, 246–7, 249, 263, 290
cost minimization 355
and marginal product 246–7
price discrimination 336–7
profit maximization 267, 268, 269, 270, 271, 273
equality to marginal revenue 267, 279, 335
relationship to average cost 289–90, 290
and total cost 287–9, 373–4
- marginal probability 437–8
- marginal product
and average product 283–7, 290, 290–2
and marginal cost 246–7
and marginal revenue 245–6
unconstrained optimization 338–40, 341
- marginal product of capital 238, 321–2, 339–40, 341
- marginal product of labour 232, 238, 245, 250–1, 287–8, 321–2, 339–40, 341
- marginal propensity to consume (mpc) 113, 120, 238, 247, 430
multiplier 318
- marginal propensity to save 238, 248
- marginal rate of substitution 326, 356
- marginal revenue 237, 238, 242–4, 249, 264
and marginal product 245–6
price discrimination 336–7
profit maximization 267, 268, 269, 270, 271, 272, 273
equality to marginal cost 267, 279, 335
- marginal revenue product (MRP) 245–6, 249
- marginal utility 238, 323–4, 327
- marginal utility of income 356
- marginality 237, 238, 249
- market equilibrium 98–100, 107, 369
difference equations 424–6
effect of an excise tax 103–4, 175–6
matrix algebra 173–4, 175–6
- partial differentiation 315–17
- quadratic functions 209–10
- market models 81, 96–110
- dynamic model 424–6
- elasticity *see* elasticity
- excise tax in a competitive market 101–4, 107, 175–6
- market demand and supply 96–8
- matrix algebra 171–6, 181
- partial equilibrium model 98–100, 171–4, 314–17
- mathematical dependency 42–3
- mathematics
economic theory, economic models and 6–8
need for in economics 3–6
- matrices 131–2, 140
bordered Hessian matrix 359–60
dimensions of a matrix 132
Hessian matrix 342–4
identity matrix 133, 140, 150, 151, 152–4
null matrix 133, 140
transpose matrix 133, 140, 145–8
- matrix algebra 81, 130–48
addition 134, 143–5, 146
application to economic models 82, 171–85
market model 171–6, 181
national income model 177–81, 182, 182–4
matrix multiplication 135–8, 147, 148
multiplication by a scalar 135
subtraction 135
using Excel 143–8
using to represent economic models 138–9
vocabulary 131–2
- matrix inverse 82, 149–70, 172
calculating 152–4
using determinants 158–9, 168–70
Cramer's rule 161–2, 163
using Excel 166–7
using a matrix inverse 150–1
- maxima 209, 254–6, 256–7, 257–8, 263
global 259–60, 263
local 259–60, 262, 263
unconstrained optimization 330–5, 340, 343
see also first-order conditions; second-order conditions

- maximization 253
- MDETERM function 166, 167
- mean, arithmetic 440–1, 443
- minima 209, 256–7, 257–8, 263
 - global 259–60, 263
 - local 259–60, 262, 263
 - unconstrained optimization 330–5, 340, 343*see also* first-order conditions; second-order conditions
- minimization 253
- minors of a matrix 155, 163, 172–3
- MINVERSE function 166
- MMULT function 147, 148
- models 7
 - economic *see* economic models
- modulus 240
- monetary sector (financial sector) 124–6, 182–4
- money
 - demand for 125
 - LM schedule (liquidity–money schedule) 125–6
- money supply 2
 - effect of changes in on national income equilibrium 125–6, 182–4
- monopoly 243, 249
 - profit maximization 270–2, 273
- monotonic functions 228
- multiple brackets 19–20, 30
- multiplication
 - dealing with exponents 32, 33
 - fractions 23
 - inequalities 22
 - matrices
 - matrix multiplication 135–8
 - multiplication by a scalar 135
 - multiplying two matrices 137–8, 147, 148
 - multiplying a vector and a matrix 136
- Multiplication Law 433–5, 449
- multipliers 123, 127
 - derivatives 248–9
 - effect of changes in the tax rate 248–9
 - government expenditure 122, 180–1, 318, 328
 - input–output analysis 194
 - investment 115–17, 178–9, 318
 - matrix algebra 178–9, 180–1, 182
 - partial differentiation and 318–19, 327
 - multivariable economic models 293, 314–29
 - elasticity of demand 319–20, 327
 - national income model 317–19, 327, 327–8
 - partial market equilibrium 314–17
 - production functions 320–3, 327
 - utility functions 323–6, 327
- multivariable functions 293, 295–313
 - differentials 304–6
 - implicit functions 309–10
 - partial differentiation 295–304, 310, 312–13
 - total derivative 307–9, 310
- mutually exclusive events 433
- named ranges method 144–5, 146
- national income models 81, 111–29, 130
 - analysis using derivatives 248–9
 - closed economy 111–20
 - diagram form 118–20
 - difference equations model 413–24, 427–8
 - matrix algebra 177–81, 182, 182–4
 - multipliers *see* multipliers
 - partial differentiation 317–19, 327, 327–8
 - proxy measures 449–51
 - with foreign trade 123–4, 127, 127–9, 195–8
 - with a government sector *see* government sector
 - with a monetary sector 124–6, 182–4
- natural logarithms 60, 227, 229
- negative exponents 32
- negative values 22
 - in graphs 39–40
 - inequalities 30
- net present value (NPV) 386
 - of an annuity 399, 411
 - interest rates and the price of government bonds 388–90
 - internal rate of return 386–8, 391
- nominal interest rate 381–2
- non-differentiable functions 228–9, 230
- non-linear functions 11, 55–80
 - definite integrals and areas under curves 367–8, 369
 - derived from linear functions 58–9
 - exponential functions *see* exponential functions
 - functions with more than one independent variable 66–9, 70
 - gradient (slope) 216–21
 - graphs of 57–8, 59, 70, 75–80
 - inverse functions 69–70, 70, 71
 - logarithms and logarithmic functions 60–2, 70
 - polynomial functions 56–8, 70, 75–80, 201–2
- non-singularity 150, 159–60
- non-stationary inflection points 262
- normal goods 320
- normative expected utility theory 448
- notation
 - algebraic 14
 - functional 44
 - integration 363–4
 - matrices 131–2
- null matrix 133, 140
- numerator 23
- numerical values 6
- objective function 346–8, 348
- observed (frequentist) probability 432–3, 439, 448
- opportunity cost 350, 357
- optimization 199–200
 - constrained *see* constrained optimization
 - example of 253–7
 - in general 257–8
 - maxima *see* maxima
 - minima *see* minima
 - points of inflection 260–2
 - principles of 200, 253–65
 - in production theory 200, 283–92
 - profit maximization *see* profit maximization
 - tax revenue maximization 107–9, 211–12, 276–8
 - unconstrained *see* unconstrained optimization

- ordinary annuities 397, 410
 - present value 399
 - sum 398
- origin of a graph 36
- oscillations 424
 - damped 419–20, 421, 427
 - explosive 420–1, 422, 426, 427
- ‘other things being equal’ (*ceteris paribus*) assumption 5
- output 186–7
 - breakeven analysis *see* breakeven analysis
 - constraint and cost minimization 354–5
 - exponential growth model 410
 - input–output analysis *see* input–output analysis
 - maximization subject to a cost constraint 352–3
 - profit-maximizing *see* profit maximization
 - total cost as a function of 44–6
- parameters 4, 5–6
 - consumption function 112–14
 - exponential functions 62, 64
 - linear functions 45, 83–6
 - quadratic functions 203–4, 205
- partial derivatives 298–9, 307–8
 - cross-partial derivatives 300–2, 303, 303–4, 310
 - first-order 298–9, 300, 303, 310, 311
 - general principles of
 - unconstrained optimization 330–4
 - interpretation 303
 - Lagrange multipliers 348–9, 350–1
 - second-order 299–302, 303, 310, 311
- partial differentiation 295–304, 310
 - analysis of multivariable economic models 293, 314–29
 - generalization to n -variable functions 302–4
 - rules for 312–13
- partial market equilibrium model 98–100
 - matrix algebra 171–4
 - multivariable model 314–17
- particular solution 416, 418, 427
- partition of the sample space 437
- perfect competition 243, 249
 - profit maximization 267–70
- perpetual annuities 397, 410
- planes 68–9
- planned consumption and investment 118–19
- point elasticities 238–41, 249
- points of inflection 260–2
- polynomial functions 56–8, 70, 201–2
 - graphs of 57–8, 75–80
- positive expected utility theory 448
- posterior probability 438
- power rule
 - differentiation 223–4, 229
 - integration 364
- powers (exponents) 31–3, 62–4
- precautionary demand for money 125
- present value 384–5, 391, 396
 - annuities 399, 411
 - investment appraisal 385–6
 - net *see* net present value (NPV)
- price 14
 - breakeven analysis 92, 93, 94
 - equilibrium price 98–100, 103–4, 107, 174, 176, 210, 315–17
 - consumer’s and producer’s surpluses 369–71, 372
 - impact of a tax on short-haul flights 1–2, 3–6
 - and profit maximization 279–81
 - relationship with quantity 35, 37–9, 235–6
 - sales, profit and 140–1, 163–5
 - sales, revenue and 138–9, 140–1, 150–1
- price discrimination 336–8
- price elasticity of demand 105–7, 238–40, 249, 319, 327
- cross-elasticity 241, 320, 327
- direct 319, 327
- and revenue 243–4
- primary factors of production 186–8, 193–4
 - see also* capital; input–output analysis; labour
- principal sum 379–80
- prior probability 437–8
- probability 362, 430–53
 - assigning 432–3
 - basic rules 433–7
- Bayes’ theorem 437–9, 449, 449–51
- decision-making under uncertainty 444–8
- uncertainty and 431
- understanding 432–3
- probability density function 442
- probability distributions 439–43, 449
 - continuous random variables 441–3
 - discrete random variables 439–41
 - expected value 440–1, 443
- probability-equivalence method 446–8
- probability function 440, 441–2
- probability utility 446–7
- producer’s surplus 371, 372
- product launches 65–6, 438
- product rule for differentiation 225, 229
- partial differentiation 312–13
- production
 - analysing using derivatives 244–6
 - constrained optimization
 - cost minimization subject to an output constraint 354–5
 - output maximization subject to a cost constraint 352–3
 - profit maximization 345–8, 357–8
 - elasticity of 322–3
 - maximization and the number of workers 230–2, 249–51
 - optimization in production theory 200, 283–92
 - relationship between the cost functions 289–90
 - theory of costs 287–9
 - unconstrained optimization 338–40, 341
- production functions 61–2, 283–7, 290–2
 - Cobb–Douglas 61–2, 320–3, 327, 410
 - partial differentiation 320–3, 327
- productivity of capital 424
- profit 264
 - sales, price and 140–1, 163–5
- profit function 48, 71–3, 208–9
 - and its derivative 221–3

- profit maximization 71–3, 208–9, 253–4, 266–75, 279, 279–81
 constrained 345–8, 357–8
 effect of tax on 272–5, 279
 monopoly 270–2, 273
 perfect competition 267–70
 and price discrimination 336–8
 unconstrained 335, 338–40, 341
- profit tax 274–5, 279
- project appraisal 384, 385–8
- proxy measures 449–51
- quadrants 39–40
- quadratic functions 56–7, 70, 199, 201–14
 breakeven analysis 205–9
 characteristics 202–5, 211
 graphs in Excel 77–80
 market equilibrium 209–10
 roots of a quadratic equation 207–8, 209, 211
 derivation of the roots formula 213–14
 turning points 254–7
 with no real roots 210–11
- quantity
 equilibrium quantity 98–100, 103–4, 107, 174, 176, 209–10, 315–16
 consumer's and producer's surpluses 369–71, 372
 relationship with price 35, 37–9, 235–6
- quarterly interest 404, 406
- quotient rule for differentiation 226, 229
- partial differentiation 313
- random experiments 432
- random variables 439–43, 449
 continuous 441–3
 discrete 439–41
- real input price 339
- reducing-balance method 383
- repayment annuities 400, 401
- retirement income planning 378, 379
- returns to scale effect 61
- revenue
 analysing using derivatives 242–4
 average 242, 249, 270, 271
 marginal *see* marginal revenue
- sales, price and 138–9, 140–1, 150–1
 total *see* total revenue
- risk, attitudes to 445–8
- risk aversion 446, 447, 448
- risk neutrality 448
- risk seeking 446, 448
- row vectors 132
- saddle point 334, 341
- sales
 price, profit and 140–1, 163–5
 price, revenue and 138–9, 140–1, 150–1
- sales tax *see* excise tax
- sample point 432
- sample space 432
 partition of 437
- savings 113, 118, 127, 247–8, 325
 Harrod–Domar growth model 424
 IS schedule (investment and savings schedule) 125–6
- scalars 132, 140
 multiplication of a matrix by a scalar 135
- scales, in graphs 37, 38
- second derivative 222–3, 236
 and optimization 255–6, 263
- second-order conditions 256–7, 257–8, 259, 268, 330–1
 bordered Hessian matrix 359–60
 Hessian matrix 342–4
 optimization of production 286–7
 unconstrained optimization 334, 339, 340, 342–4
- second-order determinants 154
- second-order difference equations 414
- second-order partial derivatives 299–302, 303, 310, 311
- semi-annual interest 404, 405
- shadow price 350
- simple interest 379–80, 391
- simplification of fractions 24–5, 27
- simultaneous equations 89–90, 130, 160
 Cramer's rule 161–2, 163
- singular matrices 150, 159–60
- sinking funds 400–1, 410
- slope *see* gradient (slope)
- special matrices 133, 140
- speculative demand for money 125
- square matrices 132, 150
- stability over time 418–22, 423, 427
- standard deviation 441, 443
- states of nature 444
- stationary points 222, 254
 points of inflection 260–1
 turning points *see* maxima; minima; turning points
 unconstrained optimization 330–4
- straight-line depreciation 382
- straight-line graphs *see* linear graphs
- subjective probability 432–3, 439, 448
- substitution rule for integration 365–6
- subtraction
 fractions 25–6, 30
 inequalities 21, 22
 matrices 135
- summation operator (sigma) 156–7
- sums/differences
 derivatives of 225, 229
 integrals of 365
- supply
 of bonds 390
 elasticity of 241
 market models 96–8
 money supply 2, 125–6, 182–4
- surplus
 consumer's 369–70, 371, 372
 government budget surplus 127–9, 327–8
 producer's 371, 372
- tangent lines 216–17, 300–2
- tax
 bankers' bonuses 2
 effect of changes in the tax rate on the multiplier 248–9
 effect on profit maximization 272–5, 279
 excise tax *see* excise tax
 multiplier 318, 319
 national income models 120–2, 127–9, 179–81
 short-haul air travel tax 1–2, 3–6
 revenue maximization 107–9, 276–8
 tax rate and 211–12
 term of an annuity 397

- three-dimensional graphs 67–9, 295–8
- time preference 377, 384, 390
- total cost
 - average 59
 - breakeven analysis 48–9, 87–8, 205–6
 - linear function 44–6, 47–9, 83–6
 - and marginal cost 287–9, 373–4
 - monopoly 270, 271
 - non-linear function 56–7, 59
 - perfect competition 268, 269
 - total revenue, profit and 71–2, 73, 266–7
- total demand for primary factors of production 189, 193–4, 197
- total derivative 307–9, 310
- total differentials 305–6, 307, 310, 311, 351
- total output vector 189, 191, 192–4, 196–7
- Total Probability, Law of 437, 449
- total revenue 242, 264
 - breakeven analysis 48–9, 87–8, 205–6
 - monopoly 270, 271, 272, 273
 - perfect competition 268, 269
 - price elasticity of demand 106
 - from quantity demanded function 69–70
 - total cost, profit and 71–2, 73, 266–7
- trade, foreign 123–4, 127, 127–9, 195–8
- transactions demand for money 125
- TRANSPOSE function 145–8
- transpose matrix 133, 140, 145–8
- transposition of an expression 27–9
- trendlines 77–80
- trials 432
- turning points 254–8
 - general approach to finding 257–8
 - local and global 259–60, 262, 263
 - maxima *see* maxima
 - minima *see* minima
 - quadratic functions 208, 209, 211, 212
- uncertainty 430–1
 - decision-making under 444–8
 - and probability 431
- unconditional probability 437–8
- unconstrained optimization 293, 330–44
 - general principles 330–4
 - Hessian matrix 342–4
 - price discrimination 336–8
 - profit maximization 335, 338–40, 341
- unitary elasticity 106, 107, 240, 244, 249
- utility 446–7
 - decision-making under uncertainty 445–8
 - marginal 238, 323–4, 327
 - marginal utility of income 356
 - maximization subject to a budget constraint 355–6
 - partial differentiation of a utility function 323–6, 327
 - unconstrained optimization 341
- value of a future amount 379–81, 396, 397–8
- variable costs 45–6
 - average variable cost 264
 - breakeven analysis 92, 93–4
- variables 4, 14
 - dependent 35, 36, 37, 42–3, 47
 - endogenous 112–14
 - exogenous 112, 114–18, 127
 - independent *see* independent variable
 - random 439–43, 449
- variance 440–1, 443
- vectors 132, 140
 - multiplying a vector and a matrix 136
- x axis 36–7, 39–40, 47
- y axis 36–7, 39–40, 47
- Young's theorem 300, 333