NPR College of Engineering and Technology
(AN AUTONOMOUS INSTITUTION)
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Affiliated to Anna University, Chennai
Natham, Dindigul (Dist.) - 624 401, Tamil Nadu.

Department of Science & Humanities
25MA101- Engineering Mathematics
Question Bank - Internal test 2

Unit 3 – Mathematical Modelling of Power and Polynomial Functions

Comparison test, Ratio test:

1) a) A communication engineer models the probability of losing the n-th packet in a long transmission as $P_n=\frac{1}{n^3+2}$ .To estimate the total expected loss, they want to know if $\sum_{n=1}^{\infty} P_n$. (5 marks)

   b) A medical researcher studies how the concentration of a drug diffuses through tissue. The concentration error at the n-th diffusion cycle is modeled as $E_n = \frac{3^n}{n!}$. To check whether the total accumulated error *i*s safely bounded, the researcher wants to know whether the infinite series $\sum_{n=1}^{\infty} E_n = \sum_{n=1}^{\infty} \frac{3^n}{n!}$ converges. (5 marks)

2) a) Heat loss at the n-th minute is approximated by $H_n = \frac{3}{n^2+5n}$ .The engineer needs to know if the cumulative heat loss $\sum_{n=1}^{\infty} H_n$ is finite. (5 marks)

   b) A biomedical engineer is studying how a complex enzyme unfolds inside a cell. At the n-th stage of unfolding, the probability that the enzyme reaches that exact configuration is modeled as $\left(\frac{1}{3}\right)^2 + \left(\frac{1.2}{3.5}\right)^2 + \left(\frac{1.2.3}{3.5.7}\right)^2 + \cdots$ The numerator represents the increasing number of ways the molecule can stretch, while the denominator represents the rapidly growing energy barriers at each stage.Test to determine whether this series converges or diverges. (5 marks)

3) a) An AI algorithm's error probability at iteration n is given by $E_n = \frac{5}{\sqrt{n}+10}$. The developer wants to know if the total expected error $\sum_{n=1}^{\infty} E_n$ is finite. (5 marks)

   b) A data scientist is analyzing the computational cost of a deep-learning algorithm. At the n-th training layer, the cost of feature expansion is modeled by $\frac{1^2 2^2}{1!} + \frac{2^2 3^2}{2!} + \frac{3^2 4^2}{3!} + \ldots$ Here the numerator represents the rapid growth of feature interactions in consecutive layers and the denominator represents the computational efficiency improvements due to optimized matrix operations. Test to determine whether the total complexity is finite or diverges. (5 marks)

4) a) The inflow rate into the tank decreases as $F_n = \frac{4}{n+n^2}$. Use the Comparison Test to check convergence of $\sum_{n=1}^{\infty} F_n$. (5 marks)

   b) A company launches a loyalty program where customers earn reward points each month. In the first month, every customer receives $P_1 = \frac{1}{3}$, the second month customer receives $P_2 = \frac{1}{2.3^2}$, third month customer receives $P_3 = \frac{1}{3.3^3}$ and so on the reward points decrease according to the rule $\sum_{n=1}^{\infty} P_n = \frac{1}{3} + \frac{1}{2.3^2} + \frac{1}{3.3^3} + \cdots$. The company wants to know whether the total reward points given to each customer over infinite months is finite. (5 marks)

5) a) A risk analyst models the risk factor at the n-th time period as $R_n = \frac{\log(n)}{n^2}$. He wants to check if the total risk $\sum_{n=1}^{\infty} R_n$. (5 marks)

b) A company is testing a new robotic arm that performs tasks each day. The robot improves rapidly, and the number of tasks it completes on day n is modeled by $T_n = \frac{n!}{7^n}$. Because the robot becomes faster every day, Use the Ratio Test to determine whether the total number of tasks done by the robot over all days is finite or infinite. (5 marks)

6) a) A reliability engineer monitors a sensor's daily bias over many days. On day n the bias (absolute value) is modeled by $B_n = \frac{1}{\log n}$ for $n \geq 2$, because improvements slow down as the system ages. The engineer wants to know whether the total accumulated bias $\sum_{n=2}^{\infty} B_n = \sum_{n=2}^{\infty} \frac{1}{\log n}$ remains finite. (5 marks)

b) A software company tests an AI bot that completes a certain number of tasks every hour. The number of tasks completed in hour n is modeled as $T_n = \frac{5^n}{2^n n}$. Use Ratio test the company wants to know whether the total tasks completed over infinite hours is a finite or infinite amount. (5 marks)

7) a) A data scientist is evaluating the total error produced by a machine-learning algorithm over many iterations. The error at the n-th iteration is modeled as $E_n = \frac{1}{n^2} + \frac{1}{n+1}$. Using comparison test the scientist wants to know whether the total accumulated error $\sum_{n=1}^{\infty} E_n = \sum_{n=1}^{\infty} \left(\frac{1}{n^2} + \frac{1}{n+1}\right)$ is finite. (5 marks)

b) A construction company uses an experimental robot. The number of bricks laid by the robot in hour n grows according to $B_n = \frac{n!}{10^n}$. The total bricks laid is represented by $\sum_{n=1}^{\infty} B_n = \sum_{n=1}^{\infty} \frac{n!}{10^n}$. Use the Ratio Test to determine whether the robot's total brick-laying capacity over time is bounded or whether it grows without limit. (5 marks)

Unit-4 Mathematical Modelling of Exponential Functions

Fitting of Exponential

1)A biotechnology company is studying bacterial growth over time. The population (in millions) at various times (in hours) is recorded as follows:

Time x (hours)	0	0.5	1	1.5	2	2.5
Population y (millions)	0.10	0.45	2.15	9.15	40.35	180.75

The scientists assume the growth follows an exponential law $y = ae^{bx}$.

a) Write the normal equations. (3 marks)
b) To determine a and b using the Least Squares Method. (4 marks)
c) Use the fitted model to estimate the bacterial population at x=3hours. (3 marks)

2) A pharmaceutical company is studying the decay of a drug concentration in the bloodstream over time (in hours). The measured drug concentrations (y) at different times (x) are shown below:

Time x (hours)	0	5	8	12	20
Drug concentration y (mg/L)	3.0	1.5	1.0	0.55	0.18

The scientists assume the decay follows an exponential law $y = ae^{bx}$.

a) Write the normal equations. (3 marks)
b) To determine a and b using the Least Squares Method. (4 marks)
c) Use the fitted model to estimate the drug concentration at x=15 hours. (3 marks)

3) A startup company is analyzing the growth of website traffic (in thousands of visits) over several days. The recorded traffic data is as follows:

Day x	0	1	2	3
Visits y (thousands)	1.05	2.10	3.85	8.30

The marketing team assumes the growth follows an exponential law $y = ae^{bx}$.

a) Write the normal equations. (3 marks)
b) To determine a and b using the Least Squares Method. (4 marks)
c) Use the fitted model to estimate the website traffic on day x=4. (3 marks)

4) A materials science lab is testing the growth of a chemical reaction product over time (in hours). The measured product yield (y) at different times (x) is recorded as follows:

Time x (hours)	0	2	4
Product yield y	5.012	10	31.62

The scientists assume the growth follows an exponential law $y = ae^{bx}$.

a) Write the normal equations. (3 marks)
b) To determine a and b using the Least Squares Method. (4 marks)
c) Use the fitted model to predict the product yield at x=6. (3 marks)

5) A small tech company is analyzing the adoption of its new app over several weeks. The number of active users (in thousands) is recorded as follows:

Week x	1	2	3	4
Active users y (thousands)	1.65	2.70	4.50	7.35

The marketing team assumes the growth follows an exponential law $y = ae^{bx}$.

a) Write the normal equations. (3 marks)
b) To determine a and b using the Least Squares Method. (4 marks)
c) Use the fitted model to estimate the number of active users in week 5. (3 marks)

Exponential growth and decay

6) A country currently has a population of 5 million people. The government observes that the population grows at a constant annual rate of 4%.

a) Identify the initial population P_0 and growth rate r. (2 marks)
b) Write the population growth by an exponential growth equation. (2 marks)
c) Using the exponential growth model, calculate the population of the country after 15 years. (3 marks)
d) calculate the population of the country after 24 years. (3 marks)

7) A catfish weighs 0.1 pound initially. During the next 8 weeks, its weight increases by 23% each week.

a) Identify the initial population P_0 and growth rate r. (2 marks)
b) Write the catfish's weight growth by an exponential growth equation. (2 marks)
c) Calculate the weight after 3 months. (3 marks)
d) Calculate the weight after 5 months. (3 marks)

8) The count in a culture was 400 after 2 hours and 25600 after 6 hours.

a) What is the relative growth of bacteria population. (3 marks)
b) Find the initial value of the culture. (1 mark)
c) Find the function that models the number of bacteria after t hours. (2 marks)
d) Find the number of bacteria after 4.5 hours. (2 marks)
e) When will the number of bacteria be 50000. (2 marks)

9) The inaugural attendance of an annual music festival is 150,000. The attendance y increases by 8% each year.

a) Identify initial attendance and growth rate. (2 marks)
b) Write an exponential growth function that represents the attendance after t years. (4 marks)
c) How many people will attend the festival in the fifth year? Round your answer to the nearest thousand. (4 marks)

10) Consider the population of bacteria described earlier. This population grows according to the function $f(t)=200e^{0.02t}$, where t is measured in minutes.

a) How many bacteria are present in the population after 5 hours? (5 marks)
b) when does the population reach 100000 bacteria? (5 marks)

Exponential series

11) a) Expand e^{2x} as an exponential series. (2 marks)
b) Write the general term of e^{2x}. (2 marks)
c) Find the coefficient of x^6 in this expansion. (3 marks)
d) Find the coefficient of x^{10} in this expansion. (3 marks)

12) a) Expand e^{-x} as an exponential series. (2 marks)
b) Write the general term of e^{-x}. (2 marks)
c) Find the coefficient of x^2 in this expansion. (3 marks)
d) Find the coefficient of x^5 in this expansion. (3 marks)

13) a) Write e^{3x} as an exponential series for the first four terms. (2 marks)
b) Write the general term of e^{3x}. (2 marks)
c) Find the coefficient of x^5 in this expansion. (2 marks)
d) Expand e^x as an exponential series. (2 marks)
e) Find the 7th term for the exponential series e^x. (2 marks)

Calculus of exponential functions – (i)Solving of exponential formula

14) a) A tech company is tracking the number of users on two of its servers. The user counts follow exponential growth based on the time x in days:
- Server A: 5^x users
- Server B: 5^{x+3} users

The IT manager wants to find the difference in users between Server A and Server B at any given day x. (5 marks)

b) A research lab is studying the intensity of a laser beam used in an experiment. The intensity increases according to powers of 4 depending on two different settings:
- Setting A produces an intensity of $(4)^3 * (4)^{x+5}$
- Setting B produces an intensity $(4)^{2x+12}$

To maintain safety, the researchers must operate the machine only when both settings produce the same intensity. Find the value of x at which the two laser intensities are equal. (5 marks)

(ii)Exponential Derivative and Integration

15) a) A chemical reaction produces a certain concentration of a substance over time. The concentration y (in mg/L) at time x (in hours) is modeled by $y = 5^{-2x^3}$. Scientists want to know how fast the concentration is changing at any moment. To find this, they need the derivative of y with respect to x. (5 marks)

b) A radioactive substance decays according to the rate $\frac{dy}{dt} = -5e^{-0.2t}$, t in hours, y in grams. Scientists want to find the total amount of substance after time t. Find y(t) by integrating the rate. (5 marks)

16) a) A bacteria population grows over time according to the model $y = -6e^{3x}$, where y represents the negative change in bacteria concentration (in millions per hour) at time x hours. Scientists want to know how quickly the concentration is changing at any moment, so they need the derivative of y. (5 marks)

b) A factory emits a chemical into a tank, and the rate of accumulation of the chemical (in kg per hour) at time x hours is modeled by $R(x)=e^{1-x}$. The engineers want to know the total amount of chemical accumulated in the tank between 1 hour and 2 hours. Compute the total accumulation using the definite integral. (5 marks)

<u>Unit-5 Mathematical Modelling of Multi Variable Fuctions</u>

Partial derivatives, Jacobian

1) a) Suppose you are analyzing the shape of a 3-D surface defined by $z(x, y) = x^3 + y^3 - 3axy$, where a is a constant. To understand how the surface bends and changes, you need to compute the following
   i) First-order partial derivatives. (2 marks)
   ii) Second-order partial derivatives. (2 marks)

b) A physics researcher is studying a system whose behavior depends only on the differences between three variables x, y, z. The potential energy of the system is u=(x−y) (y−z) (z−x). Since shifting all three variables by the same amount does not change the differences (and hence does not change the energy), you are asked to show that the function is translation-invariant. To test this, you must show that the sum of its partial derivatives is zero.(6 marks)

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2) Imagine you are analyzing a 3-variable function that remains symmetric even if you swap the variables. The function is $u = \log\left(x^3 + y^3 + z^3 - 3xyz\right)$. This expression appears in geometry, polynomial theory, and multivariable calculus. To understand how the function changes when all three variables rise together, you apply the operator $D = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$

a) Find the first derivative of u. (5 marks)

b) Show that the second derivative of u is $\frac{-9}{(x+y+z)^2}$. (5 marks)

3) a) A researcher is studying a model where two variables x and y represent output and input of a machine. The efficiency of the machine is defined as $u = x^y$. To check whether the efficiency behaves smoothly when both variables change, the researcher needs to verify that the mixed partial derivatives of u are equal, ie. $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$. (5 marks)

b) A space agency is designing navigation software for a new deep-space probe. The probe tracks the position of nearby asteroids using spherical coordinates $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$. For the conversion between spherical and Cartesian coordinates given above, compute the Jacobian $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$. (5 marks)

4) a) An engineer is analyzing the stability of a mechanical system where the performance index is defined as $u = \frac{y}{z} + \frac{z}{x}$. Each variable x, y, z represents a geometric dimension of the system. To check whether the system maintains scale-invariance (i.e., the performance does not change when all dimensions are scaled), the engineer must test the condition $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$, to verify this mathematically. (5 marks)

b) In a mechanical system, the coordinates (x, y) represent the horizontal and vertical displacement of a moving block. Engineers define two new quantities $u = xy$, $v = x^2$. To understand how small changes in displacement (x, y) affect these new quantities (u, v), they need to compute the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$. (5 marks)

5) a) A data scientist is analyzing a SoftMax-like function in machine learning, defined by $z = \log\left(e^x + e^y\right)$ which is used to compute probabilities. To understand the curvature of this function (for optimization or Hessian analysis), the scientist computes the second-order partial derivatives $r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$. (5 marks)

b) A physics researcher is studying the motion of a particle in a plane. The particle's rectangular coordinates (x, y) are expressed in polar form as $x = r\cos\theta$, $y = r\sin\theta$. Two important physical quantities are defined $u = 2xy$, $v = x^2$. The researcher needs to compute the Jacobian $\frac{\partial(u,v)}{\partial(r,\theta)}$. (5 marks)

6) A satellite imaging system scans the Earth's surface using two different coordinate systems. For ground analysis, engineers convert each point (r, θ) in polar coordinates (used by the satellite) into Cartesian coordinates (x, y) used by mapping software. The conversion is defined by $x = r\cos\theta$, $y = r\sin\theta$.

a) compute the Jacobian $\frac{\partial(x,y)}{\partial(r,\theta)}$. (5 marks)

b) Verify the rule $JJ' = 1$. (5 marks)

7) a) Write the Jacobian formula for u, v, w with respect to x, y, z. (3 marks)
   b) In a chemical processing unit, the concentrations of two interacting substances depend on temperature and pressure. Let x = temperature (in °C), y = pressure (in atm). Chemists define two new measurement variables $u = \frac{x^2}{y}, v = \frac{y^2}{x}$. To understand how temperature and pressure can be recovered from the experimentally measured values (u, v), the scientists need to compute the Jacobian of $\frac{\partial(x,y)}{\partial(u,v)}$. (7 marks)

Total Derivatives

8) A company manufactures three types of connected components x, y, and z for a machine. The efficiency u of the machine depends on the ratios of these components rather than their absolute amounts $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$. The manager wants to understand how changes in the individual component quantities affect the overall efficiency. Using the given function, prove that proportionally scaling all three components together. Mathematically, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$. (10 marks)

9) A research lab is studying temperature differences between three connected metal plates: Plate A, Plate B, and Plate C. Their temperatures are x, y, and z degrees Celsius respectively. The stability of the system is measured by a function u=f (x−y, y−z, z−x). This means the stability depends only on the temperature differences between the plates, not the actual temperatures. Mathematically, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$. (10 marks)

10) A factory uses a robotic arm to pick and place objects. The lifting force generated by the robot at any moment is given by z = xy, where x is the strength factor of the motor and y is an efficiency factor of the gears. Both depend on time t, $x = 2t^2$, y = sin t. The engineer wants to know how fast the lifting force is changing with time $\frac{dz}{dt}$ so she can adjust the control system. (10 marks)

With Constraints

11) A packaging company wants to design an open-top rectangular box to ship a new product. For each box, they can use only 108 cm² of cardboard. If the box has length = *l* cm, width = w cm, height = h cm. What should be the dimensions *l*, w, h of the open-top box so that its volume is maximized? (10 marks)

12) A materials scientist is analyzing a spherical crystal. The temperature at any point (x, y, z) inside the crystal is given by $T(x, y, z) = 400xyz^2$. Because the crystal is perfectly shaped, all points on its outer surface satisfy $x^2+y^2+z^2 = 1$.
   a) Find the points (x, y, z) on the sphere where this highest temperature occurs. (5 marks)
   b) Determine the highest temperature on the surface of the sphere. (5 marks)

13) A company wants to design an open-top rectangular box to hold a liquid product.
The box must have a volume of 32 cm³. Since the manufacturing cost depends on the amount of material used, the company wants the total surface area of the box to be as small as possible. If the box has length = *l* cm, width = w cm, height = h cm. Find the dimensions *l*, w, h of the open-top box. (10 marks)