



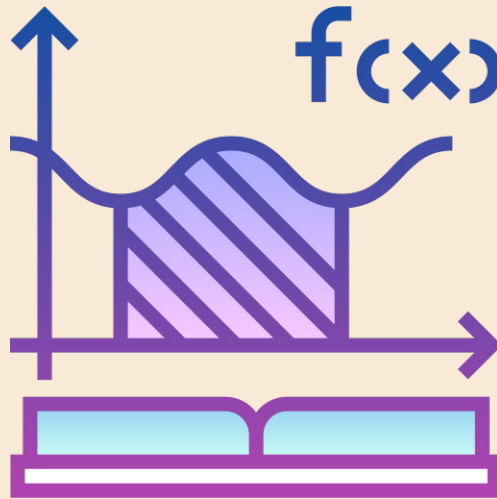
MATHS102

Calculus II

Lesson (1)

(4.4) L'Hopital's Rule

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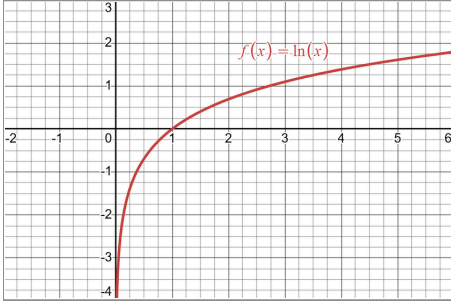
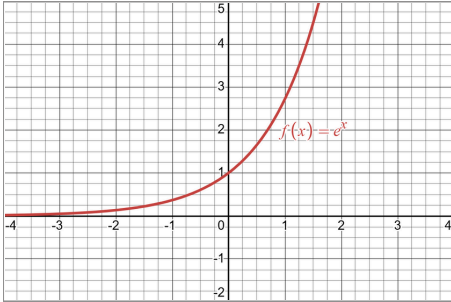
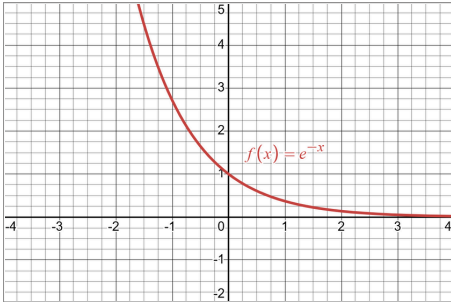
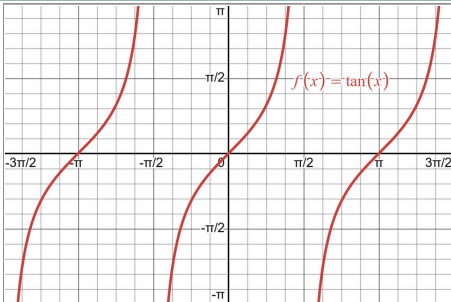
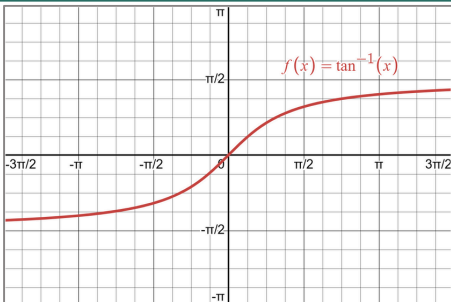
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Function	Domain	Range	End Behavior
	$(0, \infty)$	$(-\infty, \infty)$	$\lim_{x \rightarrow \infty} \ln(x) = \infty$ $\lim_{x \rightarrow 0^+} \ln(x) = -\infty$
	$(-\infty, \infty)$	$(0, \infty)$	$\lim_{x \rightarrow \infty} e^x = \infty$ $\lim_{x \rightarrow -\infty} e^x = 0$
	$(-\infty, \infty)$	$(0, \infty)$	$\lim_{x \rightarrow \infty} e^{-x} = 0$ $\lim_{x \rightarrow -\infty} e^{-x} = \infty$
	$x \neq \frac{\pi}{2} + n\pi$	$(-\infty, \infty)$	$\lim_{x \rightarrow \frac{\pi}{2}^-} \tan^{-1} x = \frac{\pi}{2}$ $\lim_{x \rightarrow -\frac{\pi}{2}^+} \tan^{-1} x = -\frac{\pi}{2}$
	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$\lim_{x \rightarrow \pm \frac{\pi}{2}^+} \tan x = -\infty$ $\lim_{x \rightarrow \pm \frac{\pi}{2}^-} \tan x = \infty$
Exponential Functions $b^x, b > 1$	$(-\infty, \infty)$	$(0, \infty)$	$\lim_{x \rightarrow \infty} b^x = \infty$ $\lim_{x \rightarrow -\infty} b^x = 0$
Exponential Functions $b^x, 0 < b < 1$	$(-\infty, \infty)$	$(0, \infty)$	$\lim_{x \rightarrow \infty} b^x = 0$ $\lim_{x \rightarrow -\infty} b^x = \infty$

Indeterminate Forms

Sometimes, direct substitution in a limit does not provide a clear answer. Instead, it leads to forms that require special techniques. These forms are called indeterminate forms. In this section, we will discuss these forms and how to evaluate the limits for each one correctly.

Indeterminate
Fractions

$$\left(\frac{0}{0}, \frac{\infty}{\infty}\right)$$

Indeterminate
Products

$$(0 \cdot \infty)$$

Indeterminate
Differences

$$(\infty - \infty)$$

Indeterminate
Powers

$$(0^0, \infty^0, 1^\infty)$$

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Indeterminate Fractions $\left(\frac{0}{0}, \frac{\infty}{\infty}\right)$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \longrightarrow \text{is of the form } \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

can be any real number
or ∞



Apply L'Hospital's Rule: $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$



Notice that, we can apply L'Hospital's Rule more than once.

Examples

① $\lim_{x \rightarrow 0} \frac{\sin x}{x} \longrightarrow \frac{0}{0}$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = \cos(0) = 1$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

② $\lim_{x \rightarrow \infty} \frac{e^x}{x^2} \longrightarrow \frac{\infty}{\infty}$

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} \longrightarrow \frac{\infty}{\infty}$$

$$= \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\textcircled{3} \quad \lim_{x \rightarrow 0} \frac{x \cdot 3^x}{3^x - 1} \mapsto \frac{0}{0}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x \cdot 3^x}{3^x - 1} &= \lim_{x \rightarrow 0} \frac{3^x + x \cdot 3^x \ln(3)}{3^x \ln(3)} \\ &= \lim_{x \rightarrow 0} \frac{3^x (1 + x \ln(3))}{3^x \ln(3)} \\ &= \lim_{x \rightarrow 0} \frac{1 + x \ln(3)}{\ln(3)} = \frac{1}{\ln(3)} \end{aligned}$$

$$\frac{d}{dx}(b^x) = b^x \ln(b), b > 0$$

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Indeterminate Products ($0 \cdot \infty$)

$\lim_{x \rightarrow a} f(x) \cdot g(x) \mapsto$ is of the form $0 \cdot \infty$

can be any real number
or ∞

✓ **Rewrite the Product as a Quotient:** $f(x) \cdot g(x) \rightarrow \frac{f(x)}{\frac{1}{g(x)}} \text{ or } \frac{g(x)}{\frac{1}{f(x)}}$

✓ **Apply L'Hospital's Rule:** $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$

Examples

$$\textcircled{4} \quad \lim_{x \rightarrow \frac{\pi}{2}^-} \tan x \ln(\sin x) \mapsto 0 \cdot \infty$$

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \tan x \ln(\sin x) = \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\ln(\sin x)}{\cot x}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\frac{\cos x}{\sin x}}{-\csc^2 x}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\cos x}{\sin x} \cdot -\sin^2 x$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^-} (-\cos x \sin x) = -\cos\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{2}\right) = 0$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$$



Indeterminate Differences ($\infty - \infty$)

$\lim_{x \rightarrow a} f(x) - g(x) \longrightarrow$ is of the form $\infty - \infty$

can be any real number
or ∞

✓ Rewrite the Difference as a Quotient or a Product:

$$f(x) - g(x) \rightarrow \frac{A}{B} \text{ or } A \cdot B$$

✓ You will get one of the previous forms

Examples

⑤ $\lim_{x \rightarrow 0^+} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) \longrightarrow \infty - \infty$

$$\begin{aligned} \lim_{x \rightarrow 0^+} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) &= \lim_{x \rightarrow 0^+} \frac{x - e^x + 1}{x(e^x - 1)} \\ &= \lim_{x \rightarrow 0^+} \frac{1 - e^x}{e^x + xe^x - 1} \\ &= \lim_{x \rightarrow 0^+} \frac{-e^x}{e^x + e^x + xe^x} \\ &= \lim_{x \rightarrow 0^+} \frac{-e^x}{e^x(2 + x)} = \lim_{x \rightarrow 0^+} \frac{-1}{2 + x} = -\frac{1}{2} \end{aligned}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(f \cdot g) = f' \cdot g + f \cdot g'$$



Indeterminate Powers $(0^0, \infty^0, 1^\infty)$

$\lim_{x \rightarrow a} [f(x)]^{g(x)}$ ----- is of the form 0^0 or ∞^0 or 1^∞

can be any real number
or ∞



Transform the power to logarithm: $[f(x)]^{g(x)} \rightarrow y = \ln [f(x)]^{g(x)}$
 $y = g(x) \ln f(x)$



Evaluate the limit of y and the main limit:

$$\lim_{x \rightarrow a} y = L \Rightarrow \lim_{x \rightarrow a} [f(x)]^{g(x)} = e^L$$

Examples

⑥ $\lim_{x \rightarrow \infty} (x)^{\frac{1}{x}} \mapsto \infty^0$

$$y = \ln \left(x^{\frac{1}{x}} \right) = \frac{1}{x} \ln x$$

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} \frac{1}{x} \ln x \mapsto 0 \cdot \infty$$

$$= \lim_{x \rightarrow \infty} \frac{\ln x}{x} \mapsto \frac{\infty}{\infty}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = \frac{1}{\infty} = 0$$

$$\Rightarrow \lim_{x \rightarrow \infty} (x)^{\frac{1}{x}} = e^0 = 1$$

$$\frac{d}{dx} (\ln(f(x))) = \frac{f'(x)}{f(x)}$$

$$\frac{d}{dx} (x^n) = nx^{n-1}$$

$$\ln(f^g) = g \ln(f)$$

$$\textcircled{7} \quad \lim_{x \rightarrow 1^-} (1-x)^{\ln x} \mapsto 0^0$$

$$y = \ln(1-x)^{\ln x} = \ln x \ln(1-x)$$

$$\lim_{x \rightarrow 1^-} y = \lim_{x \rightarrow 1^-} \ln x \ln(1-x) \mapsto 0 \cdot \infty$$

$$= \lim_{x \rightarrow 1^-} \frac{\ln(1-x)}{\frac{1}{\ln x}} \mapsto \frac{\infty}{\infty}$$

$$= \lim_{x \rightarrow 1^-} \frac{-\frac{1}{1-x}}{-\frac{1}{x(\ln x)^2}}$$

$$= \lim_{x \rightarrow 1^-} \frac{x(\ln x)^2}{1-x} \mapsto \frac{0}{0}$$

$$= \lim_{x \rightarrow 1^-} \frac{(\ln x)^2 + 2 \ln x}{-1} = 0$$

$$\Rightarrow \lim_{x \rightarrow 1^-} (1-x)^{\ln x} = e^0 = 1$$

$$\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\ln(f^g) = g \ln(f)$$

$$\frac{d}{dx}(f \cdot g) = f' \cdot g + f \cdot g'$$

$$\textcircled{8} \quad \lim_{x \rightarrow 0^+} (1+x)^{\frac{1}{x}} \mapsto 1^\infty$$

$$y = \ln(1+x)^{\frac{1}{x}} = \frac{1}{x} \ln(1+x)$$

$$\lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} \frac{1}{x} \ln(1+x) \mapsto 0 \cdot \infty$$

$$= \lim_{x \rightarrow 0^+} \frac{\ln(1+x)}{x} \mapsto \frac{0}{0}$$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{1+x}}{1} = \frac{1}{1} = 1$$

$$\Rightarrow \lim_{x \rightarrow 0^+} (1+x)^{\frac{1}{x}} = e^1 = e$$

$$\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\ln(f^g) = g \ln(f)$$

Exercises

$$\textcircled{1} \quad \lim_{x \rightarrow 0^+} \frac{\tan^{-1}(2x)}{\ln x}$$

$$\lim_{x \rightarrow 0^+} \frac{\tan^{-1}(2x)}{\ln x} = \frac{\tan^{-1}(0)}{-\infty} = \frac{0}{-\infty} = 0$$

$$\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\ln(f^g) = g \ln(f)$$

$$\textcircled{2} \quad \lim_{x \rightarrow 0^+} \ln x - \ln(\sin x) \mapsto \infty - \infty$$

$$\lim_{x \rightarrow 0^+} \ln x - \ln(\sin x) = \lim_{x \rightarrow 0^+} \ln\left(\frac{x}{\sin x}\right)$$

$$\lim_{x \rightarrow 0^+} \frac{x}{\sin x} \mapsto \frac{0}{0}$$

$$\lim_{x \rightarrow 0^+} \frac{x}{\sin x} = \lim_{x \rightarrow 0^+} \frac{1}{\cos x} = \frac{1}{\cos(0)} = 1$$

$$\Rightarrow \lim_{x \rightarrow 0^+} \ln x - \ln(\sin x) = \ln(1) = 0$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\ln(f) - \ln(g) = \ln\left(\frac{f}{g}\right)$$

$$\textcircled{3} \quad \lim_{x \rightarrow \infty} \frac{\log_2 x}{\log_3(x+3)} \mapsto \frac{\infty}{\infty}$$

$$\lim_{x \rightarrow \infty} \frac{\log_2 x}{\log_3(x+3)} = \lim_{x \rightarrow \infty} \frac{\frac{1}{\ln(2)} x}{\frac{1}{\ln(3)}(x+3)}$$

$$= \lim_{x \rightarrow \infty} \frac{\ln(3)}{\ln(2)} \cdot \frac{x+3}{x} \mapsto \frac{\infty}{\infty}$$

$$= \lim_{x \rightarrow \infty} \frac{\ln(3)}{\ln(2)} \cdot \frac{1}{1} = \frac{\ln(3)}{\ln(2)}$$

$$\frac{d}{dx}(\log_b[f(x)]) = \frac{f'(x)}{\ln(b) f(x)}$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\textcircled{4} \quad \lim_{x \rightarrow 0} \frac{e^{ax} - 1}{\sin x} \mapsto \frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{e^{ax} - 1}{\sin x} = \lim_{x \rightarrow 0} \frac{ae^{ax}}{\cos x} = \frac{ae^0}{\cos(0)} = \frac{a}{1} = a$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(e^{f(x)}) = f'(x)e^{f(x)}$$

5 If $\lim_{x \rightarrow 0} \frac{\sin(ax) + bx^3 - 2x}{x^3} = 0$.

Find a and b .

$$\lim_{x \rightarrow 0} \frac{\sin(ax) + bx^3 - 2x}{x^3} \rightarrow \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{a \cos(ax) + 3bx^2 - 2}{3x^2} \rightarrow \frac{0}{0}$$

$$\Rightarrow a \cos(0) + 3b(0)^2 - 2 = 0$$

$$\Rightarrow a - 2 = 0$$

$$\Rightarrow a = 2$$

$$= \lim_{x \rightarrow 0} \frac{2 \cos(2x) + 3bx^2 - 2}{3x^2} \rightarrow \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{-4 \sin(2x) + 6bx}{6x} \rightarrow \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{-8 \cos(2x) + 6b}{6} = 0$$

$$\Rightarrow \frac{-8 \cos(0) + 6b}{6} = 0$$

$$\Rightarrow -8 \cos(0) + 6b = 0$$

$$\Rightarrow -8 + 6b = 0$$

$$\Rightarrow 6b = 8$$

$$\Rightarrow b = \frac{8}{6} = \frac{4}{3}$$