INEXACT STOCHASTIC MIRROR DESCENT FOR TWO-STAGE NONLINEAR STOCHASTIC PROGRAMS

Vincent Guigues
School of Applied Mathematics, FGV
Praia de Botafogo, Rio de Janeiro, Brazil
vguigues@fgv.br

Abstract. We introduce an inexact variant of Stochastic Mirror Descent (SMD), called Inexact Stochastic Mirror Descent (ISMD), to solve nonlinear two-stage stochastic programs where the second stage problem has linear and nonlinear coupling constraints and a nonlinear objective function which depends on both first and second stage decisions. Given a candidate first stage solution and a realization of the second stage random vector, each iteration of ISMD combines a stochastic subgradient descent using a prox-mapping with the computation of approximate (instead of exact for SMD) primal and dual second stage solutions. We provide two convergence analysis of ISMD, under two sets of assumptions. The first convergence analysis is based on the formulas for inexact cuts of value functions of convex optimization problems shown recently in [6]. The second convergence analysis provides a convergence rate (the same as SMD) and relies on new formulas that we derive for inexact cuts of value functions of convex optimization problems assuming that the dual function of the second stage problem for all fixed first stage solution and realization of the second stage random vector, is strongly concave. We show that this assumption of strong concavity is satisfied for some classes of problems and present the results of numerical experiments on two simple two-stage problems which show that solving approximately the second stage problem for the first iterations of ISMD can help us obtain a good approximate first stage solution quicker than with SMD.

Keywords: Inexact cuts for value functions and Inexact Stochastic Mirror Descent and Strong Concavity of the dual function and Stochastic Programming.

AMS subject classifications: 90C15, 90C90.

1. Introduction

We are interested in inexact solution methods for two-stage nonlinear stochastic programs of form

$$\left\{ \begin{array}{l} \min \ f(x_1) := f_1(x_1) + \mathcal{Q}(x_1) \\ x_1 \in X_1 \end{array} \right. \tag{1.1}$$

with $X_1 \subset \mathbb{R}^n$ a convex, nonempty, and compact set, and $\mathcal{Q}(x_1) = \mathbb{E}_{\xi_2}[\mathfrak{Q}(x_1, \xi_2)]$ where $\mathbb{E}$ is the expectation operator, ξ_2 is a random vector with probability distribution P on $\Xi \subset \mathbb{R}^k$, and

$$\mathfrak{Q}(x_1, \xi_2) = \left\{ \begin{array}{l} \min_{x_2} \ f_2(x_2, x_1, \xi_2) \\ x_2 \in X_2(x_1, \xi_2) := \{x_2 \in \mathcal{X}_2 : \ Ax_2 + Bx_1 = b, \ g(x_2, x_1, \xi_2) \leq 0\}. \end{array} \right. \tag{1.2}$$

In the problem above vector ξ_2 contains in particular the random elements in matrices A, B, and vector b. Problem (1.1) is the first stage problem while problem (1.2) is the second stage problem which has abstract constraints ($x_2 \in \mathcal{X}_2$), and linear ($Ax_2 + Bx_1 = b$) and nonlinear ($g(x_2, x_1, \xi_2) \leq 0$) constraints both of which couple first stage decision x_1 and second stage decision x_2. Our solution methods are suited for the following framework:

a) first stage problem (1.1) is convex;
b) second stage problem (1.2) is convex, i.e., $\mathcal{X}_2$ is convex and for every $\xi_2 \in \Xi$ functions $f_2(\cdot, \cdot, \xi_2)$ and $g(\cdot, \cdot, \xi_2)$ are convex;
c) for every realization $\tilde{\xi}_2$ of ξ_2, the primal second stage problem obtained replacing ξ_2 by $\tilde{\xi}_2$ in (1.2) with optimal value $\mathfrak{Q}(x_1, \tilde{\xi}_2)$ and its dual (obtained dualizing coupling constraints) are solved approximately.

There is a large literature on solution methods for two-stage risk-neutral stochastic programs. Essentially, these methods can be cast in two categories: (A) decomposition methods based on sampling and cutting plane approximations of $\mathcal{Q}$ (which date back to [3],[8]) and their variants with regularization such as [17] and (B) Robust Stochastic Approximation [15] and its variants such as stochastic Primal-Dual subgradient methods [9], Stochastic Mirror Descent (SMD) [13], [10], or Multistep Stochastic Mirror Descent (MSMD) [5]. These methods have been extended to solve multistage problems, for instance Stochastic Dual Dynamic Programming [14], belonging to class (A), and recently Dynamic Stochastic Approximation [11], belonging to class (B).

However, for all these methods, it is assumed that second stage problems are solved exactly. This latter assumption is not satisfied when the second stage problem is nonlinear since in this setting only approximate solutions are available. On top of that, for the first iterations, we still have crude approximations of the first stage solution and it may be useful to solve inexactly, with less accuracy, the second stage problem for these iterations and to increase the accuracy of the second stage solutions computed when the algorithm progresses in order to decrease the overall computational bulk.

Therefore the objective of this paper is to fill a gap considering the situation when second stage problems are nonlinear and solved approximately (both primal and dual, see Assumption c) above). More precisely, to account for Assumption (c), as an extension of the methods from class (B) we derive an Inexact Stochastic Mirror Descent (ISMD) algorithm, designed to solve problems of form (1.1). This inexact solution method is based on an inexact black box for the objective in (1.1). To this end, we compute inexact cuts (affine lower bounding functions) for value function $\mathfrak{Q}(\cdot, \xi_2)$ in (1.2). For this analysis, we first need formulas for exact cuts (cuts based on exact primal and dual solutions). We had shown such formulas in [4, Lemma 2.1] using convex analysis tools, in particular standard calculus on normal and tangeant cones. We derive in Proposition 3.2 a proof for these formulas based purely on duality. This is an adaptation of the proof of the formulas we gave in [6, Proposition 2.7] for inexact cuts, considering exact solutions instead of inexact solutions. To our knowledge, the computation of inexact cuts for value functions has only been discussed in [6] so far (see Proposition 3.7). We propose in Section 3 new formulas for computing inexact cuts based in particular on the strong concavity of the dual function. In Section 2, we provide, for several classes of problems, conditions ensuring that the dual function of an optimization problem is strongly concave and give formulas for computing the corresponding constant of strong concavity when possible. It turns out that our results improve Theorem 10 in [19] (the only reference we are aware of on the strong concavity of the dual function) which proves the strong concavity of the dual function under stronger assumptions. The tools developped in Sections 2 and 3 allow us to build the inexact black boxes necessary for the Inexact Stochastic Mirror Descent (ISMD) algorithm and its convergence analysis presented in Section 4. Finally, in Section 5 we report the results of numerical tests comparing the performance of SMD and ISMD on two simple two-stage nonlinear stochastic programs.

Throughout the paper, we use the following notation:

- The domain $\text{dom}(f)$ of a function $f : X \to \bar{\mathbb{R}}$ is the set of points in X such that f is finite: $\text{dom}(f) = \{x \in X : -\infty < f(x) < +\infty\}$.
- The largest (resp. smallest) eigenvalue of a matrix Q having real-valued eigenvalues is denoted by $\lambda_{\max}(Q)$ (resp. $\lambda_{\min}(Q)$).
- The $\|\cdot\|_2$ of a matrix A is given by $\|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$.
- $\text{Diag}(x_1, x_2, \ldots, x_n)$ is the $n \times n$ diagonal matrix whose entry (i,i) is x_i.
- For a linear application $\mathcal{A}$, $\text{Ker}(\mathcal{A})$ is its kernel and $\text{Im}(\mathcal{A})$ its image.
- $\langle \cdot, \cdot, \rangle$ is the usual scalar product in $\mathbb{R}^n$: $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ which induces the norm $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$.
- Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be an extended real-valued function. The Fenchel conjugate f^* of f is the function given by $f^*(x^*) = \sup_{x\in\mathbb{R}^n} \langle x^*, x\rangle - f(x)$.
- For functions $f : X \to Y$ and $g : Y \to Z$, the function $g \circ f : X \to Z$ is the composition of functions g and f given by $(g \circ f)(x) = g(f(x))$ for every $x \in X$.

2. On the strong concavity of the dual function of an optimization problem

The study of the strong concavity of the dual function of an optimization problem on some set has applications in numerical optimization. For instance, the strong concavity of the dual function and the knowledge of the associated constant of strong concavity are used by the Drift-Plus-Penalty algorithm in [19] and by the (convergence proof of) Inexact SMD algorithm presented in Section 4 when inexact cuts are computed using Proposition 3.8.

The only paper we are aware of providing conditions ensuring this strong concavity property is [19]. In this section, we prove similar results under weaker assumptions and study an additional class of problems (quadratic with a quadratic constraint, see Proposition 2.8).

2.1. Preliminaries.

In what follows, $X \subset \mathbb{R}^n$ is a nonempty convex set.

Definition 2.1 (Strongly convex functions)**.** *Function $f : X \to \mathbb{R} \cup \{+\infty\}$ is strongly convex with constant of strong convexity $\alpha > 0$ with respect to norm $\|\cdot\|$ if for every $x, y \in$ dom(f) we have*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - \frac{\alpha t(1-t)}{2}\|y - x\|^2,$$

for all $0 \le t \le 1$.

We can deduce the following well known characterizations of strongly convex functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ (see for instance [7]):

Proposition 2.2. *(i) Function $f : X \to \mathbb{R} \cup \{+\infty\}$ is strongly convex with constant of strong convexity $\alpha > 0$ with respect to norm $\|\cdot\|$ if and only if for every $x, y \in$ dom(f) we have*

$$f(y) \ge f(x) + s^T(y - x) + \frac{\alpha}{2}\|y - x\|^2, \ \forall s \in \partial f(x).$$

(ii) Function $f : X \to \mathbb{R} \cup \{+\infty\}$ is strongly convex with constant of strong convexity $\alpha > 0$ with respect to norm $\|\cdot\|$ if and only if for every $x, y \in$ dom(f) we have

$$f(y) \ge f(x) + f'(x; y - x) + \frac{\alpha}{2}\|y - x\|^2,$$

where $f'(x; y - x)$ denotes the derivative of f at x in the direction $y - x$.

(iii) Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be differentiable. Then f is strongly convex with constant of strong convexity $\alpha > 0$ with respect to norm $\|\cdot\|$ if and only if for every $x, y \in$ dom(f) we have

$$(\nabla f(y) - \nabla f(x))^T(y - x) \ge \alpha\|y - x\|^2.$$

(iv) Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be twice differentiable. Then f is strongly convex on $X \subset \mathbb{R}^n$ with constant of strong convexity $\alpha > 0$ with respect to norm $\|\cdot\|$ if and only if for every $x \in$ dom(f) we have

$$h^T\nabla^2 f(x)h \ge \alpha\|h\|^2, \forall h \in \mathbb{R}^n.$$

Definition 2.3 (Strongly concave functions)**.** *$f : X \to \mathbb{R} \cup \{-\infty\}$ is strongly concave with constant of strong concavity $\alpha > 0$ with respect to norm $\|\cdot\|$ if and only if $-f$ is strongly convex with constant of strong convexity $\alpha > 0$ with respect to norm $\|\cdot\|$.*

The following propositions are immediate and will be used in the sequel:

Proposition 2.4. *If $f : X \to \mathbb{R} \cup \{+\infty\}$ is strongly convex with constant of strong convexity $\alpha > 0$ with respect to norm $\|\cdot\|$ and $\ell : \mathbb{R}^n \to \mathbb{R}$ is linear then $f + \ell$ is strongly convex on X with constant of strong convexity $\alpha > 0$ with respect to norm $\|\cdot\|$.*

Proposition 2.5. *Let $X \subset \mathbb{R}^m, Y \subset \mathbb{R}^n$, be two nonempty convex sets. Let $\mathcal{A} : X \to Y$ be a linear operator and let $f : Y \to \mathbb{R} \cup \{+\infty\}$ be a strongly convex function with constant of strong convexity $\alpha > 0$ with respect to a norm $\|\cdot\|_n$ on $\mathbb{R}^n$ induced by scalar product $\langle\cdot,\cdot\rangle_n$ on $\mathbb{R}^n$. Assume that Ker$(\mathcal{A}^* \circ \mathcal{A}) = \{0\}$. Then $g = f \circ \mathcal{A}$ is strongly convex on X with constant of strong convexity $\alpha\lambda_{\min}(\mathcal{A}^* \circ \mathcal{A})$ with respect to norm $\|\cdot\|_m$.*

Proof. For every $x, y \in X$, using Proposition 2.2-(ii) we have

$$f(\mathcal{A}(y)) \geq f(\mathcal{A}(x)) + f'(\mathcal{A}(x); \mathcal{A}(y-x)) + \frac{\alpha}{2}\|\mathcal{A}(y-x)\|_n^2$$

and since $g'(x; y-x) = f'(\mathcal{A}(x); \mathcal{A}(y-x))$, we get

$$g(y) \geq g(x) + g'(x; y-x) + \frac{1}{2}\alpha\lambda_{\min}(\mathcal{A}^* \circ \mathcal{A})\|y-x\|_m^2$$

with $\alpha\lambda_{\min}(\mathcal{A}^* \circ \mathcal{A}) > 0$ ($\lambda_{\min}(\mathcal{A}^* \circ \mathcal{A})$ is nonnegative because $\mathcal{A}^* \circ \mathcal{A}$ is self-adjoint and it cannot be zero because $\mathcal{A}^* \circ \mathcal{A}$ is nondegenerate). $\square$

In the rest of this section, we fix $\|\cdot\| = \|\cdot\|_2$ and provide, under some assumptions, the constant of strong concavity of the dual function of an optimization problem for this norm.[1]

2.2. **Problems with linear constraints.** Consider the optimization problem

$$\left\{\begin{array}{l} \inf f(x) \\ Ax \leq b \end{array}\right. \tag{2.3}$$

where $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $b \in \mathbb{R}^q$, and A is a $q \times n$ real matrix.

We will use the following known fact, see for instance [16]:

Proposition 2.6. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function. Then f^* is strongly convex with constant of strong convexity $\alpha > 0$ for norm $\|\cdot\|_2$ if and only if f is differentiable and ∇f is Lipschitz continuous with constant $1/\alpha$ for norm $\|\cdot\|_2$.*

Proposition 2.7. *Let θ be the dual function of* (2.3) *given by*

$$\theta(\lambda) = \inf_{x \in \mathbb{R}^n} \{f(x) + \lambda^T(Ax - b)\}, \tag{2.4}$$

for $\lambda \in \mathbb{R}^q$. Assume that the rows of matrix A are independent, that f is convex, differentiable, and ∇f is Lipschitz continuous with constant $L \geq 0$ with respect to norm $\|\cdot\|_2$. Then dual function θ is strongly concave on $\mathbb{R}^q$ with constant of strong concavity $\frac{\lambda_{\min}(AA^T)}{L}$ with respect to norm $\|\cdot\|_2$ on $\mathbb{R}^q$.

Proof. The dual function of (2.3) can be written

$$\begin{array}{rcl} \theta(\lambda) & = & \inf_{x \in \mathbb{R}^n} \{f(x) + \lambda^T(Ax - b)\} = -\lambda^T b - \sup_{x \in \mathbb{R}^n} \{-x^T A^T \lambda - f(x)\} \\ & = & -\lambda^T b - f^*(-A^T\lambda) \text{ by definition of } f^*. \end{array} \tag{2.5}$$

Since the rows of A are independent, matrix AA^T is invertible and $\mathrm{Ker}(AA^T) = \{0\}$. The result follows from the above representation of θ and Propositions 2.4, 2.5, and 2.6. $\square$

The strong concavity of the dual function of (2.3) was shown in Corollary 5 in [19] assuming that f is second-order continuously differentiable and strongly convex. Therefore Proposition 2.7 (whose proof is very short), which only assumes that f is convex, differentiable, and has Lipschitz continuous gradient, improves existing results (neither second-order differentiability nor strong convexity is required).

2.3. **Problems with quadratic objective and a quadratic constraint.** We now consider the following quadratically constrained quadratic optimization problem

$$\left\{\begin{array}{l} \inf_{x \in \mathbb{R}^n} f(x) := \frac{1}{2}x^T Q_0 x + a_0^T x + b_0 \\ g_1(x) := \frac{1}{2}x^T Q_1 x + a_1^T x + b_1 \leq 0, \end{array}\right. \tag{2.6}$$

with Q_0 positive definite and Q_1, positive semidefinite. The dual function θ of this problem is known in closed-form: for $\mu \geq 0$, we have

$$\theta(\mu) = \inf_{x \in \mathbb{R}^n} \{f(x) + \mu g_1(x)\} = -\frac{1}{2}\mathcal{A}(\mu)^T \mathcal{Q}(\mu)^{-1}\mathcal{A}(\mu) + \mathcal{B}(\mu) \tag{2.7}$$

where

$$\mathcal{A}(\mu) = a_0 + \mu a_1, \ \mathcal{Q}(\mu) = Q_0 + \mu Q_1, \ \text{ and } \mathcal{B}(\mu) = b_0 + \mu_i b_1.$$

[1]Using the equivalence between norms in $\mathbb{R}^n$, we can derive a valid constant of strong concavity for other norms, for instance $\|\cdot\|_\infty$ and $\|\cdot\|_1$.

We can show, under some assumptions, that dual function θ is strongly concave on some set and compute analytically the corresponding constant of strong concavity:

Proposition 2.8. *Consider optimization problem* (2.6)*. Assume that Q_0, Q_1, are positive definite, that there exists x_0 such that $g_1(x_0) < 0$, and that $a_0 \neq Q_0 Q_1^{-1} a_1$. Let $\mathcal{L}$ be any lower bound on the optimal value of* (2.6) *and let $\bar{\mu} = (\mathcal{L} - f(x_0))/g_1(x_0) \geq 0$. Then the optimal solution of the dual problem*

$$\max_{\mu \geq 0} \theta(\mu)$$

is contained in the interval $[0, \bar{\mu}]$ and the dual function θ given by (2.7) *is strongly concave on the interval $[0, \bar{\mu}]$ with constant of strong concavity $\alpha_D = (Q_1^{-1/2}(a_0 - Q_0 Q_1^{-1} a_1))^T (Q_1^{-1/2} Q_0 Q_1^{-1/2} + \bar{\mu} I_n)^{-3} Q_1^{-1/2}(a_0 - Q_0 Q_1^{-1} a_1) > 0$.*

Proof. Making the change of variable $x = y - Q_1^{-1} a_1$, we can rewrite (2.6) without linear terms in g_1 under the form:

$$\left\{ \begin{array}{l} \inf_{x \in \mathbb{R}^n} \frac{1}{2} x^T Q_0 x + (a_0 - Q_0 Q_1^{-1} a_1)^T x + b_0 + \frac{1}{2} a_1^T Q_1^{-1} Q_0 Q_1^{-1} a_1 - a_0^T Q_1^{-1} a_1 \\ \frac{1}{2} x^T Q_1 x + b_1 - \frac{1}{2} a_1^T Q_1^{-1} a_1 \leq 0, \end{array} \right.$$

with corresponding dual function given by

$$\theta(\mu) = -\frac{1}{2} \bar{a}_0^T (Q_0 + \mu Q_1)^{-1} \bar{a}_0 + (b_1 - \frac{1}{2} a_1^T Q_1^{-1} a_1)\mu + b_0 - a_0^T Q_1^{-1} a_1 + \frac{1}{2} a_1^T Q_1^{-1} Q_0 Q_1^{-1} a_1$$

where we have set $\bar{a}_0 = a_0 - Q_0 Q_1^{-1} a_1$ (see (2.7)).

Using [7, Remark 2.3.3, p.313] we obtain that the optimal dual solutions are contained in the interval $[0, \bar{\mu}]$. Setting $\tilde{a}_0 = Q_1^{-1/2} \bar{a}_0$ and $A = Q_1^{-1/2} Q_0 Q_1^{-1/2}$, we compute the first and second derivatives of the nonlinear term $\theta_q(\mu) = -\frac{1}{2} \bar{a}_0^T (Q_0 + \mu Q_1)^{-1} \bar{a}_0 = -\frac{1}{2} \tilde{a}_0^T (A + \mu I_n)^{-1} \tilde{a}_0$ of θ on $[0, \bar{\mu}]$:

$$\theta_q'(\mu) = \tfrac{1}{2} \tilde{a}_0^T (A + \mu I_n)^{-2} \tilde{a}_0 \ \text{ and } \theta_q''(\mu) = -\tilde{a}_0^T (A + \mu I_n)^{-3} \tilde{a}_0.$$

For these computations we have used the fact that for $\mathcal{F} : \mathcal{I} \to \mathrm{GL}_n(\mathbb{R})$ differentiable on $\mathcal{I} \subset \mathbb{R}$, we have $\frac{d\mathcal{F}(t)^{-1}}{dt} = -\mathcal{F}(t)^{-1} \frac{d\mathcal{F}(t)}{dt} \mathcal{F}(t)^{-1}$. Since $-\theta_q''(\mu)$ is decreasing on $[0, \bar{\mu}]$, we get $-\theta_q''(\mu) \geq \alpha_D = -\theta_q''(\bar{\mu})$ on $[0, \bar{\mu}]$. This computation, together with Proposition 2.2-(iv), shows that θ is strongly concave on $[0, \bar{\mu}]$ with constant of strong concavity α_D. □

2.4. **General case: problems with linear and nonlinear constraints.** Let us add to problem (2.3) nonlinear constraints. More precisely, given $f : \mathbb{R}^n \to \mathbb{R}$, a $q \times n$ real matrix A, $b \in \mathbb{R}^q$, and $g : \mathbb{R}^n \to \mathbb{R}^p$ with convex component functions $g_i, i = 1, \ldots, p$, we consider the optimization problem

$$\left\{ \begin{array}{l} \inf f(x) \\ x \in X, Ax \leq b, g(x) \leq 0. \end{array} \right. \tag{2.8}$$

Let v be the value function of this problem given by

$$v(c) = v(c_1, c_2) = \left\{ \begin{array}{l} \inf f(x) \\ x \in X, Ax - b + c_1 \leq 0, g(x) + c_2 \leq 0, \end{array} \right. \tag{2.9}$$

for $c_1 \in \mathbb{R}^q, c_2 \in \mathbb{R}^p$. In the next lemma, we relate the conjugate of v to the dual function

$$\theta(\lambda, \mu) = \left\{ \begin{array}{l} \inf f(x) + \lambda^T (Ax - b) + \mu^T g(x) \\ x \in X, \end{array} \right.$$

of this problem:

Lemma 2.9. *If v^* is the conjugate of the value function v then $v^*(\lambda, \mu) = -\theta(\lambda, \mu)$ for every $(\lambda, \mu) \in \mathbb{R}_+^q \times \mathbb{R}_+^p$.*

Proof. For $(\lambda, \mu) \in \mathbb{R}_+^q \times \mathbb{R}_+^p$, we have

$$\begin{array}{rcl} -v^*(\lambda,\mu) & = & -\sup\limits_{(c_1,c_2)\in\mathbb{R}^q\times\mathbb{R}^p} \lambda^T c_1 + \mu^T c_2 - v(c_1,c_2) \\ & = & \left\{ \begin{array}{l} \inf -\lambda^T c_1 - \mu^T c_2 + f(x) \\ x \in X, Ax - b + c_1 \leq 0, g(x) + c_2 \leq 0, \\ c_1 \in \mathbb{R}^q, c_2 \in \mathbb{R}^p, \end{array} \right. \\ & = & \left\{ \begin{array}{l} \inf f(x) + \lambda^T(Ax-b) + \mu^T g(x) \\ x \in X, \end{array} \right. \\ & = & \theta(\lambda,\mu). \end{array}$$

□

From Lemma 2.9 and Proposition 2.6, we obtain that dual function θ of problem (2.8) is strongly concave with constant α with respect to norm $\|\cdot\|_2$ on $\mathbb{R}^{p+q}$ if and only if the value function v given by (2.9) is differentiable and ∇v is Lipschitz continuous with constant $1/\alpha$ with respect to norm $\|\cdot\|_2$ on $\mathbb{R}^{p+q}$. Using Lemma 2.1 in [4] the subdifferential of the value function is the set of optimal dual solutions of (2.9). Therefore θ is strongly concave with constant α with respect to norm $\|\cdot\|_2$ on $\mathbb{R}^{p+q}$ if and only if the value function is differentiable and the dual solution of (2.9) seen as a function of (c_1, c_2) is Lipschitz continuous with Lipschitz constant $1/\alpha$ with respect to norm $\|\cdot\|_2$ on $\mathbb{R}^{p+q}$.

We now provide conditions ensuring that the dual function is strongly concave in a neighborhood of the optimal dual solution.

Theorem 2.10. *Consider the optimization problem*

$$\inf_{x\in\mathbb{R}^n} \{f(x) : Ax \leq b, g_i(x) \leq 0, i = 1, \ldots, p\}. \tag{2.10}$$

We assume that

- (A1) $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *is strongly convex and has Lipschitz continuous gradient;*
- (A2) $g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, i = 1, \ldots, p,$ *are convex and have Lipschitz continuous gradients;*
- (A3) *if* x_* *is the optimal solution of* (2.10) *then the rows of matrix* $\begin{pmatrix} A \\ J_g(x_*) \end{pmatrix}$ *are linearly independent where* $J_g(x)$ *denotes the Jacobian matrix of* $g(x) = (g_1(x), \ldots, g_p(x))$ *at* x*;*
- (A4) *there is* $x_0 \in ri(\{g \leq 0\})$ *such that* $Ax_0 \leq b$.

Let θ *be the dual function of this problem:*

$$\theta(\lambda,\mu) = \left\{ \begin{array}{l} \inf\ f(x) + \lambda^T(Ax-b) + \mu^T g(x) \\ x \in \mathbb{R}^n. \end{array} \right. \tag{2.11}$$

Let $(\lambda_*, \mu_*) \geq 0$ *be an optimal solution of the dual problem*

$$\sup_{\lambda\geq 0,\mu\geq 0} \theta(\lambda,\mu).$$

Then there is some neighborhood $\mathcal{N}$ *of* (λ_*, μ_*) *such that* θ *is strongly concave on* $\mathcal{N} \cap \mathbb{R}_+^{p+q}$.

Proof. Due to (A1) the optimization problem (2.11) has a unique optimal solution that we denote by $x(\lambda, \mu)$. Assumptions (A2) and (A3) imply that there is some neighborhood $\mathcal{V}_\varepsilon(x_*) = \{x \in \mathbb{R}^n : \|x - x_*\|_2 \leq \varepsilon\}$ of x_* for some $\varepsilon > 0$ such that the rows of matrix $\begin{pmatrix} A \\ J_g(x) \end{pmatrix}$ are independent for x in $\mathcal{V}_\varepsilon(x_*)$.

We argue that $(\lambda, \mu) \to x(\lambda, \mu)$ is continuous on $\mathbb{R}^q \times \mathbb{R}^p$. Indeed, let $(\bar\lambda, \bar\mu) \in \mathbb{R}^q \times \mathbb{R}^p$ and take a sequence (λ_k, μ_k) converging to $(\bar\lambda, \bar\mu)$. We want to show that $x(\lambda_k, \mu_k)$ converges to $x(\bar\lambda, \bar\mu)$. Take an arbitrary accumulation point $\bar x$ of the sequence $x(\lambda_k, \mu_k)$, i.e., $\bar x = \lim_{k\to+\infty} x(\lambda_{\sigma(k)}, \mu_{\sigma(k)})$ for some subsequence $x(\lambda_{\sigma(k)}, \mu_{\sigma(k)})$ of $x(\lambda_k, \mu_k)$. Then by definition of $x(\lambda_{\sigma(k)}, \mu_{\sigma(k)})$, for every $x \in \mathbb{R}^n$ and every $k \geq 1$ we have

$$f(x(\lambda_{\sigma(k)}, \mu_{\sigma(k)})) + \lambda_{\sigma(k)}^T(Ax(\lambda_{\sigma(k)}, \mu_{\sigma(k)}) - b) + \mu_{\sigma(k)}^T g(x(\lambda_{\sigma(k)}, \mu_{\sigma(k)})) \leq f(x) + \lambda_{\sigma(k)}^T(Ax - b) + \mu_{\sigma(k)}^T g(x).$$

Passing to the limit in the inequality above and using the continuity of f and g_i we obtain for all $x \in \mathbb{R}^n$:

$$f(\bar x) + \bar\lambda^T(A\bar x - b) + \bar\mu^T g(\bar x) \leq f(x) + \bar\lambda^T(Ax - b) + \bar\mu^T g(x),$$

which shows that $\bar{x} = x(\bar{\lambda}, \bar{\mu})$. Therefore there is only one accumuation point $\bar{x} = x(\bar{\lambda}, \bar{\mu})$ for the sequence $x(\lambda_k, \mu_k)$ which shows that this sequence converges to $x(\bar{\lambda}, \bar{\mu})$. Consequently, we have shown that $(\lambda, \mu) \to x(\lambda, \mu)$ is continuous on $\mathbb{R}^q \times \mathbb{R}^p$. This implies that there is a neighborhood $\mathcal{N}(\lambda_*, \mu_*)$ of (λ_*, μ_*) such that for $(\lambda, \mu) \in \mathcal{N}(\lambda_*, \mu_*)$ we have $\|x(\lambda, \mu) - x(\lambda_*, \mu_*)\|_2 \leq \varepsilon$. Moreover, due to (A4), we have $x(\lambda_*, \mu_*) = x_*$. It follows that for $(\lambda, \mu) \in \mathcal{N}(\lambda_*, \mu_*)$ we have $\|x(\lambda, \mu) - x(\lambda_*, \mu_*)\|_2 = \|x(\lambda, \mu) - x_*\|_2 \leq \varepsilon$ which in turn implies that the rows of matrix $\begin{pmatrix} A \\ J_g(x(\lambda, \mu)) \end{pmatrix}$ are independent. We now show that θ is strongly concave on $\mathcal{N}(\lambda_*, \mu_*) \cap \mathbb{R}_+^{p+q}$.

Take (λ_1, μ_1), (λ_2, μ_2) in $\mathcal{N}(\lambda_*, \mu_*) \cap \mathbb{R}_+^{p+q}$ and denote $x_1 = x(\lambda_1, \mu_1)$ and $x_2 = x(\lambda_2, \mu_2)$. The optimality conditions give

$$\begin{array}{c} \nabla f(x_1) + A^T \lambda_1 + J_g(x_1)^T \mu_1 = 0, \\ \nabla f(x_2) + A^T \lambda_2 + J_g(x_2)^T \mu_2 = 0. \end{array} \tag{2.12}$$

Recall that (2.11) has a unique solution and therefore θ is differentiable. The gradient of θ is given by (see for instance Lemma 2.1 in [4])

$$\nabla \theta(\lambda, \mu) = \begin{pmatrix} Ax(\lambda, \mu) - b \\ g(x(\lambda, \mu)) \end{pmatrix}$$

and we obtain, using the notation $\langle x, y \rangle = x^T y$:

$$-\left\langle \nabla \theta(\lambda_2, \mu_2) - \nabla \theta(\lambda_1, \mu_1), \begin{pmatrix} \lambda_2 - \lambda_1 \\ \mu_2 - \mu_1 \end{pmatrix} \right\rangle = -\langle A(x_2 - x_1), \lambda_2 - \lambda_1 \rangle - \langle g(x_2) - g(x_1), \mu_2 - \mu_1 \rangle. \tag{2.13}$$

By convexity of constraint functions we can write for $i = 1, \ldots, p$:

$$\begin{array}{ll} g_i(x_2) \geq g_i(x_1) + \langle \nabla g_i(x_1), x_2 - x_1 \rangle & (a) \\ g_i(x_1) \geq g_i(x_2) + \langle \nabla g_i(x_2), x_1 - x_2 \rangle. & (b) \end{array} \tag{2.14}$$

Multiplying (2.14)-(a) by $\mu_1(i) \geq 0$ and (2.14)-(b) by $\mu_2(i) \geq 0$ we obtain

$$-\langle g(x_2) - g(x_1), \mu_2 - \mu_1 \rangle \geq \langle J_g(x_1)^T \mu_1 - J_g(x_2)^T \mu_2, x_2 - x_1 \rangle. \tag{2.15}$$

Recalling (A1), we can find $0 \leq L(f) < +\infty$ such that for all $x, y \in \mathbb{R}^n$:

$$\|\nabla f(y) - \nabla f(x)\|_2 \leq L(f) \|y - x\|_2. \tag{2.16}$$

Using (2.13) and (2.15) and denoting by $\alpha > 0$ the constant of strong convexity of f with respect to norm $\|\cdot\|_2$ we get:

(2.17)

$$\begin{array}{lll} -\left\langle \nabla \theta(\lambda_2, \mu_2) - \nabla \theta(\lambda_1, \mu_1), \begin{pmatrix} \lambda_2 - \lambda_1 \\ \mu_2 - \mu_1 \end{pmatrix} \right\rangle & \geq & -\langle x_2 - x_1, A^T(\lambda_2 - \lambda_1) \rangle + \langle J_g(x_1)^T \mu_1 - J_g(x_2)^T \mu_2, x_2 - x_1 \rangle, \\ & \stackrel{(2.12)}{=} & \langle x_2 - x_1, \nabla f(x_2) - \nabla f(x_1) \rangle \\ & \geq & \alpha \|x_2 - x_1\|_2^2 \text{ by strong convexity of } f, \\ & \geq & \frac{\alpha}{L(f)^2} \|\nabla f(x_2) - \nabla f(x_1)\|_2^2 \text{ using (2.16)}, \\ & \stackrel{(2.12)}{=} & \frac{\alpha}{L(f)^2} \| \underbrace{\begin{pmatrix} A^T \ J_g(x_2)^T \end{pmatrix} \begin{pmatrix} \lambda_2 - \lambda_1 \\ \mu_2 - \mu_1 \end{pmatrix}}_{a} + \underbrace{(J_g(x_2) - J_g(x_1))^T \mu_1}_{b} \|_2^2. \end{array}$$

Now recall that for every $x \in \mathcal{V}_\varepsilon(x_*)$ the rows of the matrix $\begin{pmatrix} A \\ J_g(x) \end{pmatrix}$ are independent and therefore the matrix $\begin{pmatrix} A \\ J_g(x) \end{pmatrix} \begin{pmatrix} A \\ J_g(x) \end{pmatrix}^T$ is invertible. Moreover, the function $x \to \lambda_{\min} \left(\begin{pmatrix} A \\ J_g(x) \end{pmatrix} \begin{pmatrix} A \\ J_g(x) \end{pmatrix}^T \right)$ is continuous (due to (A2)) and positive on the compact set $\mathcal{V}_\varepsilon(x_*)$. It follows that we can define

$$\underline{\lambda}_\varepsilon(x_*) = \min_{x \in \mathcal{V}_\varepsilon(x_*)} \lambda_{\min} \left(\begin{pmatrix} A \\ J_g(x) \end{pmatrix} \begin{pmatrix} A \\ J_g(x) \end{pmatrix}^T \right),$$

and $\underline{\lambda}_\varepsilon(x_*) > 0$. Since $x_2 \in \mathcal{V}_\varepsilon(x_*)$, we deduce that

$$\|a\|_2 \geq \sqrt{\underline{\lambda}_\varepsilon(x_*)} \left\| \begin{pmatrix} \lambda_2 - \lambda_1 \\ \mu_2 - \mu_1 \end{pmatrix} \right\|_2. \tag{2.18}$$

Recalling that (λ_1, μ_1) is in $\mathcal{N}(\lambda_*, \mu_*)$, there is $\eta > 0$ such that

$$\|\mu_1\|_1 \leq U_\eta(\mu_*) := \|\mu_*\|_1 + \eta. \tag{2.19}$$

Due to (A2), there is $L(g) \geq 0$ such that for every $x, y \in \mathbb{R}^n$, we have

$$\|\nabla g_i(y) - \nabla g_i(x)\|_2 \leq L(g)\|y - x\|_2, 1, \ldots, p.$$

Combining this relation with (2.19), we get

$$\|b\|_2 \leq \|\mu_1\|_1 L(g)\|x_2 - x_1\|_2 \leq L(g)U_\eta(\mu_*)\|x_2 - x_1\|_2. \tag{2.20}$$

Therefore

$$\|a + b\|_2 \geq \|a\|_2 - \|b\|_2 \geq \sqrt{\underline{\lambda}_\varepsilon(x_*)} \left\| \begin{pmatrix} \lambda_2 - \lambda_1 \\ \mu_2 - \mu_1 \end{pmatrix} \right\|_2 - L(g)U_\eta(\mu_*)\|x_2 - x_1\|_2$$

and combining this relation with (2.17) we obtain

$$\|x_2 - x_1\|_2 \geq \frac{1}{L(f)} \left[\sqrt{\underline{\lambda}_\varepsilon(x_*)} \left\| \begin{pmatrix} \lambda_2 - \lambda_1 \\ \mu_2 - \mu_1 \end{pmatrix} \right\|_2 - L(g)U_\eta(\mu_*)\|x_2 - x_1\|_2 \right]$$

which gives

$$\|x_2 - x_1\|_2 \geq \frac{\sqrt{\underline{\lambda}_\varepsilon(x_*)}}{L(f) + L(g)U_\eta(\mu_*)} \left\| \begin{pmatrix} \lambda_2 - \lambda_1 \\ \mu_2 - \mu_1 \end{pmatrix} \right\|_2. \tag{2.21}$$

Plugging (2.21) into (2.17) we get

$$-\left\langle \nabla\theta(\lambda_2, \mu_2) - \nabla\theta(\lambda_1, \mu_1), \begin{pmatrix} \lambda_2 - \lambda_1 \\ \mu_2 - \mu_1 \end{pmatrix} \right\rangle \quad \geq \quad \tfrac{\alpha \underline{\lambda}_\varepsilon(x_*)}{(L(f)+L(g)U_\eta(\mu_*))^2} \left\| \begin{pmatrix} \lambda_2 - \lambda_1 \\ \mu_2 - \mu_1 \end{pmatrix} \right\|_2^2.$$

Using Proposition 2.2-(iii), the relation above shows that θ is strongly concave on $\mathcal{N}(\lambda_*, \mu_*) \cap \mathbb{R}_+^{p+q}$ with constant of strong concavity $\frac{\alpha \underline{\lambda}_\varepsilon(x_*)}{(L(f)+L(g)U_\eta(\mu_*))^2}$ with respect to norm $\|\cdot\|_2$. □

The local strong concavity of the dual function of (2.10) was shown recently in Theorem 10 in [19] assuming (A3), assuming instead of (A1) that f is strongly convex and second-order continuously differentiable (which is stronger than (A1)), and assuming instead of (A2) that $g_i, i = 1, \ldots, p$, are convex second-order continuously differentiable, which is stronger than (A2).[2] Therefore Theorem 2.10 gives a new proof of the local strong concavity of the dual function and improves existing results.

3. Computing inexact cuts for value functions of convex optimization problems

3.1. **Preliminaries.** Let $\mathcal{Q} : X \to \mathbb{R} \cup \{+\infty\}$ be the value function given by

$$\mathcal{Q}(x) = \left\{ \begin{array}{l} \inf_{y \in \mathbb{R}^n} \; f(y, x) \\ y \in S(x) := \{y \in Y \;:\; Ay + Bx = b, \; g(y, x) \leq 0\}. \end{array} \right. \tag{3.22}$$

Here, and in all this section, $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ are nonempty, compact, and convex sets, and A and B are respectively $q \times n$ and $q \times m$ real matrices. We will make the following assumptions:[3]

- (H1) $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, proper, and convex.
- (H2) For $i = 1, \ldots, p$, the i-th component of function $g(y, x)$ is a convex lower semicontinuous function $g_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$.

In what follows, we say that $\mathcal{C}$ is a cut for $\mathcal{Q}$ on X if $\mathcal{C}$ is an affine function of x such that $\mathcal{Q}(x) \geq \mathcal{C}(x)$ for all $x \in X$. We say that the cut is exact at $\bar{x}$ if $\mathcal{Q}(\bar{x}) = \mathcal{C}(\bar{x})$. Otherwise, the cut is said to be inexact at $\bar{x}$.

In this section, our basic goal is, given $\bar{x} \in X$ and ε-optimal primal and dual solutions of (3.22) written for $x = \bar{x}$, to derive an inexact cut $\mathcal{C}(x)$ for $\mathcal{Q}$ at $\bar{x}$, i.e., an affine lower bounding function for $\mathcal{Q}$ such that the distance $\mathcal{Q}(\bar{x}) - \mathcal{C}(\bar{x})$ between the values of $\mathcal{Q}$ and of the cut at $\bar{x}$ is bounded from above by a known function of the problem parameters. Of course, when $\varepsilon = 0$, we will check that $\mathcal{Q}(\bar{x}) = \mathcal{C}(\bar{x})$.

[2] Note that we used (A4) to ensure that $x(\lambda_*, \mu_*) = x_*$, which is also used in the proof of Theorem 10 in [19].

[3] Note that (H1) and (H2) imply the convexity of $\mathcal{Q}$ given by (3.22). Indeed, let $x_1, x_2 \in X$, $0 \leq t \leq 1$, and $y_1 \in S(x_1), y_2 \in S(x_2)$, such that $\mathcal{Q}(x_1) = f(y_1, x_1)$ and $\mathcal{Q}(x_2) = f(y_2, x_2)$. By convexity of g and Y, we have that have $ty_1 + (1-t)y_2 \in S(tx_1 + (1-t)x_2)$ and therefore $\mathcal{Q}(tx_1 + (1-t)x_2) \leq f(ty_1 + (1-t)y_2, tx_1 + (1-t)x_2) \leq tf(y_1, x_1) + (1-t)f(y_2, x_2) = t\mathcal{Q}(x_1) + (1-t)\mathcal{Q}(x_2)$ where for the last inequality we have used the convexity of f.

We first provide in Proposition 3.2 below a characterization of the subdifferential of value function $\mathcal{Q}$ at $\bar{x} \in X$ when optimal primal and dual solutions for (3.22) written for $x = \bar{x}$ are available (computation of exact cuts).

Consider for problem (3.22) the Lagrangian dual problem

$$\sup_{(\lambda,\mu) \in \mathbb{R}^q \times \mathbb{R}_+^p} \theta_x(\lambda, \mu) \tag{3.23}$$

for the dual function

$$\theta_x(\lambda, \mu) = \inf_{y \in Y} L_x(y, \lambda, \mu) \tag{3.24}$$

where

$$L_x(y, \lambda, \mu) = f(y, x) + \lambda^T (Ay + Bx - b) + \mu^T g(y, x).$$

We denote by $\Lambda(x)$ the set of optimal solutions of the dual problem (3.23) and we use the notation

$$\mathrm{Sol}(x) := \{y \in S(x) : f(y, x) = \mathcal{Q}(x)\}$$

to indicate the solution set to (3.22).

Lemma 3.1 (Lemma 2.1 in [4])**.** *Consider the value function $\mathcal{Q}$ given by (3.22) and take $\bar{x} \in X$ such that $S(\bar{x}) \neq \emptyset$. Let Assumptions (H1) and (H2) hold and assume the Slater-type constraint qualification condition:*

$$\textit{there exists } (x_*, y_*) \in X \times \mathrm{ri}(Y) \textit{ such that } Ay_* + Bx_* = b \textit{ and } (y_*, x_*) \in \mathrm{ri}(\{g \leq 0\}).$$

Then $s \in \partial \mathcal{Q}(\bar{x})$ if and only if

$$\begin{aligned} (0, s) \in \partial f(\bar{y}, \bar{x}) &+ \Big\{ [A^T; B^T]\lambda \ : \ \lambda \in \mathbb{R}^q \Big\} \\ &+ \Big\{ \sum_{i \in I(\bar{y}, \bar{x})} \mu_i \partial g_i(\bar{y}, \bar{x}) \ : \ \mu_i \geq 0 \Big\} + \mathcal{N}_Y(\bar{y}) \times \{0\}, \end{aligned} \tag{3.25}$$

where $\bar{y}$ is any element in the solution set Sol($\bar{x}$) and with

$$I(\bar{y}, \bar{x}) = \Big\{ i \in \{1, \ldots, p\} \ : \ g_i(\bar{y}, \bar{x}) = 0 \Big\}.$$

In particular, if f and g are differentiable, then

$$\partial \mathcal{Q}(\bar{x}) = \Big\{ \nabla_x f(\bar{y}, \bar{x}) + B^T \lambda + \sum_{i \in I(\bar{y}, \bar{x})} \mu_i \nabla_x g_i(\bar{y}, \bar{x}) \ : \ (\lambda, \mu) \in \Lambda(\bar{x}) \Big\}. \tag{3.26}$$

The proof of Lemma 3.1 is given in [4] using calculus on normal and tangeant cones. In Proposition 3.2 below, we show how to obtain an exact cut for $\mathcal{Q}$ at $\bar{x} \in X$ using convex duality when f and g are differentiable.

Proposition 3.2. *Consider the value function $\mathcal{Q}$ given by (3.22) and take $\bar{x} \in X$ such that $S(\bar{x}) \neq \emptyset$. Let Assumptions (H1) and (H2) hold and assume the following constraint qualification condition: there exists $y_0 \in ri(Y) \cap ri(\{g(\cdot, \bar{x}) \leq 0\})$ such that $Ay_0 + B\bar{x} = b$. Assume that f and g are differentiable on $Y \times X$. Let $(\bar{\lambda}, \bar{\mu})$ be an optimal solution of dual problem (3.23) written with $x = \bar{x}$ and let*

$$s(\bar{x}) = \nabla_x f(\bar{y}, \bar{x}) + B^T \bar{\lambda} + \sum_{i \in I(\bar{y}, \bar{x})} \bar{\mu}_i \nabla_x g_i(\bar{y}, \bar{x}), \tag{3.27}$$

where $\bar{y}$ is any element in the solution set Sol($\bar{x}$) and with

$$I(\bar{y}, \bar{x}) = \Big\{ i \in \{1, \ldots, p\} \ : \ g_i(\bar{y}, \bar{x}) = 0 \Big\}.$$

Then $s(\bar{x}) \in \partial \mathcal{Q}(\bar{x})$.

Proof. The constraint qualification condition implies that there is no duality gap and therefore

$$f(\bar{y}, \bar{x}) = \mathcal{Q}(\bar{x}) = \theta_{\bar{x}}(\bar{\lambda}, \bar{\mu}). \tag{3.28}$$

Moreover, $\bar{y}$ is an optimal solution of $\inf\{L_{\bar{x}}(y, \bar{\lambda}, \bar{\mu}) : y \in Y\}$ which gives

$$\langle \nabla_y L_{\bar{x}}(\bar{y}, \bar{\lambda}, \bar{\mu}), y - \bar{y} \rangle \geq 0 \ \forall y \in Y,$$

and therefore

$$\min_{y\in Y}\langle \nabla_y L_{\bar x}(\bar y,\bar\lambda,\bar\mu), y-\bar y\rangle = 0. \tag{3.29}$$

Using the convexity of the function which associates to (x,y) the value $L_x(y,\bar\lambda,\bar\mu)$ we obtain for every $x\in X$ and $y\in Y$ that

$$L_x(y,\bar\lambda,\bar\mu) \geq L_{\bar x}(\bar y,\bar\lambda,\bar\mu) + \langle \nabla_x L_{\bar x}(\bar y,\bar\lambda,\bar\mu), x-\bar x\rangle + \langle \nabla_y L_{\bar x}(\bar y,\bar\lambda,\bar\mu), y-\bar y\rangle. \tag{3.30}$$

By definition of θ_x, for any $x\in X$ we get

$$\mathcal{Q}(x) \geq \theta_x(\bar\lambda,\bar\mu)$$

which combined with (3.30) gives

$$\begin{array}{lll}\mathcal{Q}(x) & \geq & L_{\bar x}(\bar y,\bar\lambda,\bar\mu) + \langle \nabla_x L_{\bar x}(\bar y,\bar\lambda,\bar\mu), x-\bar x\rangle + \min_{y\in Y}\langle \nabla_y L_{\bar x}(\bar y,\bar\lambda,\bar\mu), y-\bar y\rangle \\ & \stackrel{(3.29)}{=} & L_{\bar x}(\bar y,\bar\lambda,\bar\mu) + \langle \nabla_x f(\bar y,\bar x) + B^T\bar\lambda + \displaystyle\sum_{i=1}^p \bar\mu_i \nabla_x g_i(\bar y,\bar x), x-\bar x\rangle, \\ & = & \mathcal{Q}(\bar x) + \langle s(\bar x), x-\bar x\rangle\end{array}$$

where the last equality follows from (3.28), $A\bar y + B\bar x = b$ (feasibility of $\bar y$), $\langle \bar\mu, g(\bar y,\bar x)\rangle = 0$, and $\bar\mu_i = 0$ if $i\notin I(\bar y,\bar x)$(complementary slackness for $\bar y$). □

3.2. **Inexact cuts with fixed feasible set.** As a special case of (3.22), we first consider value functions where the argument only appears in the objective of optimization problem (3.22):

$$\mathcal{Q}(x) = \left\{\begin{array}{l}\inf_{y\in\mathbb{R}^n}\ f(y,x)\\ y\in Y.\end{array}\right. \tag{3.31}$$

We fix $\bar x\in X$ and denote by $\bar y\in Y$ an optimal solution of (3.31) written for $x=\bar x$:

$$\mathcal{Q}(\bar x) = f(\bar y,\bar x). \tag{3.32}$$

If f is differentiable, using Proposition 3.2, we have that $\nabla_x f(\bar y,\bar x)\in\partial\mathcal{Q}(\bar x)$ and

$$\mathcal{C}(x) := \mathcal{Q}(\bar x) + \langle \nabla_x f(\bar y,\bar x), x-\bar x\rangle$$

is an exact cut for $\mathcal{Q}$ at $\bar x$. If instead of an optimal solution $\bar y$ of (3.31), we only have at hand an approximate ε-optimal solution $\hat y(\varepsilon)$, Proposition 3.3 below gives an inexact cut for $\mathcal{Q}$ at $\bar x$:

Proposition 3.3 (Proposition 2.2 in [6])**.** *Let $\bar x\in X$ and let $\hat y(\varepsilon)\in Y$ be an ϵ-optimal solution for problem (3.31) written for $x=\bar x$ with optimal value $\mathcal{Q}(\bar x)$, i.e., $\mathcal{Q}(\bar x)\geq f(\hat y(\varepsilon),\bar x)-\varepsilon$. Assume that f is convex and differentiable on $Y\times X$. Then setting $\eta(\varepsilon,\bar x)=\ell_1(\hat y(\varepsilon),\bar x)$ where $\ell_1: Y\times X\to\mathbb{R}_+$ is the function given by*

$$\ell_1(\hat y,\bar x) = -\min_{y\in Y}\langle \nabla_y f(\hat y,\bar x), y-\hat y\rangle = \max_{y\in Y}\langle \nabla_y f(\hat y,\bar x), \hat y - y\rangle, \tag{3.33}$$

the affine function

$$\mathcal{C}(x) := f(\hat y(\varepsilon),\bar x) - \eta(\varepsilon,\bar x) + \langle \nabla_x f(\hat y(\varepsilon),\bar x), x-\bar x\rangle \tag{3.34}$$

is a cut for $\mathcal{Q}$ at $\bar x$, i.e., for every $x\in X$ we have $\mathcal{Q}(x)\geq\mathcal{C}(x)$ and the quantity $\eta(\varepsilon,\bar x)$ is an upper bound for the distance $\mathcal{Q}(\bar x)-\mathcal{C}(\bar x)$ between the values of $\mathcal{Q}$ and of the cut at $\bar x$.

Remark 3.4. *If $\varepsilon=0$ then $\hat y(\varepsilon)$ is an optimal solution of problem (3.31) written for $x=\bar x$, $\eta(\varepsilon,\bar x)=\ell_1(\hat y(\varepsilon),\bar x)=0$ and the cut given by Proposition 3.3 is exact. Otherwise it is inexact.*

In Proposition 3.5 below, we derive inexact cuts with an additional assumption of strong convexity on f:

(H3) f is convex and differentiable on $Y\times X$ and for every $x\in X$ there exists $\alpha(x)>0$ such that the function $f(\cdot,x)$ is strongly convex on Y with constant of strong convexity $\alpha(x)>0$ for $\|\cdot\|_2$:

$$f(y_2,x)\geq f(y_1,x)+(y_2-y_1)^T\nabla_y f(y_1,x)+\frac{\alpha(x)}{2}\|y_2-y_1\|_2^2,\ \forall x\in X,\ \forall\, y_1,y_2\in Y.$$

We will also need the following assumption, used to control the error on the gradients of f:

(H4) For every $y\in Y$ the function $f(y,\cdot)$ is differentiable on X and for every $x\in X$ there exists $0\leq M_1(x)<+\infty$ such that for every $y_1,y_2\in Y$, we have

$$\|\nabla_x f(y_2,x)-\nabla_x f(y_1,x)\|_2\leq M_1(x)\|y_2-y_1\|_2.$$

Proposition 3.5. *Let $\bar{x} \in X$ and let $\hat{y}(\varepsilon) \in Y$ be an ϵ-optimal solution for problem (3.31) written for $x = \bar{x}$ with optimal value $\mathcal{Q}(\bar{x})$, i.e., $\mathcal{Q}(\bar{x}) \geq f(\hat{y}(\varepsilon), \bar{x}) - \varepsilon$. Let Assumptions (H3) and (H4) hold. Then setting*

$$\eta(\varepsilon, \bar{x}) = \varepsilon + M_1(\bar{x}) \mathrm{Diam}(X) \sqrt{\frac{2\varepsilon}{\alpha(\bar{x})}}, \tag{3.35}$$

the affine function

$$\mathcal{C}(x) := f(\hat{y}(\varepsilon), \bar{x}) - \eta(\varepsilon, \bar{x}) + \langle \nabla_x f(\hat{y}(\varepsilon), \bar{x}), x - \bar{x} \rangle \tag{3.36}$$

is a cut for $\mathcal{Q}$ at $\bar{x}$, i.e., for every $x \in X$ we have $\mathcal{Q}(x) \geq \mathcal{C}(x)$ and the distance $\mathcal{Q}(\bar{x}) - \mathcal{C}(\bar{x})$ between the values of $\mathcal{Q}$ and of the cut at $\bar{x}$ is at most $\eta(\varepsilon, \bar{x})$, or, equivalently, $\nabla_x f(\hat{y}, \bar{x}) \in \partial_{\eta(\varepsilon,\bar{x})} \mathcal{Q}(\bar{x})$.

Proof. For short, we use the notation $\hat{y}$ instead of $\hat{y}(\varepsilon)$. Using the fact that $\hat{y} \in Y$, the first order optimality conditions for $\bar{y}$ imply $(\hat{y} - \bar{y})^T \nabla_y f(\bar{y}, \bar{x}) \geq 0$, which combined with Assumption (H3), gives

$$\begin{array}{lll} f(\hat{y}, \bar{x}) & \geq & f(\bar{y}, \bar{x}) + (\hat{y} - \bar{y})^T \nabla_y f(\bar{y}, \bar{x}) + \frac{\alpha(\bar{x})}{2} \|\hat{y} - \bar{y}\|_2^2 \\ & \geq & \mathcal{Q}(\bar{x}) + \frac{\alpha(\bar{x})}{2} \|\hat{y} - \bar{y}\|_2^2, \end{array}$$

yielding

$$\|\bar{y} - \hat{y}\|_2 \leq \sqrt{\frac{2}{\alpha(\bar{x})} \Big(f(\hat{y}, \bar{x}) - \mathcal{Q}(\bar{x}) \Big)} \leq \sqrt{\frac{2\varepsilon}{\alpha(\bar{x})}}. \tag{3.37}$$

Now recalling that $\nabla_x f(\bar{y}, \bar{x}) \in \partial \mathcal{Q}(\bar{x})$, we have for every $x \in X$,

$$\begin{array}{lll} \mathcal{Q}(x) & \geq & \mathcal{Q}(\bar{x}) + (x - \bar{x})^T \nabla_x f(\bar{y}, \bar{x}) \\ & \geq & f(\hat{y}, \bar{x}) - \varepsilon + (x - \bar{x})^T \nabla_x f(\bar{y}, \bar{x}) \\ & = & f(\hat{y}, \bar{x}) - \varepsilon + (x - \bar{x})^T \nabla_x f(\hat{y}, \bar{x}) + (x - \bar{x})^T \Big(\nabla_x f(\bar{y}, \bar{x}) - \nabla_x f(\hat{y}, \bar{x}) \Big) \\ & \geq & f(\hat{y}, \bar{x}) - \varepsilon + (x - \bar{x})^T \nabla_x f(\hat{y}, \bar{x}) - M_1(\bar{x}) \|\hat{y} - \bar{y}\|_2 \|x - \bar{x}\|_2 \\ & \underset{(3.37)}{\geq} & f(\hat{y}, \bar{x}) - \varepsilon - M_1(\bar{x}) \mathrm{Diam}(X) \sqrt{\frac{2\varepsilon}{\alpha(\bar{x})}} + (x - \bar{x})^T \nabla_x f(\hat{y}, \bar{x}), \end{array} \tag{3.38}$$

where for the third inequality we have used Cauchy-Schwartz inequality and Assumption (H4). Finally, observe that $\mathcal{C}(\bar{x}) = f(\hat{y}, \bar{x}) - \eta(\varepsilon, \bar{x}) \geq \mathcal{Q}(\bar{x}) - \eta(\varepsilon, \bar{x})$. □

Remark 3.6. *As expected, if $\varepsilon = 0$ then $\eta(\varepsilon, \bar{x}) = 0$ and the cut given by Proposition 3.5 is exact. Otherwise it is inexact. The error term $\eta(\varepsilon, \bar{x})$ is the sum of the upper bound ε on the error on the optimal value and of the error term $M_1(\bar{x}) \mathrm{Diam}(X) \sqrt{\frac{2\varepsilon}{\alpha(\bar{x})}}$ which accounts for the error on the subgradients of $\mathcal{Q}$.*

3.3. **Inexact cuts with variable feasible set.** For $x \in X$, recall that for problem (3.22) the Lagrangian function is

$$L_x(y, \lambda, \mu) = f(y, x) + \lambda^T (Bx + Ay - b) + \mu^T g(y, x),$$

and the dual function is given by

$$\theta_x(\lambda, \mu) = \inf_{y \in Y} \; L_x(y, \lambda, \mu). \tag{3.39}$$

Define $\ell_2 : Y \times X \times \mathbb{R}^q \times \mathbb{R}_+^p \to \mathbb{R}_+$ by

$$\ell_2(\hat{y}, \bar{x}, \hat{\lambda}, \hat{\mu}) = -\min_{y \in Y} \langle \nabla_y L_{\bar{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}), y - \hat{y} \rangle = \max_{y \in Y} \langle \nabla_y L_{\bar{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}), \hat{y} - y \rangle. \tag{3.40}$$

We make the following assumption which ensures no duality gap for (3.22) for any $x \in X$:

(H5) if Y is polyhedral then for every $x \in X$ there exists $y_x \in Y$ such that $Bx + Ay_x = b$ and $g(y_x, x) < 0$ and if Y is not polyhedral then for every $x \in X$ there exists $y_x \in \mathrm{ri}(Y)$ such that $Bx + Ay_x = b$ and $g(y_x, x) < 0$.

The following proposition, proved in [6], provides an inexact cut for $\mathcal{Q}$ given by (3.22):

Proposition 3.7. *[Proposition 2.7 in [6]] Let $\bar{x} \in X$, let $\hat{y}(\epsilon)$ be an ϵ-optimal feasible primal solution for problem* (3.22) *written for $x = \bar{x}$ and let $(\hat{\lambda}(\epsilon), \hat{\mu}(\epsilon))$ be an ϵ-optimal feasible solution of the corresponding dual problem, i.e., of problem* (3.23) *written for $x = \bar{x}$. Let Assumptions (H1), (H2), and (H5) hold. If additionally f and g are differentiable on $Y \times X$ then setting $\eta(\varepsilon, \bar{x}) = \ell_2(\hat{y}(\epsilon), \bar{x}, \hat{\lambda}(\epsilon), \hat{\mu}(\epsilon))$, the affine function*

$$\mathcal{C}(x) := L_{\bar{x}}(\hat{y}(\epsilon), \hat{\lambda}(\epsilon), \hat{\mu}(\epsilon)) - \eta(\varepsilon, \bar{x}) + \langle \nabla_x L_{\bar{x}}(\hat{y}(\epsilon), \hat{\lambda}(\epsilon), \hat{\mu}(\epsilon)), x - \bar{x} \rangle \tag{3.41}$$

with

$$\nabla_x L_{\bar{x}}(\hat{y}(\epsilon), \hat{\lambda}(\epsilon), \hat{\mu}(\epsilon)) = \nabla_x f(\hat{y}(\epsilon), \bar{x}) + B^T \hat{\lambda}(\epsilon) + \sum_{i=1}^{p} \hat{\mu}_i(\epsilon) \nabla_x g_i(\hat{y}(\epsilon), \bar{x}),$$

is a cut for $\mathcal{Q}$ at $\bar{x}$ and the distance $\mathcal{Q}(\bar{x}) - \mathcal{C}(\bar{x})$ between the values of $\mathcal{Q}$ and of the cut at $\bar{x}$ is at most $\varepsilon + \ell_2(\hat{y}(\epsilon), \bar{x}, \hat{\lambda}(\epsilon), \hat{\mu}(\epsilon))$.

In Proposition 3.8 below, we derive another formula for inexact cuts with an additional assumption of strong convexity:

(H6) Strong concavity of the dual function: for every $x \in X$ there exists $\alpha_D(x) > 0$ and a set D_x containing the set of optimal solutions of dual problem (3.23) such that the dual function θ_x is strongly concave on D_x with constant of strong concavity $\alpha_D(x)$ with respect to $\| \cdot \|_2$.

We refer to Section 2 for conditions on the problem data ensuring Assumption (H6).

If the constants $\alpha(\bar{x})$ and $\alpha_D(\bar{x})$ in Assumptions (H3) and (H6) are sufficiently large and n is small then the cuts given by Proposition 3.8 are better than the cuts given by Proposition 3.7, i.e., $\mathcal{Q}(\bar{x}) - \mathcal{C}(\bar{x})$ is smaller. We refer to Section 3.4 for numerical tests comparing the cuts given by Propositions 3.7 and 3.8 on quadratic programs.

To proceed, take an optimal primal solution $\bar{y}$ of problem (3.22) written for $x = \bar{x}$ and an optimal dual solution $(\bar{\lambda}, \bar{\mu})$ of the corresponding dual problem, i.e., problem (3.23) written for $x = \bar{x}$.

With this notation, using Proposition 3.2, we have that $\nabla_x L_{\bar{x}}(\bar{y}, \bar{\lambda}, \bar{\mu}) \in \partial \mathcal{Q}(\bar{x})$. Since we only have approximate primal and dual solutions, $\hat{y}(\varepsilon)$ and $(\hat{\lambda}(\epsilon), \hat{\mu}(\epsilon))$ respectively, we will use the approximate subgradient $\nabla_x L_{\bar{x}}(\hat{y}(\epsilon), \hat{\lambda}(\epsilon), \hat{\mu}(\epsilon))$ instead of $\nabla_x L_{\bar{x}}(\bar{y}, \bar{\lambda}, \bar{\mu})$. To control the error on this subgradient, we assume differentiability of the constraint functions and that the gradients of these functions are Lipschitz continuous. More precisely, we assume:

(H7) g is differentiable on $Y \times X$ and for every $x \in X$ there exists $0 \leq M_2(x) < +\infty$ such that for all $y_1, y_2 \in Y$, we have

$$\|\nabla_x g_i(y_1, x) - \nabla_x g_i(y_2, x)\|_2 \leq M_2(x) \|y_1 - y_2\|_2, \; i = 1, \ldots, p.$$

If Assumptions (H1)-(H7) hold, the following proposition provides an inexact cut for $\mathcal{Q}$ at $\bar{x}$:

Proposition 3.8. *Let $\bar{x} \in X$, let $\hat{y}(\varepsilon)$ be an ϵ-optimal feasible primal solution for problem* (3.22) *written for $x = \bar{x}$ and let $(\hat{\lambda}(\epsilon), \hat{\mu}(\epsilon))$ be an ϵ-optimal feasible solution of the corresponding dual problem, i.e., of problem* (3.23) *written for $x = \bar{x}$. Let Assumptions (H1), (H2), (H3), (H4), (H5), (H6), and (H7) hold. Assume that $(\hat{\lambda}(\epsilon), \hat{\mu}(\epsilon)) \in D_{\bar{x}}$ where $D_{\bar{x}}$ is defined in (H6) and let*

$$U = \max_{i=1,\ldots,p} \|\nabla_x g_i(\hat{y}(\varepsilon), \bar{x})\|_2. \tag{3.42}$$

Let also $\mathcal{L}_{\bar{x}}$ be any lower bound on $\mathcal{Q}(\bar{x})$. Define

$$\mathcal{U}_{\bar{x}} = \frac{f(y_{\bar{x}}, \bar{x}) - \mathcal{L}_{\bar{x}}}{\min(-g_i(y_{\bar{x}}, \bar{x}), i = 1, \ldots, p)} \tag{3.43}$$

and

$$\eta(\varepsilon, \bar{x}) = \varepsilon + \Big((M_1(\bar{x}) + M_2(\bar{x}) \mathcal{U}_{\bar{x}}) \sqrt{\frac{2}{\alpha(\bar{x})}} + \frac{2 \max(\|B^T\|, \sqrt{p} U)}{\sqrt{\alpha_D(\bar{x})}} \Big) \mathrm{Diam}(X) \sqrt{\varepsilon}.$$

Then

$$\mathcal{C}(x) := f(\hat{y}(\varepsilon), \bar{x}) - \eta(\epsilon, \bar{x}) + \langle \nabla_x L_{\bar{x}}(\hat{y}(\epsilon), \hat{\lambda}(\epsilon), \hat{\mu}(\epsilon)), x - \bar{x} \rangle$$

where

$$\nabla_x L_{\bar{x}}(\hat{y}(\epsilon), \hat{\lambda}(\epsilon), \hat{\mu}(\epsilon)) = \nabla_x f(\hat{y}(\epsilon), \bar{x}) + B^T \hat{\lambda}(\epsilon) + \sum_{i=1}^{p} \hat{\mu}_i(\epsilon) \nabla_x g_i(\hat{y}(\epsilon), \bar{x}),$$

is a cut for $\mathcal{Q}$ at $\bar{x}$ and the distance $\mathcal{Q}(\bar{x}) - \mathcal{C}(\bar{x})$ between the values of $\mathcal{Q}$ and of the cut at $\bar{x}$ is at most $\eta(\epsilon, \bar{x})$.

Proof. For short, we use the notation $\hat{y}, \hat{\lambda}, \hat{\mu}$ instead of $\hat{y}(\varepsilon), \hat{\lambda}(\varepsilon), \hat{\mu}(\varepsilon)$. Since $\nabla_x L_{\bar{x}}(\bar{y}, \bar{\lambda}, \bar{\mu}) \in \partial \mathcal{Q}(\bar{x})$, we have

$$\mathcal{Q}(x) \geq \mathcal{Q}(\bar{x}) + \langle \nabla_x L_{\bar{x}}(\bar{y}, \bar{\lambda}, \bar{\mu}), x - \bar{x} \rangle \geq f(\hat{y}, \bar{x}) - \varepsilon + \langle \nabla_x L_{\bar{x}}(\bar{y}, \bar{\lambda}, \bar{\mu}), x - \bar{x} \rangle. \tag{3.44}$$

Next observe that

$$\begin{array}{rcl} \|\nabla_x L_{\bar{x}}(\bar{y}, \bar{\lambda}, \bar{\mu}) - \nabla_x L_{\bar{x}}(\hat{y}, \hat{\lambda}, \hat{\mu})\| & \leq & M_1(\bar{x})\|\bar{y} - \hat{y}\| + \|B^T\| \|\bar{\lambda} - \hat{\lambda}\| \\ & & + \| \sum_{i=1}^{p} \bar{\mu}(i) \Big(\nabla_x g_i(\bar{y}, \bar{x}) - \nabla_x g_i(\hat{y}, \bar{x}) \Big) \| \\ & & + \| \sum_{i=1}^{p} \Big(\bar{\mu}(i) - \hat{\mu}(i) \Big) \nabla_x g_i(\hat{y}, \bar{x}) \| \\ & \leq & M_1(\bar{x})\|\bar{y} - \hat{y}\| + \|B^T\| \|\bar{\lambda} - \hat{\lambda}\| + M_2(\bar{x}) \|\bar{\mu}\|_1 \|\bar{y} - \hat{y}\| + U\sqrt{p}\|\bar{\mu} - \hat{\mu}\| \\ (3.45) & \leq & (M_1(\bar{x}) + M_2(\bar{x})\|\bar{\mu}\|_1)\|\bar{y} - \hat{y}\| + \sqrt{2} \max(\|B^T\|, U\sqrt{p}) \sqrt{\|\hat{\lambda} - \bar{\lambda}\|^2 + \|\hat{\mu} - \bar{\mu}\|^2}. \end{array}$$

Using Remark 2.3.3, p.313 in [7] and Assumption (H5) we have for $\|\bar{\mu}\|_1$ the upper bound

$$\|\bar{\mu}\|_1 \leq \frac{f(y_{\bar{x}}, \bar{x}) - \mathcal{Q}(\bar{x})}{\min(-g_i(y_{\bar{x}}, \bar{x}), i = 1, \ldots, p)} \leq \mathcal{U}_{\bar{x}}. \tag{3.46}$$

Using Assumptions (H3) and (H6), we also get

$$\|\hat{y} - \bar{y}\|^2 \leq \frac{2\varepsilon}{\alpha(\bar{x})} \text{ and } \|\hat{\lambda} - \bar{\lambda}\|^2 + \|\hat{\mu} - \bar{\mu}\|^2 \leq \frac{2\varepsilon}{\alpha_D(\bar{x})}. \tag{3.47}$$

Combining (3.45), (3.46), and (3.47), we get

$$\|\nabla_x L_{\bar{x}}(\bar{y}, \bar{\lambda}, \bar{\mu}) - \nabla_x L_{\bar{x}}(\hat{y}, \hat{\lambda}, \hat{\mu})\| \leq \frac{\eta(\varepsilon, \bar{x}) - \varepsilon}{\text{Diam}(X)}. \tag{3.48}$$

Plugging the above relation into (3.44) and using Cauchy-Schwartz inequality, we get

$$\begin{array}{rcl} \mathcal{Q}(x) & \geq & f(\hat{y}, \bar{x}) - \varepsilon + \langle \nabla_x L_{\bar{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}), x - \bar{x} \rangle + \langle \nabla_x L_{\bar{x}}(\bar{y}, \bar{\lambda}, \bar{\mu}) - \nabla_x L_{\bar{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}), x - \bar{x} \rangle \\ & \geq & f(\hat{y}, \bar{x}) - \varepsilon - \|\nabla_x L_{\bar{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}) - \nabla_x L_{\bar{x}}(\bar{y}, \bar{\lambda}, \bar{\mu})\| \text{Diam}(X) + \langle \nabla_x L_{\bar{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}), x - \bar{x} \rangle \\ & \geq & f(\hat{y}, \bar{x}) - \eta(\varepsilon, \bar{x}) + \langle \nabla_x L_{\bar{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}), x - \bar{x} \rangle. \end{array} \tag{3.49}$$

Finally, since $\hat{y} \in \mathcal{S}(\bar{x})$ we check that $\mathcal{Q}(\bar{x}) - \mathcal{C}(\bar{x}) = \mathcal{Q}(\bar{x}) - f(\hat{y}, \bar{x}) + \eta(\varepsilon, \bar{x}) \leq \eta(\varepsilon, \bar{x})$, which achieves the proof of the proposition. □

Observe that the "slope" $\nabla_x L_{\bar{x}}(\hat{y}(\epsilon), \hat{\lambda}(\epsilon), \hat{\mu}(\epsilon))$ of the cut given by Proposition 3.7 is the same as the "slope" of the cut given by Proposition 3.8.

Remark 3.9. *If $\hat{y}(\varepsilon)$ and $(\hat{\lambda}(\varepsilon), \hat{\mu}(\varepsilon))$ are respectively optimal primal and dual solutions, i.e., $\varepsilon = 0$, then Proposition 3.8 gives, as expected, an exact cut for $\mathcal{Q}$ at $\bar{x}$.*

As shown in Corollary 3.10, the formula for the inexact cuts given in Proposition 3.8 can be simplified depending if there are nonlinear coupling constraints or not, if f is separable (sum of a function of x and of a function of y) or not, and if g is separable.

Corollary 3.10. *Consider the value functions $\mathcal{Q} : X \to \mathbb{R}$ where $\mathcal{Q}(x)$ is given by the optimal value of the following optimization problems:*

$$
(3.50)\quad
\begin{array}{lll}
(a)\left\{\begin{array}{l}\min_y f(y,x)\\ Ay+Bx=b,\\ h(y)+k(x)\le 0,\\ y\in Y,\end{array}\right. &
(b)\left\{\begin{array}{l}\min_y f_0(y)+f_1(x)\\ Ay+Bx=b,\\ g(y,x)\le 0,\\ y\in Y,\end{array}\right. &
(c)\left\{\begin{array}{l}\min_y f_0(y)+f_1(x)\\ Ay+Bx=b,\\ h(y)+k(x)\le 0,\\ y\in Y,\end{array}\right.\\
(d)\left\{\begin{array}{l}\min_y f(y,x)\\ g(y,x)\le 0,\\ y\in Y,\end{array}\right. &
(e)\left\{\begin{array}{l}\min_y f(y,x)\\ h(y)+k(x)\le 0,\\ y\in Y,\end{array}\right. &
(f)\left\{\begin{array}{l}\min_y f_0(y)+f_1(x)\\ g(y,x)\le 0,\\ y\in Y,\end{array}\right.\\
(g)\left\{\begin{array}{l}\min_y f_0(y)+f_1(x)\\ h(y)+k(x)\le 0,\\ y\in Y,\end{array}\right. &
(h)\left\{\begin{array}{l}\min_y f(y,x)\\ Ay+Bx=b,\\ y\in Y,\end{array}\right. &
(i)\left\{\begin{array}{l}\min_y f_0(y)+f_1(x)\\ Ay+Bx=b,\\ y\in Y.\end{array}\right.
\end{array}
$$

For problems (b),(c),(f),(g), (i) above define $f(y,x) = f_0(y)+f_1(x)$ and for problems (a), (c), (e), (g) define $g(y,x) = h(y)+k(x)$. With this notation, assume that (H1), (H2), (H3), (H4), (H5), (H6), and (H7) hold for these problems. If g is defined, let $L_x(y,\lambda,\mu) = f(y,x)+\lambda^T(Bx+Ay-b)+\mu^T g(y,x)$ be the Lagrangian and define

$$U = \max_{i=1,\ldots,p} \|\nabla_x g_i(\hat{y}(\varepsilon),\bar{x})\| \text{ and } \mathcal{U}_{\bar{x}} = \frac{f(y_{\bar{x}},\bar{x})-\mathcal{L}_{\bar{x}}}{\min(-g_i(y_{\bar{x}},\bar{x}), i=1,\ldots,p)}$$

where $\mathcal{L}_{\bar{x}}$ is any lower bound on $\mathcal{Q}(\bar{x})$. If g is not defined, define $L_x(y,\lambda) = f(y,x)+\lambda^T(Bx+Ay-b)$.

Let $\bar{x} \in X$, let $\hat{y}$ be an ϵ-optimal feasible primal solution for problem (3.22) *written for $x=\bar{x}$ and let $(\hat{\lambda},\hat{\mu})$ be an ϵ-optimal feasible solution of the corresponding dual problem, i.e., of problem* (3.23) *written for $x=\bar{x}$.*

Then $\mathcal{C}(x) = f(\hat{y},\bar{x})-\eta(\varepsilon,\bar{x})+\langle s(\bar{x}), x-\bar{x}\rangle$ is an inexact cut for $\mathcal{Q}$ at $\bar{x}$ where the formulas for $\eta(\varepsilon,\bar{x})$ and $s(\bar{x})$ in each of cases (a)-(i) above are the following:

$$
(3.51)\quad
\begin{array}{l}
(a)\left\{\begin{array}{l}\eta(\varepsilon,\bar{x}) = \varepsilon+\Big(M_1(\bar{x})\frac{1}{\sqrt{\alpha(\bar{x})}}+\sqrt{2}\max(\|B^T\|,\sqrt{p}U)\frac{1}{\sqrt{\alpha_D(\bar{x})}}\Big)\mathrm{Diam}(X)\sqrt{2\varepsilon},\\ s(\bar{x}) = \nabla_x f(\hat{y},\bar{x})+B^T\hat{\lambda}+\sum_{i=1}^p \hat{\mu}_i\nabla_x k_i(\bar{x}),\end{array}\right.\\
(b)\left\{\begin{array}{l}\eta(\varepsilon,\bar{x}) = \varepsilon+\Big(M_2(\bar{x})\mathcal{U}_{\bar{x}}\frac{1}{\sqrt{\alpha(\bar{x})}}+\sqrt{2}\max(\|B^T\|,\sqrt{p}U)\frac{1}{\sqrt{\alpha_D(\bar{x})}}\Big)\mathrm{Diam}(X)\sqrt{2\varepsilon},\\ s(\bar{x}) = \nabla_x f_1(\bar{x})+B^T\hat{\lambda}+\sum_{i=1}^p \hat{\mu}_i\nabla_x g_i(\hat{y},\bar{x}),\end{array}\right.\\
(c)\left\{\begin{array}{l}\eta(\varepsilon,\bar{x}) = \varepsilon+2\max(\|B^T\|,\sqrt{p}U)\mathrm{Diam}(X)\sqrt{\frac{\varepsilon}{\alpha_D(\bar{x})}},\\ s(\bar{x}) = \nabla_x f_1(\bar{x})+B^T\hat{\lambda}+\sum_{i=1}^p \hat{\mu}_i\nabla_x k_i(\bar{x}),\end{array}\right.\\
(d)\left\{\begin{array}{l}\eta(\varepsilon,\bar{x}) = \varepsilon+\Big((M_1(\bar{x})+M_2(\bar{x})\mathcal{U}_{\bar{x}})\frac{1}{\sqrt{\alpha(\bar{x})}}+U\sqrt{\frac{p}{\alpha_D(\bar{x})}}\Big)\mathrm{Diam}(X)\sqrt{2\varepsilon},\\ s(\bar{x}) = \nabla_x f(\hat{y},\bar{x})+\sum_{i=1}^p \hat{\mu}_i\nabla_x g_i(\hat{y},\bar{x}),\end{array}\right.\\
(e)\left\{\begin{array}{l}\eta(\varepsilon,\bar{x}) = \varepsilon+\Big(\frac{M_1(\bar{x})}{\sqrt{\alpha(\bar{x})}}+\sqrt{\frac{p}{\alpha_D(\bar{x})}}U\Big)\mathrm{Diam}(X)\sqrt{2\varepsilon},\\ s(\bar{x}) = \nabla_x f(\hat{y},\bar{x})+\sum_{i=1}^p \hat{\mu}_i\nabla_x k_i(\bar{x}),\end{array}\right.\\
(f)\left\{\begin{array}{l}\eta(\varepsilon,\bar{x}) = \varepsilon+\Big(\frac{M_2(\bar{x})}{\sqrt{\alpha(\bar{x})}}\mathcal{U}_{\bar{x}}+U\sqrt{\frac{p}{\alpha_D(\bar{x})}}\Big)\mathrm{Diam}(X)\sqrt{2\varepsilon},\\ s(\bar{x}) = \nabla_x f_1(\bar{x})+\sum_{i=1}^p \hat{\mu}_i\nabla_x g_i(\hat{y},\bar{x}),\end{array}\right.\\
(g)\left\{\begin{array}{l}\eta(\varepsilon,\bar{x}) = \varepsilon+\mathrm{Diam}(X)\sqrt{\frac{2\varepsilon p}{\alpha_D(\bar{x})}}U,\\ s(\bar{x}) = \nabla_x f_1(\bar{x})+\sum_{i=1}^p \hat{\mu}_i\nabla_x k_i(\bar{x}),\end{array}\right.\\
(h)\left\{\begin{array}{l}\eta(\varepsilon,\bar{x}) = \varepsilon+\Big(\frac{M_1(\bar{x})}{\sqrt{\alpha(\bar{x})}}+\frac{\|B^T\|}{\sqrt{\alpha_D(\bar{x})}}\Big)\mathrm{Diam}(X)\sqrt{2\varepsilon},\\ s(\bar{x}) = \nabla_x f(\hat{y},\bar{x})+B^T\hat{\lambda},\end{array}\right.\\
(i)\left\{\begin{array}{l}\eta(\varepsilon,\bar{x}) = \varepsilon+\|B^T\|\sqrt{\frac{2\varepsilon}{\alpha_D(\bar{x})}}\mathrm{Diam}(X),\\ s(\bar{x}) = \nabla_x f_1(\bar{x})+B^T\hat{\lambda}.\end{array}\right.
\end{array}
$$

Proof. It suffices to follow the proof of Proposition 3.8, specialized to cases (a)-(i). For instance, let us check the formulas in case (g). For (g), $s(\bar{x}) = \nabla_x L_{\bar{x}}(\hat{y}, \hat{\mu}) = \nabla_x f_1(\bar{x}) + \sum_{i=1}^{p} \hat{\mu}_i \nabla_x k_i(\bar{x})$ and

$$\begin{array}{rcl} \|\nabla_x L_{\bar{x}}(\hat{y}, \hat{\mu}) - \nabla_x L_{\bar{x}}(\bar{y}, \bar{\mu})\| & = & \| \sum_{i=1}^{p} (\hat{\mu}_i - \bar{\mu}_i) \nabla_x k_i(\bar{x})\| \leq U \|\hat{\mu} - \bar{\mu}\|_1 \\ & \leq & U\sqrt{p}\|\hat{\mu} - \bar{\mu}\| \leq U\sqrt{p}\sqrt{\frac{2\varepsilon}{\alpha_D(\bar{x})}}. \end{array} \tag{3.52}$$

It then suffices to combine (3.44) and (3.52). □

3.4. **Numerical results.**

3.4.1. *Argument of the value function in the objective only.* Let $S = \begin{pmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{pmatrix}$ be a positive definite matrix, let $c_1 \in \mathbb{R}^m, c_2 \in \mathbb{R}^n$ be vectors of ones, and let $\mathcal{Q}$ be the value function given by

$$\begin{array}{rcl} \mathcal{Q}(x) & = & \left\{ \begin{array}{l} \min_{y \in \mathbb{R}^n} \; f(y,x) = \frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^T S \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix} \\ y \in Y := \{y \in \mathbb{R}^n : y \geq 0, \; \sum_{i=1}^{n} y_i = 1\}, \end{array} \right. \\ & = & \left\{ \begin{array}{l} \min_{y \in \mathbb{R}^n} \; c_1^T x + c_2^T y + \frac{1}{2} x^T S_1 x + x^T S_2 y + \frac{1}{2} y^T S_3 y \\ y \geq 0, \; \sum_{i=1}^{n} y_i = 1. \end{array} \right. \end{array} \tag{3.53}$$

Clearly, Assumption (H3) is satisfied with $\alpha(x) = \lambda_{\min}(S_3)$, and

$$\|\nabla_x f(y_2, x) - \nabla_x f(y_1, x)\| = \|S_2(y_2 - y_1)\|_2 \leq \|S_2\|_2 \|y_2 - y_1\|_2$$

implying that Assumption (H4) is satisfied with $M_1(\bar{x}) = \|S_2\|_2 = \sigma(S_2)$ where $\sigma(S_2)$ is the largest singular value of S_2. We take $X = Y$ with $\text{Diam}(X) = \max_{x_1, x_2 \in X} \|x_2 - x_1\|_2 \leq \sqrt{2}$. With this notation, if $\hat{y}$ is an ϵ-optimal solution of (3.53) written for $x = \bar{x}$, we compute at $\bar{x}$ the cut $\mathcal{C}(x) = f(\hat{y}, \bar{x}) - \eta(\varepsilon, \bar{x}) + \langle \nabla_x f(\hat{y}, \bar{x}), x - \bar{x} \rangle = f(\hat{y}, \bar{x}) - \eta(\varepsilon, \bar{x}) + \langle c_1 + S_1 \bar{x} + S_2 \hat{y}, x - \bar{x} \rangle$ where

- $\eta(\varepsilon, \bar{x}) = \eta_1(\varepsilon, \bar{x}) = \varepsilon + 2M_1(\bar{x})\sqrt{\frac{\varepsilon}{\alpha(\bar{x})}}$ using Proposition 3.5;
- $\eta(\varepsilon, \bar{x})$ is given by

$$\eta(\varepsilon, \bar{x}) = \eta_2(\varepsilon, \bar{x}) = \left\{ \begin{array}{l} \max \; \langle \nabla_y f(\hat{y}, \bar{x}), \hat{y} - y \rangle \\ y \geq 0, \; \sum_{i=1}^{n} y_i = 1, \end{array} \right. = \left\{ \begin{array}{l} \max \; \langle c_2 + S_2^T \bar{x} + S_3 \hat{y}, \hat{y} - y \rangle \\ y \geq 0, \; \sum_{i=1}^{n} y_i = 1, \end{array} \right.$$

  using Proposition 3.3.

We compare in Table 1 the values of $\eta_1(\varepsilon, \bar{x})$ and $\eta_2(\varepsilon, \bar{x})$ for several values of $m = n$, ε, and $\alpha(\bar{x})$. In these experiments S is of the form $AA^T + \lambda I_{2n}$ for some $\lambda > 0$ and A has random entries in $[-20, 20]$.

Optimization problems were solved using Mosek optimization toolbox [1], setting Mosek parameter MSK_DPAR_INTPNT_QO_TOL _REL_GAP which corresponds to the relative error ε_r on the optimal value to 0.1, 0.5, and 1. In each run, ε was estimated computing the duality gap (the difference between the approximate optimal values of the dual and the primal). Though $\eta_1(\varepsilon, \bar{x})$ does not depend on $\bar{x}$ (because on this example α and M_1 do not depend on $\bar{x}$), the absolute error ε depends on the run (for a fixed ε_r, different runs corresponding to different $\bar{x}$ yield different errors $\varepsilon, \eta_1(\varepsilon, \bar{x})$ and $\eta_2(\varepsilon, \bar{x})$). Therefore, for each fixed $(\varepsilon_r, \alpha(\bar{x}), n)$, the values ε, $\eta_1(\varepsilon, \bar{x})$, and $\eta_2(\varepsilon, \bar{x})$ reported in the table correspond to the mean values of ε, $\eta_1(\varepsilon, \bar{x})$, and $\eta_2(\varepsilon, \bar{x})$ obtained taking randomly 50 points in X. We see that the cuts computed by Proposition 3.5 are much more conservative on nearly all combinations of parameters, except on three of these combinations when $n = 10$ and $\alpha(\bar{x}) = 10^6$ is very large.

3.4.2. *Argument of the value function in the objective and constraints.* We close this section comparing the error terms in the cuts given by Propositions 3.7 and 3.8 on a very simple problem with a quadratic objective and a quadratic constraint.

Let $S = \begin{pmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{pmatrix}$ be a positive definite matrix, let $c_1, c_2 \in \mathbb{R}^n$, and let $\mathcal{Q} : X \to \mathbb{R}$ be the value function given by

$$\mathcal{Q}(x) = \min_{y \in \mathbb{R}^n} \; \{f(y,x) : g_1(y,x) \leq 0\}, \tag{3.54}$$

ε	$\alpha(\bar{x})$	n	η_1	η_2	ε	$\alpha(\bar{x})$	n	η_1	η_2
0.0024	102.9	10	1.76	0.025	0.0061	190.2	10	2.73	0.026
0.0080	10 087	10	0.86	0.054	0.0024	10^6	10	0.076	0.354
0.016	129.0	10	9.81	0.047	0.0084	174.5	10	4.85	0.037
0.029	10054	10	2.49	0.128	0.002	10^6	10	0.09	0.342
0.008	112.3	10	8.07	0.043	0.008	150.0	10	6.36	0.022
0.018	10 090	10	1.29	0.078	0.0019	10^6	10	0.06	0.442
0.15	531.9	100	175.6	0.3	0.18	665.3	100	183.5	0.3
0.23	10 687	100	44.5	0.2	0.03	10^6	100	2.1	0.9
0.17	676.2	100	185.7	0.2	0.09	734.3	100	106.5	0.2
0.11	10 638	100	37.9	0.2	0.02	10^6	100	1.7	0.3
0.05	660	100	106.7	0.2	0.40	777	100	253.8	0.4
0.07	10 585	100	32.6	0.2	0.02	10^6	100	1.3	0.4
6.78	6017.9	1000	4177.8	9.5	2.69	5991.4	1000	2778.8	6.8
8.12	15 722	1000	3059.5	11.1	0.99	10^6	1000	132.1	3.2
7.40	5799	1000	4160.2	9.8	7.83	6020	1000	4590.7	9.3
12.5	15860	1000	4001.6	14.6	1.3	10^6	1000	153.6	3.47
9.9	6065	1000	4996.4	11.8	8.3	5955	1000	4034.9	8.3
7.2	15 895	1000	2564.3	3.4	9.7	10^6	1000	117.2	1.8

TABLE 1. Values of $\eta(\varepsilon, \bar{x}) = \eta_1(\varepsilon, \bar{x})$ (resp. $\eta(\varepsilon, \bar{x}) = \eta_2(\varepsilon, \bar{x})$) for the inexact cuts given by Proposition 3.5 (resp. Proposition 3.3) for value function (3.53) for various values of n (problem dimension), $\alpha(\bar{x}) = \lambda_{\min}(S_3)$, and ε.

where

$$
\begin{array}{lll}
f(y,x) & = & \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}^T S \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix} \\
 & = & c_1^T x + c_2^T y + \frac{1}{2}x^T S_1 x + x^T S_2 y + \frac{1}{2} y^T S_3 y, \\
g_1(y,x) & = & \frac{1}{2}\|y-y_0\|_2^2 + \frac{1}{2}\|x-x_0\|_2^2 - \frac{R^2}{2}, \\
X & = & \{x \in \mathbb{R}^n : \|x-x_0\|_2 \le 1\}.
\end{array} \tag{3.55}
$$

In what follows, we take $R = 5$ and $x_0, y_0 \in \mathbb{R}^n$ given by $x_0(i) = y_0(i) = 10, i = 1, \ldots, n$. Clearly, for fixed $\bar{x} \in X$ and any feasible y for (3.54), (3.55) written for $x = \bar{x}$, we have

$$
\left\| \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| + R \ge \left\| \begin{pmatrix} \bar{x} \\ y \end{pmatrix} \right\| \ge \left\| \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| - R.
$$

Knowing that with our problem data $\left\| \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| - R > 0$, we obtain the bound $\mathcal{Q}(\bar{x}) \ge \mathcal{L}_{\bar{x}}$ where

$$
\mathcal{L}_{\bar{x}} = \frac{1}{2}\lambda_{\min}(S)\Big(\left\| \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| - R\Big)^2 - \Big(\left\| \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| + R\Big) \left\| \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right\|_2 .
$$

Next, for every $\bar{x} \in X$ we have $g_1(y_0, \bar{x}) < 0$ which gives the upper bound

$$
\mathcal{U}_{\bar{x}} = \frac{\mathcal{L}_{\bar{x}} - f(y_0, \bar{x})}{g_1(y_0, \bar{x})} \tag{3.56}
$$

for any optimal dual solution $\bar{\mu} \ge 0$ of the dual of (3.54), (3.55) written for $x = \bar{x}$. Making the change of variable $z = y - y_0$, we can express (3.54) under the form (2.6) where

$$
\begin{array}{l}
Q_0 = S_3, a_0 = a_0(x) = c_2 + S_2^T x + S_3 y_0, b_0 = b_0(x) = \frac{1}{2}x^T S_1 x + c_1^T x + y_0^T(c_2 + S_2^T x) + \frac{1}{2} y_0^T S_3 y_0, \\
Q_1 = I_n, a_1 = 0, b_1 = b_1(x) = \frac{1}{2}(\|x-x_0\|_2^2 - R^2).
\end{array} \tag{3.57}
$$

Therefore, using Proposition 2.8, we have that dual function $\theta_{\bar{x}}$ for (3.54) is given by

$$
\theta_{\bar{x}}(\mu) = -\frac{1}{2}a_0(\bar{x})^T(S_3 + \mu I_n)^{-1}a_0(\bar{x}) + b_0(\bar{x}) + \mu b_1(\bar{x}) \tag{3.58}
$$

with a_0, b_0, b_1 given by (3.57) and setting

$$\alpha_D(\bar{x}) = a_0(\bar{x})^T (S_3 + \mathcal{U}_{\bar{x}} I_n)^{-3} a_0(\bar{x}),$$

if $a_0(\bar{x}) \neq 0$ then $\theta_{\bar{x}}$ is strongly concave on the interval $[0, \mathcal{U}_{\bar{x}}]$ with constant of strong concavity $\alpha_D(\bar{x})$ where $\mathcal{U}_{\bar{x}}$ is given by (3.56). Let $\hat{y}$ be an ε-optimal primal solution of (3.54) written for $x = \bar{x}$ and let $\hat{\mu}$ be an ε-optimal solution of its dual. If $a_0(\bar{x}) \neq 0$, we obtain for $\mathcal{Q}$ the cut

(3.59)

$$\left\{ \begin{array}{lcl} \mathcal{C}_1(x) & = & f(\hat{y}, \bar{x}) - \eta_1(\varepsilon, \bar{x}) + \langle \nabla_x L_{\bar{x}}(\hat{y}, \hat{\mu}), x - \bar{x} \rangle \text{ where} \\ \eta_1(\varepsilon, \bar{x}) & = & \varepsilon + D(X)\sqrt{2\varepsilon}\Big(\frac{M_1(\bar{x})}{\sqrt{\alpha(\bar{x})}} + \frac{\|\bar{x} - x_0\|}{\sqrt{\alpha_D(\bar{x})}} \Big) \text{ with } D(X) = 2, M_1(\bar{x}) = \|S_2\|_2, \alpha(\bar{x}) = \lambda_{\min}(S_3), \\ \nabla_x L_{\bar{x}}(\hat{y}, \hat{\mu}) & = & S_1 \bar{x} + c_1 + S_2 \hat{y} + \hat{\mu}(\bar{x} - x_0). \end{array} \right.$$

We now apply Proposition 3.7 to obtain another inexact cut for $\mathcal{Q}$ at $\bar{x} \in X$ rewriting (3.54) under the form (3.22) with Y the compact set $Y = \{y \in \mathbb{R}^n : \|y - y_0\|_2 \le R\}$:

$$\mathcal{Q}(x) = \min_{y \in \mathbb{R}^n} \; \{ f(y, x) : g_1(y, x) \le 0, \|y - y_0\|_2 \le R \}. \tag{3.60}$$

Applying Proposition 3.7 to reformulation (3.60) of (3.54), we obtain for $\mathcal{Q}$ the inexact cut $\mathcal{C}_2$ at $\bar{x}$ where

$$\left\{ \begin{array}{lcl} \mathcal{C}_2(x) & = & f(\hat{y}, \bar{x}) - \eta_2(\varepsilon, \bar{x}) + \langle \nabla_x L_{\bar{x}}(\hat{y}, \hat{\mu}), x - \bar{x} \rangle \text{ with} \\ \eta_2(\varepsilon, \bar{x}) & = & -\min\{ \langle \nabla_y L_{\bar{x}}(\hat{y}, \hat{\mu}), y - \hat{y} \rangle : \|y - y_0\|_2 \le R \}, \\ & = & \langle \nabla_y L_{\bar{x}}(\hat{y}, \hat{\mu}), \hat{y} - y_0 \rangle + R \|\nabla_y L_{\bar{x}}(\hat{y}, \hat{\mu})\|_2, \\ \nabla_x L_{\bar{x}}(\hat{y}, \hat{\mu}) & = & S_1 \bar{x} + c_1 + S_2 \hat{y} + \hat{\mu}(\bar{x} - x_0), \\ \nabla_y L_{\bar{x}}(\hat{y}, \hat{\mu}) & = & S_3 \hat{y} + S_2^T \bar{x} + c_2 + \hat{\mu}(\hat{y} - y_0). \end{array} \right. \tag{3.61}$$

As in the previous example, we take S of form $S = AA^T + \lambda I_{2n}$ where the entries of A are randomly selected in the range $[-20, 20]$. We also take $c_1(i) = c_2(i) = 1, i = 1, \ldots, n$. For 8 values of the pair (n, λ), namely $(n, \lambda) \in \{(1, 1), (10, 1), (100, 1), (1000, 1), (1, 100), (10, 100), (100, 100), (1000, 100)\}$, we generate a matrix S of form $AA^T + \lambda I_{2n}$ where the entries of A are realizations of independent random variables with uniform distribution in $[-20, 20]$. In each case, we select randomly $\bar{x} \in X$ and solve (3.54), (3.55) and its dual written for $x = \bar{x}$ using Mosek interior point solver. The value of $\alpha(\bar{x}) = \lambda_{\min}(S_3)$, the dual function $\theta_{\bar{x}}(\cdot)$, and the dual iterates computed along the iterations are reported in Figure 6 in the Appendix. Figure 7 shows the plots of $\eta_1(\varepsilon_k, \bar{x})$ and $\eta_2(\varepsilon_k, \bar{x})$ as a function of iteration k where ε_k is the duality gap at iteration k.

The cuts computed by Proposition 3.8 are more conservative than cuts given by Proposition 3.7 on nearly all instances and iterations. We also see that, as expected, the error terms $\eta_1(\varepsilon_k, \bar{x})$ and $\eta_2(\varepsilon_k, \bar{x})$ go to zero when ε_k goes to zero (see the proof of Theorem 4.2 for a proof of this statement).

4. Inexact Stochastic Mirror Descent for two-stage nonlinear stochastic programs

The algorithm to be described in this section is an inexact extension of SMD [13] to solve

$$\left\{ \begin{array}{l} \min \; f(x_1) := f_1(x_1) + \mathcal{Q}(x_1) \\ x_1 \in X_1 \end{array} \right. \tag{4.62}$$

with $X_1 \subset \mathbb{R}^n$ a convex, nonempty, and compact set, and $\mathcal{Q}(x_1) = \mathbb{E}_{\xi_2}[\mathfrak{Q}(x_1, \xi_2)]$, ξ_2 is a random vector with probability distribution P on $\Xi \subset \mathbb{R}^k$, and

$$\mathfrak{Q}(x_1, \xi_2) = \left\{ \begin{array}{l} \min_{x_2} \; f_2(x_2, x_1, \xi_2) \\ x_2 \in X_2(x_1, \xi_2) := \{x_2 \in \mathcal{X}_2 : \; Ax_2 + Bx_1 = b, \; g(x_2, x_1, \xi_2) \le 0\}. \end{array} \right. \tag{4.63}$$

Recall that ξ_2 contains the random variables in (A, B, b) and eventually other sources of randomness. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and let $\omega : X_1 \to \mathbb{R}$ be a *distance-generating function*. This function should

- be convex and continuous on X_1,
- admit on $X_1^o = \{x \in X_1 : \partial\omega(x) \neq \emptyset\}$ a selection $\omega'(x)$ of subgradients, and
- be compatible with $\|\cdot\|$, meaning that $\omega(\cdot)$ is strongly convex with constant of strong convexity $\mu(\omega) > 0$ with respect to the norm $\|\cdot\|$:

$$(\omega'(x) - \omega'(y))^T (x - y) \ge \mu(\omega) \|x - y\|^2 \;\forall x, y \in X_1^o.$$

We also define

(1) the *ω-center of X_1* given by $x_{1\omega} = \text{argmin}_{x_1 \in X_1} \omega(x_1) \in X_1^o$;

(2) the *Bregman distance* or prox-function

$$V_x(y) = \omega(y) - \omega(x) - (y-x)^T \omega'(x), \tag{4.64}$$

for $x \in X_1^o$, $y \in X_1$;

(3) the *ω-radius of* X_1 defined as

$$D_{\omega, X_1} = \sqrt{2\Big[\max_{x \in X_1} \omega(x) - \min_{x \in X_1} \omega(x)\Big]}. \tag{4.65}$$

(4) *The proximal mapping*

$$\text{Prox}_x(\zeta) = \text{argmin}_{\,y \in X_1} \{\omega(y) + y^T(\zeta - \omega'(x))\} \quad [x \in X_1^o, \zeta \in \mathbb{R}^n], \tag{4.66}$$

taking values in X_1^o.

We describe below ISMD, an inexact variant of SMD for solving problem (4.62) in which primal and dual second stage problems are solved approximately.

For $x_1 \in X_1$, $\xi_2 \in \Xi$, and $\varepsilon \geq 0$, we denote by $x_2(x_1, \xi_2, \varepsilon)$ an ε-optimal feasible primal solution of (4.63), i.e., $x_2(x_1, \xi_2, \varepsilon) \in X_2(x_1, \xi_2)$ and

$$\mathfrak{Q}(x_1, \xi_2) \leq f_2(x_2, x_1, \xi_2) \leq \mathfrak{Q}(x_1, \xi_2) + \varepsilon.$$

We now define ε-optimal dual second stage solutions. For $x_1 \in X_1$ and $\xi_2 \in \Xi$ let

$$L_{x_1, \xi_2}(x_2, \lambda, \mu) = f_2(x_2, x_1, \xi_2) + \langle \lambda, Ax_2 + Bx_1 - b \rangle + \langle \mu, g(x_2, x_1, \xi_2) \rangle,$$

and let θ_{x_1, ξ_2} be the dual function given by

$$\theta_{x_1, \xi_2}(\lambda, \mu) = \left\{ \begin{array}{l} \min \; L_{x_1, \xi_2}(x_2, \lambda, \mu) \\ x_2 \in \mathcal{X}_2. \end{array} \right. \tag{4.67}$$

For $x_1 \in X_1$, $\xi_2 \in \Xi$, and $\varepsilon \geq 0$, we denote by $(\lambda(x_1, \xi_2, \varepsilon), \mu(x_1, \xi_2, \varepsilon))$ an ε-optimal feasible solution of the dual problem

$$\left\{ \begin{array}{l} \max \; \theta_{x_1, \xi_2}(\lambda, \mu) \\ \mu \geq 0, \lambda = Ax_2 + Bx_1 - b, x_2 \in \text{Aff}(\mathcal{X}_2). \end{array} \right. \tag{4.68}$$

Under Slater-type constraint qualification conditions to be specified in Theorems 4.2 and 4.4, the optimal values of primal second stage problem (4.63) and dual second stage problem (4.68) are the same and $(\lambda(x_1, \xi_2, \varepsilon), \mu(x_1, \xi_2, \varepsilon))$ satisfies:

$$\mu(x_1, \xi_2, \varepsilon) \geq 0, \lambda(x_1, \xi_2, \varepsilon) = Ax_2 + Bx_1 - b,$$

for some $x_2 \in \text{Aff}(\mathcal{X}_2)$ and

$$\mathfrak{Q}(x_1, \xi_2) - \varepsilon \leq \theta_{x_1, \xi_2}(\lambda(x_1, \xi_2, \varepsilon), \mu(x_1, \xi_2, \varepsilon)) \leq \mathfrak{Q}(x_1, \xi_2).$$

We also denote by $D_{X_1} = \max_{x, y \in X_1} \|y - x\|$ the diameter of X_1, by $s_{f_1}(x_1)$ a subgradient of f_1 at x_1, and we define

(4.69)

$$\begin{array}{l} H(x_1, \xi_2, \varepsilon) = \nabla_{x_1} f_2(x_2(x_1, \xi_2, \varepsilon), x_1, \xi_2) + B^T \lambda(x_1, \xi_2, \varepsilon) + \sum_{i=1}^p \mu_i(x_1, \xi_2, \varepsilon) \nabla_{x_1} g_i(x_2(x_1, \xi_2, \varepsilon), x_1, \xi_2), \\ G(x_1, \xi_2, \varepsilon) = s_{f_1}(x_1) + H(x_1, \xi_2, \varepsilon). \end{array}$$

Inexact Stochastic Mirror Descent (ISMD) for risk-neutral two-stage nonlinear stochastic problems.

Parameters: Sequence (ε_t) and $\theta > 0$.

For $N = 2, 3, \ldots,$

Take $x_1^{N,1} = x_{1\omega}$.

For $t = 1, \ldots, N-1$, sample a realization $\xi_2^{N,t}$ of ξ_2 (with corresponding realizations $A^{N,t}$ of A, $B^{N,t}$ of B, and $b^{N,t}$ of b), compute an ε_t-optimal solution $x_2^{N,t}$ of the problem

$$\mathfrak{Q}(x_1^{N,t}, \xi_2^{N,t}) = \left\{ \begin{array}{l} \min_{x_2} \; f_2(x_2, x_1^{N,t}, \xi_2^{N,t}) \\ A^{N,t}x_2 + B^{N,t}x_1^{N,t} = b^{N,t}, \\ g(x_2, x_1^{N,t}, \xi_2^{N,t}) \leq 0, \\ x_2 \in \mathcal{X}_2, \end{array} \right. \tag{4.70}$$

and an ε_t-optimal solution $(\lambda^{N,t}, \mu^{N,t}) = (\lambda(x_1^{N,t}, \xi_2^{N,t}, \varepsilon_t), \mu(x_1^{N,t}, \xi_2^{N,t}, \varepsilon_t))$ of the dual problem

$$\left\{ \begin{array}{l} \max \; \theta_{x_1^{N,t}, \xi_2^{N,t}}(\lambda, \mu) \\ \mu \geq 0, \lambda = A^{N,t}x_2 + B^{N,t}x_1^{N,t} - b^{N,t}, x_2 \in \mathrm{Aff}(\mathcal{X}_2) \end{array} \right. \tag{4.71}$$

used to compute $G(x_1^{N,t}, \xi_2^{N,t}, \varepsilon_t)$ given by (4.69) replacing $(x_1, \xi_2, \varepsilon)$ by $(x_1^t, \xi_2^t, \varepsilon_t)$.[4]

Compute $\gamma_t(N) = \frac{\theta}{\sqrt{N}}$ and

$$x_1^{N,t+1} \quad = \quad \mathrm{Prox}_{x_1^{N,t}}(\gamma_t(N) G(x_1^{N,t}, \xi_2^{N,t}, \varepsilon_t)). \tag{4.72}$$

Compute

$$\begin{array}{l} x_1(N) = \dfrac{1}{\Gamma_N} \displaystyle\sum_{\tau=1}^{N} \gamma_\tau(N) x_1^{N,\tau} \text{ and} \\ \hat{f}_N = \dfrac{1}{\Gamma_N} \left[\displaystyle\sum_{\tau=1}^{N} \gamma_\tau(N) \Big(f_1(x_1^{N,\tau}) + f_2(x_2^{N,\tau}, x_1^{N,\tau}, \xi_2^{N,\tau}) \Big) \right] \text{ with } \Gamma_N = \displaystyle\sum_{\tau=1}^{N} \gamma_\tau(N). \end{array} \tag{4.73}$$

End For

End For

Remark 4.1. *In practise ISMD is run fixing the number N of inner iterations, i.e., we fix N and compute $x_1(N)$ and $\hat{f}_N$.*

Convergence of Inexact Stochastic Mirror Descent for solving (4.62) can be shown when error terms (ε_t) asymptotically vanish:

Theorem 4.2 (Convergence of ISMD)**.** *Consider problem* (4.62) *and assume that (i) X_1 and $\mathcal{X}_2$ are nonempty, convex, and compact, (ii) f_1 is convex, finite-valued, and has bounded subgradients on X_1, (iii) for every $x_1 \in X_1$ and $x_2 \in \mathcal{X}_2$, $f_2(x_2, x_1, \cdot)$ and $g_i(x_2, x_1, \cdot), i = 1, \ldots, p$, are measurable, (iv) for every $\xi_2 \in \Xi$ the functions $f_2(\cdot, \cdot, \xi_2)$ and $g_i(\cdot, \cdot, \xi_2), i = 1, \ldots, p$, are convex and continuously differentiable on $\mathcal{X}_2 \times X_1$, (v) $\exists \kappa > 0$ and $r > 0$ such that for all $x_1 \in X_1$, for all $\tilde{\xi}_2 \in \Xi$, there exists $x_2 \in \mathcal{X}_2$ such that $\mathbb{B}(x_2, r) \cap \mathit{Aff}(\mathcal{X}_2) \neq \emptyset$, $\tilde{A}x_2 + \tilde{B}x_1 = \tilde{b}$, and $g(x_2, x_1, \tilde{\xi}_2) < -\kappa e$ where e is a vector of ones. If $\gamma_t = \frac{\theta}{\sqrt{N}}$ for some $\theta > 0$, if the support Ξ of ξ_2 is compact, and if $\lim_{t \to \infty} \varepsilon_t = 0$, then*

$$\lim_{N \to +\infty} \mathbb{E}[f(x_1(N))] = \lim_{N \to +\infty} \mathbb{E}[\hat{f}_N] = f_{1*}$$

where f_{1*} is the optimal value of (4.62)*.*

Proof. For fixed N, to alleviate notation, we denote vectors $x_1^{N,t}, x_2^{N,t}, \xi_2^{N,t}, A^{N,t}, B^{N,t}, b^{N,t}, \gamma_t(N), \lambda^{N,t}, \mu^{N,t}$ used to compute $x_1(N)$ and $\hat{f}_N$ by $x_1^t, x_2^t, \xi_2^t, A^t, B^t, b^t, \gamma_t, \lambda^t, \mu^t$, respectively. Let x_1^* be an optimal solution of (4.62). Standard computations on the proximal mapping give

$$\sum_{\tau=1}^{N} \gamma_\tau G(x_1^\tau, \xi_2^\tau, \varepsilon_\tau)^T (x_1^\tau - x_1^*) \leq \frac{1}{2} D_{\omega, X_1}^2 + \frac{1}{2\mu(\omega)} \sum_{\tau=1}^{N} \gamma_\tau^2 \|G(x_1^\tau, \xi_2^\tau, \varepsilon_\tau)\|_*^2. \tag{4.74}$$

Next using Proposition 3.7 we have

$$\mathfrak{Q}(x_1^*, \xi_2^\tau) \geq \mathfrak{Q}(x_1^\tau, \xi_2^\tau) - \eta_{\xi_2^\tau}(\varepsilon_\tau, x_1^\tau) + \langle H(x_1^\tau, \xi_2^\tau, \varepsilon_\tau), x_1^* - x_1^\tau \rangle \tag{4.75}$$

[4]Any optimization solver for convex nonlinear programs able to provide ε_t-optimal solutions can be used (for instance an interior point solver).

where

$$
(4.76)\qquad \eta_{\xi_2^\tau}(\varepsilon_\tau, x_1^\tau) = \left\{\begin{array}{l}\max \langle \nabla_{x_2} L_{x_1^\tau,\xi_2^\tau}(x_2^\tau,\lambda^\tau,\mu^\tau), x_2^\tau - x_2\rangle\\ x_2 \in \mathcal{X}_2\end{array}\right. = \left\{\begin{array}{l}\max \langle \nabla_{x_2} f_2(x_2^\tau,x_1^\tau,\xi_2^\tau) + (A^\tau)^T\lambda^\tau + \sum_{i=1}^p \mu_i^\tau \nabla_{x_2} g_i(x_2^\tau,x_1^\tau,\xi_2^\tau), x_2^\tau - x_2\rangle\\ x_2 \in \mathcal{X}_2.\end{array}\right.
$$

Setting $\xi_2^{1:\tau-1} = (\xi_2^1,\ldots,\xi_2^{\tau-1})$ and taking the conditional expectation $\mathbb{E}_{\xi_2^\tau}[\cdot|\xi_2^{1:\tau-1}]$ on each side of (4.75) we obtain almost surely

$$
(4.77)\qquad \mathcal{Q}(x_1^*) \ge \mathcal{Q}(x_1^\tau) - \mathbb{E}_{\xi_2^\tau}[\eta_{\xi_2^\tau}(\varepsilon_\tau,x_1^\tau)|\xi_2^{1:\tau-1}] + (\mathbb{E}_{\xi_2^\tau}[H(x_1^\tau,\xi_2^\tau,\varepsilon_\tau)|\xi_2^{1:\tau-1}])^T(x_1^*-x_1^\tau).
$$

Combining (4.74), (4.77), and using the convexity of f we get

$$
(4.78)\qquad \begin{array}{l} 0 \le \mathbb{E}[f(x_1(N)) - f(x_1^*)] \le \dfrac{1}{\Gamma_N}\displaystyle\sum_{\tau=1}^N \gamma_\tau \mathbb{E}[f(x_1^\tau) - f(x_1^*)]\\ \le \dfrac{1}{\Gamma_N}\displaystyle\sum_{\tau=1}^N \gamma_\tau \mathbb{E}[\eta_{\xi_2^\tau}(\varepsilon_\tau,x_1^\tau)] + \dfrac{1}{2\Gamma_N}\Big[D^2_{\omega,X_1} + \dfrac{1}{\mu(\omega)}\displaystyle\sum_{\tau=1}^N \gamma_\tau^2 \mathbb{E}[\|G(x_1^\tau,\xi_2^\tau,\varepsilon_\tau)\|_*^2]\Big].\end{array}
$$

We now show by contradiction that[5]

$$
(4.79)\qquad \lim_{\tau\to+\infty} \eta_{\xi_2^\tau}(\varepsilon_\tau,x_1^\tau) = 0 \text{ almost surely.}
$$

Take an arbitrary realization of ISMD. We want to show that

$$
(4.80)\qquad \lim_{\tau\to+\infty} \eta_{\xi_2^\tau}(\varepsilon_\tau,x_1^\tau) = 0
$$

for that realization. Assume that (4.80) does not hold. Let x_{2*}^t (resp. $\tilde{x}_2^\tau$) be an optimal solution of (4.70) (resp. (4.76)). Then there is $\varepsilon_0 > 0$ and $\sigma_1 : \mathbb{N} \to \mathbb{N}$ increasing such that for every $\tau \in \mathbb{N}$, we have

(4.81)

$$
\langle \nabla_{x_2} f_2(x_2^{\sigma_1(\tau)}, x_1^{\sigma_1(\tau)}, \xi_2^{\sigma_1(\tau)}) + (A^{\sigma_1(\tau)})^T\lambda^{\sigma_1(\tau)} + \sum_{i=1}^p \mu_i^{\sigma_1(\tau)} \nabla_{x_2} g_i(x_2^{\sigma_1(\tau)}, x_1^{\sigma_1(\tau)}, \xi_2^{\sigma_1(\tau)}), x_2^{\sigma_1(\tau)} - \tilde{x}_2^{\sigma_1(\tau)}\rangle \ge \varepsilon_0.
$$

By ε_t-optimality of x_2^t we obtain

$$
(4.82)\qquad f_2(x_{2*}^t, x_1^t, \xi_2^t) \le f_2(x_2^t, x_1^t, \xi_2^t) \le f_2(x_{2*}^t, x_1^t, \xi_2^t) + \varepsilon_t.
$$

Using Assumptions (i), (iii), (iv), and Proposition 3.1 in [6] we get that the sequence $(\lambda^\tau,\mu^\tau)_\tau$ is almost surely bounded. Let $\mathcal{D}$ be a compact set to which this sequence belongs. By compacity, we can find $\sigma_2 : \mathbb{N} \to \mathbb{N}$ increasing such that setting $\sigma = \sigma_1 \circ \sigma_2$ the sequence $(x_2^{\sigma(\tau)}, x_1^{\sigma(\tau)}, \lambda^{\sigma(\tau)}, \mu^{\sigma(\tau)}, \xi_2^{\sigma(\tau)})$ converges to some $(\bar{x}_2, x_{1*}, \lambda_*, \mu_*, \xi_{2*}) \in \mathcal{X}_2 \times X_1 \times \mathcal{D} \times \Xi$. We will denote by A_*, B_*, b_* the values of A, B, and b in ξ_{2*}. By continuity arguments there is $\tau_0 \in \mathbb{N}$ such that for every $\tau \ge \tau_0$:

$$
(4.83)\qquad \begin{array}{l}\Big|\langle \nabla_{x_2} f_2(x_2^{\sigma(\tau)}, x_1^{\sigma(\tau)}, \xi_2^{\sigma(\tau)}) + (A^{\sigma(\tau)})^T\lambda^{\sigma(\tau)} + \sum_{i=1}^p \mu_i^{\sigma(\tau)} \nabla_{x_2} g_i(x_2^{\sigma(\tau)}, x_1^{\sigma(\tau)}, \xi_2^{\sigma(\tau)}), x_2^{\sigma(\tau)} - \tilde{x}_2^{\sigma(\tau)}\rangle\\ - \langle \nabla_{x_2} f_2(\bar{x}_2, x_{1*}, \xi_{2*}) + A_*^T\lambda_* + \sum_{i=1}^p \mu_*(i)\nabla_{x_2} g_i(\bar{x}_2, x_{1*}, \xi_{2*}), \bar{x}_2 - \tilde{x}_2^{\sigma(\tau)}\rangle\Big| \le \varepsilon_0/2.\end{array}
$$

We deduce from (4.81) and (4.83) that for all $\tau \ge \tau_0$

$$
(4.84)\qquad \left\langle \nabla_{x_2} f_2(\bar{x}_2, x_{1*}, \xi_{2*}) + A_*^T\lambda_* + \sum_{i=1}^p \mu_*(i)\nabla_{x_2} g_i(\bar{x}_2, x_{1*}, \xi_{2*}), \bar{x}_2 - \tilde{x}_2^{\sigma(\tau)} \right\rangle \ge \varepsilon_0/2 > 0.
$$

Assumptions (i)-(iv) imply that primal problem (4.70) and dual problem (4.71) have the same optimal value and for every $x_2 \in \mathcal{X}_2$ and $\tau \ge \tau_0$ we have:

$$
\begin{array}{l}
f_2(x_2^{\sigma(\tau)}, x_1^{\sigma(\tau)}, \xi_2^{\sigma(\tau)}) + \langle A^{\sigma(\tau)}x_2^{\sigma(\tau)} + B^{\sigma(\tau)}x_1^{\sigma(\tau)} - b^{\sigma(\tau)}, \lambda^{\sigma(\tau)}\rangle + \langle \mu^{\sigma(\tau)}, g(x_2^{\sigma(\tau)}, x_1^{\sigma(\tau)}, \xi_2^{\sigma(\tau)})\rangle\\
\le f_2(x_{2*}^{\sigma(\tau)}, x_1^{\sigma(\tau)}, \xi_2^{\sigma(\tau)}) + \varepsilon_{\sigma(\tau)} \text{ by definition of } x_{2*}^{\sigma(\tau)}, x_2^{\sigma(\tau)} \text{ and since } \mu^{\sigma(\tau)} \ge 0, x_2^{\sigma(\tau)} \in X_2(x_1^{\sigma(\tau)}, \xi_2^{\sigma(\tau)}),\\
\le \theta_{x_1^{\sigma(\tau)},\xi_2^{\sigma(\tau)}}(\lambda^{\sigma(\tau)},\mu^{\sigma(\tau)}) + 2\varepsilon_{\sigma(\tau)}, [(\lambda^{\sigma(\tau)},\mu^{\sigma(\tau)}) \text{ is an } \epsilon_{\sigma(\tau)}\text{-optimal dual solution and there is no duality gap}],\\
\le f_2(x_2, x_1^{\sigma(\tau)}, \xi_2^{\sigma(\tau)}) + \langle A^{\sigma(\tau)}x_2 + B^{\sigma(\tau)}x_1^{\sigma(\tau)} - b^{\sigma(\tau)}, \lambda^{\sigma(\tau)}\rangle + \langle \mu^{\sigma(\tau)}, g(x_2, x_1^{\sigma(\tau)}, \xi_2^{\sigma(\tau)})\rangle + 2\varepsilon_{\sigma(\tau)}
\end{array}
$$

[5]The proof is similar to the proof of Proposition 4.6 in [6].

where in the last relation we have used the definition of $\theta_{x_1^{\sigma(\tau)}, \xi_2^{\sigma(\tau)}}$. Taking the limit in the above relation as $\tau \to +\infty$, we get for every $x_2 \in \mathcal{X}_2$:

$$\begin{array}{l} f_2(\bar{x}_2, x_{1*}, \xi_{2*}) + \langle A_* \bar{x}_2 + B_* x_{1*} - b_*, \lambda_* \rangle + \langle \mu_*, g(\bar{x}_2, x_{1*}, \xi_{2*}) \rangle \\ \leq f_2(x_2, x_{1*}, \xi_{2*}) + \langle A_* x_2 + B_* x_{1*} - b_*, \lambda_* \rangle + \langle \mu_*, g(x_2, x_{1*}, \xi_{2*}) \rangle. \end{array}$$

Recalling that $\bar{x}_2 \in \mathcal{X}_2$ this shows that $\bar{x}_2$ is an optimal solution of

$$\left\{ \begin{array}{l} \min f_2(x_2, x_{1*}, \xi_{2*}) + \langle A_* x_2 + B_* x_{1*} - b_*, \lambda_* \rangle + \langle \mu_*, g(x_2, x_{1*}, \xi_{2*}) \rangle \\ x_2 \in \mathcal{X}_2. \end{array} \right. \tag{4.85}$$

The first order optimality conditions for $\bar{x}_2$ can be written

$$\left\langle \nabla_{x_2} f_2(\bar{x}_2, x_{1*}, \xi_{2*}) + A_*^T \lambda_* + \sum_{i=1}^{p} \mu_*(i) \nabla_{x_2} g_i(\bar{x}_2, x_{1*}, \xi_{2*}), x_2 - \bar{x}_2 \right\rangle \geq 0 \tag{4.86}$$

for all $x_2 \in \mathcal{X}_2$. Specializing the above relation for $x_2 = \tilde{x}_2^{\sigma(\tau_0)} \in \mathcal{X}_2$, we get

$$\left\langle \nabla_{x_2} f_2(\bar{x}_2, x_{1*}, \xi_{2*}) + A_*^T \lambda_* + \sum_{i=1}^{p} \mu_*(i) \nabla_{x_2} g_i(\bar{x}_2, x_{1*}, \xi_{2*}), \tilde{x}_2^{\sigma(\tau_0)} - \bar{x}_2 \right\rangle \geq 0,$$

but the left-hand side of the above inequality is $\leq -\varepsilon_0/2 < 0$ due to (4.84) which yields the desired contradiction. Therefore we have shown (4.79) and since the sequence $\eta_{\xi_2^\tau}(\varepsilon_\tau, x_1^\tau)$ is almost surely bounded, this implies $\lim_{\tau \to +\infty} \mathbb{E}[\eta_{\xi_2^\tau}(\varepsilon_\tau, x_1^\tau)] = 0$ and consequently $\lim_{N \to +\infty} \frac{1}{\Gamma_N} \sum_{\tau=1}^{N} \gamma_\tau \mathbb{E}[\eta_{\xi_2^\tau}(\varepsilon_\tau, x_1^\tau)] = 0$. Using the boundedness of the sequence (λ^t, μ^t) and Assumption (ii) we get that $\|G(x_1^\tau, \xi_2^\tau, \varepsilon_\tau)\|_*^2$ is almost surely bounded. Combining these observations with relation (4.78) and using the definition of γ_t we have $\lim_{N \to +\infty} \mathbb{E}[f(x_1(N))] = f_{1*}$. Finally, recalling relation (4.78), to show $\lim_{N \to +\infty} \mathbb{E}[\hat{f}_N] = f_{1*}$ all we have to show is

$$\lim_{N \to +\infty} \frac{1}{\Gamma_N} \sum_{\tau=1}^{N} \gamma_\tau \mathbb{E}[\mathcal{Q}(x_1^\tau) - f_2(x_2^\tau, x_1^\tau, \xi_2^\tau)] = 0. \tag{4.87}$$

The above relation immediately follows from

$$\mathbb{E}[\mathcal{Q}(x_1^\tau)] = \mathbb{E}_{\xi_2^{1:\tau-1}}[\mathcal{Q}(x_1^\tau)] = \mathbb{E}_{\xi_2^{1:\tau-1}}[\mathbb{E}_{\xi_2^\tau}[\mathfrak{Q}(x_1^\tau, \xi_2^\tau) | \xi_2^{1:\tau-1}]] \leq \mathbb{E}_{\xi_2^{1:\tau}}[f_2(x_2^\tau, x_1^\tau, \xi_2^\tau)] \leq \mathbb{E}[\mathcal{Q}(x_1^\tau)] + \varepsilon_\tau \tag{4.88}$$

which holds since $\mathfrak{Q}(x_1^\tau, \xi_2^\tau) \leq f_2(x_2^\tau, x_1^\tau, \xi_2^\tau) \leq \mathfrak{Q}(x_1^\tau, \xi_2^\tau) + \varepsilon_\tau$ by definition of x_2^τ. ☐

Remark 4.3. *Output $\hat{f}_N$ of ISMD is a computable approximation of the optimal value f_{1*} of optimization problem* (4.62).

Theorem 4.4. *[Convergence rate for ISMD] Consider problem* (4.62) *and assume that Assumptions (i)-(iv) of Theorem 4.2 are satisfied. We alse make the following assumptions:*

(a) $\exists \alpha > 0$ *such that for every $\xi_2 \in \Xi$, for every $x_1 \in X_1$, for every $y_1, y_2 \in \mathcal{X}_2$ we have*

$$f_2(y_2, x_1, \xi_2) \geq f_2(y_1, x_1, \xi_2) + (y_2 - y_1)^T \nabla_{x_2} f_2(y_1, x_1, \xi_2) + \frac{\alpha}{2} \|y_2 - y_1\|_2^2;$$

(b) *there is $0 < M_1 < +\infty$ such that for every $\xi_2 \in \Xi$, for every $x_1 \in X_1$, for every $y_1, y_2 \in \mathcal{X}_2$ we have*

$$\|\nabla_{x_1} f_2(y_2, x_1, \xi_2) - \nabla_{x_1} f_2(y_1, x_1, \xi_2)\|_2 \leq M_1 \|y_2 - y_1\|_2;$$

(c) *there is $0 < M_2 < +\infty$ such that for every $\xi_2 \in \Xi$, for every $x_1 \in X_1$, for every $i = 1, \ldots, p$, for every $y_1, y_2 \in \mathcal{X}_2$, we have*

$$\|\nabla_{x_1} g_i(y_2, x_1, \xi_2) - \nabla_{x_1} g_i(y_1, x_1, \xi_2)\|_2 \leq M_2 \|y_2 - y_1\|_2;$$

(d) $\exists \alpha_D > 0$ *such that for every $x_1 \in X_1$, for every $\xi_2 \in \Xi$, dual function θ_{x_1, ξ_2} given by* (4.67) *is strongly concave on D_{x_1, ξ_2} with constant of strong concavity α_D where D_{x_1, ξ_2} is a set containing the set of solutions of second stage dual problem* (4.68) *such that $(\lambda^t, \mu^t) \in D_{x_1^t, \xi_2^t}$.*

(e) *There are functions G_0, M_0 such that for every $x_1 \in X_1$, for every $x_2 \in \mathcal{X}_2$ we have*

$\max(\|B^T\|, \sqrt{p} \max_{i=1,\ldots,p} \|\nabla_{x_1} g_i(x_2, x_1, \xi_2)\|_2) \leq G_0(\xi_2)$ *and* $\|\nabla_{x_1} f_2(x_2, x_1, \xi_2)\|_2 \leq M_0(\xi_2)$

with $\mathbb{E}[G_0(\xi_2)]$ and $\mathbb{E}[M_0(\xi_2)]$ finite;

(f) *There are functions* $\overline{f}_2, \underline{f}_2$ *such that for all* $x_1 \in X_1, x_2 \in \mathcal{X}_2$ *we have*

$$\underline{f}_2(\xi_2) \le f_2(x_2, x_1, \xi_2) \le \overline{f}_2(\xi_2)$$

with $\mathbb{E}[\overline{f}_2(\xi_2)]$ *and* $\mathbb{E}[\underline{f}_2(\xi_2)]$ *finite.*

(g) *There exists* $0 < L(f_2) < +\infty$ *such that for every* $\xi_2 \in \Xi$*, for every* $x_1 \in X_1$*, function* $f_2(\cdot, x_1, \xi_2)$ *is Lipschitz continuous with Lipschitz constant* $L(f_2)$*.*

Let $\mathcal{A}$ *be a compact set such that matrix* A *in* ξ_2 *almost surely belongs to* $\mathcal{A}$ *and let* $M_3 < +\infty$ *such that* $\|s_{f_1}(x_1)\|_2 \le M_3$ *for all* $x_1 \in X_1$*. Let* $V_{\mathcal{X}_2}$ *be the vector space* $V_{\mathcal{X}_2} = \{x - y : x, y \in \text{Aff}(\mathcal{X}_2)\}$*. Define the functions* ρ *and* ρ_* *by*

$$\rho(A, z) = \left\{ \begin{array}{l} \max \ t\|z\| \\ t \ge 0, \ tz \in A(\mathbb{B}(0, r) \cap V_{\mathcal{X}_2}), \end{array} \right. \quad \rho_*(A) = \left\{ \begin{array}{l} \min \ \rho(A, z) \\ \|z\| = 1, \ z \in AV_{\mathcal{X}_2}. \end{array} \right.$$

Assume that $\gamma_t = \frac{\theta_1}{\sqrt{N}}$ *and* $\varepsilon_t = \frac{\theta_2}{t^2}$ *for some* $\theta_1, \theta_2 > 0$*. Let*

$$\left\{ \begin{array}{lcl} \mathcal{U}_1 & = & (\mathbb{E}[\overline{f}_2(\xi_2)] - \mathbb{E}[\underline{f}_2(\xi_2)])/\kappa, \\ \mathcal{U}_2(r, \xi_2) & = & \frac{\overline{f}_2(\xi_2) - \underline{f}_2(\xi_2) + \theta_2 + L(f_2) r}{\min(\rho_*, \kappa/2)} \text{ with } \rho_* = \min_{A \in \mathcal{A}} \rho_*(A), \\ \overline{\mathcal{U}} & = & \Big((M_1 + M_2 \mathcal{U}_1) \sqrt{\frac{2}{\alpha}} + \frac{2\mathbb{E}[G_0(\xi_2)]}{\sqrt{\alpha_D}} \Big) \text{Diam}(\mathcal{X}_2), \\ M_*(r) & = & \sqrt{\mathbb{E}(M_3 + M_0(\xi_2) + \sqrt{2}\mathcal{U}_2(r, \xi_2) G_0(\xi_2))^2}. \end{array} \right.$$

Let $\hat{f}_N$ *computed by ISMD. Then there is* $r_0 > 0$ *such that*

$$f_{1*} \le \mathbb{E}[\hat{f}_N] \le f_{1*} + \frac{2\theta_2 + \overline{\mathcal{U}}\sqrt{\theta_2}}{N} + \frac{\overline{\mathcal{U}}\sqrt{\theta_2}\ln(N)}{N} + \frac{\frac{D^2_{\omega, X_1}}{\theta_1} + \frac{\theta_1 M_*^2(r_0)}{\mu(\omega)}}{2\sqrt{N}} \tag{4.89}$$

where f_{1*} *is the optimal value of* (4.62)*.*

Proof. Let x_1^* be an optimal solution of (4.62). Under our assumptions, we can apply Proposition 3.8 to value function $\mathfrak{Q}(\cdot, \xi_2^t)$ and $\bar{x} = x_1^t$, which gives

$$\mathfrak{Q}(x_1^*, \xi_2^t) \ge f_2(x_2^t, x_1^t, \xi_2^t) + \langle H(x_1^t, \xi_2^t, \varepsilon_t), x_1^* - x_1^t \rangle - \eta_{\xi_2^t}(\varepsilon_t, x_1^t), \tag{4.90}$$

where

$$\begin{array}{lcl} \eta_{\xi_2^t}(\varepsilon_t, x_1^t) & = & \varepsilon_t + \Big(M_1 + \frac{M_2}{\kappa}(f_2(\bar{x}_2^t, x_1^t, \xi_2^t) - \underline{f}_2(\xi_2^t)) \Big) \sqrt{\frac{2\varepsilon_t}{\alpha}} \text{Diam}(\mathcal{X}_2), \\ & & +2 \max \Big(\|(B^t)^T\|, \sqrt{p} \ \max_{i=1,\ldots,p} \|\nabla_{x_1} g_i(x_2^t, x_1^t, \xi_2^t)\|_2 \Big) \text{Diam}(\mathcal{X}_2) \sqrt{\frac{\varepsilon_t}{\alpha_D}}, \end{array}$$

for some $\bar{x}_2^t \in \mathcal{X}_2$ depending on $\xi_2^{1:t}$. Taking the conditional expectation $\mathbb{E}_{\xi_2^t}[\cdot | \xi_2^{1:t-1}]$ in (4.90) and using (e)-(f), we get

$$\mathcal{Q}(x_1^*) \ge \mathbb{E}_{\xi_2^t}[f_2(x_2^t, x_1^t, \xi_2^t) | \xi_2^{1:t-1}] + \mathbb{E}_{\xi_2^t}[\langle H(x_1^t, \xi_2^t, \varepsilon_t), x_1^* - x_1^t \rangle | \xi_2^{1:t-1}] - (\varepsilon_t + \overline{\mathcal{U}}\sqrt{\varepsilon_t}). \tag{4.91}$$

Summing (4.91) with the relation

$$f_1(x_1^*) \ge f_1(x_1^t) + \langle s_{f_1}(x_1^t), x_1^* - x_1^t \rangle$$

and taking the expectation operator $\mathbb{E}_{\xi_2^{1:t-1}}[\cdot]$ on each side of the resulting inequality gives

$$f(x_1^*) \ge \mathbb{E}[f_2(x_2^t, x_1^t, \xi_2^t) + f_1(x_1^t)] + \mathbb{E}[\langle G(x_1^t, \xi_2^t, \varepsilon_t), x_1^* - x_1^t \rangle] - (\varepsilon_t + \overline{\mathcal{U}}\sqrt{\varepsilon_t}). \tag{4.92}$$

From (4.92), we deduce

$$\mathbb{E}[\hat{f}_N - f_{1*}] \le \frac{1}{\Gamma_N} \sum_{t=1}^{N} \gamma_t (\varepsilon_t + \overline{\mathcal{U}}\sqrt{\varepsilon_t}) + \frac{1}{\Gamma_N} \sum_{t=1}^{N} \gamma_t \mathbb{E}[\langle G(x_1^t, \xi_2^t, \varepsilon_t), x_1^t - x_1^* \rangle]. \tag{4.93}$$

Using Proposition 3.1 in [6] and our assumptions, we can find $r_0 > 0$ such that $M_*^2(r_0)$ is an upper bound for $\mathbb{E}[\|G(x_1^t, \xi_2^t, \varepsilon_t)\|_*^2]$. Using this observation, (4.93), and (4.90) (which still holds), we get

$$\begin{array}{lll}\mathbb{E}[\hat{f}_N - f_{1*}] & \leq & \frac{1}{N}\Big(\theta_2\Big(1+\int_1^N \frac{dx}{x^2}\Big) + \overline{\mathcal{U}}\sqrt{\theta_2}\Big(1+\int_1^N \frac{dx}{x}\Big)\Big) + \frac{1}{2\theta_1\sqrt{N}}\Big(D_{\omega,X_1}^2 + \frac{M_*^2(r_0)\theta_1^2}{\mu(\omega)}\Big)\\ & \leq & \frac{2\theta_2 + \overline{\mathcal{U}}\sqrt{\theta_2}}{N} + \frac{\overline{\mathcal{U}}\sqrt{\theta_2}\ln(N)}{N} + \frac{\frac{D_{\omega,X_1}^2}{\theta_1} + \frac{\theta_1 M_*^2(r_0)}{\mu(\omega)}}{2\sqrt{N}}.\end{array} \tag{4.94}$$

Finally

$$\begin{array}{lll} 0 & \stackrel{(4.78)}{\leq} & \frac{1}{\Gamma_N}\sum_{\tau=1}^N \gamma_\tau \mathbb{E}[f(x_1^\tau)] - f_{1*}\\ & = & \frac{1}{\Gamma_N}\sum_{\tau=1}^N \gamma_\tau \mathbb{E}[f_1(x_1^\tau) + \mathcal{Q}(x_1^\tau)] - f_{1*}\\ & \stackrel{(4.88)}{\leq} & \frac{1}{\Gamma_N}\sum_{\tau=1}^N \gamma_\tau \mathbb{E}[f_1(x_1^\tau) + f_2(x_2^\tau, x_1^\tau, \xi_2^\tau)] - f_{1*} = \mathbb{E}[\hat{f}_N - f_{1*}].\end{array} \tag{4.95}$$

Combining (4.94) and (4.95) we obtain (4.89). □

5. Numerical experiments

We compare the performances of SMD, ISMD, SAA (Sample Average Approximation, see [18]), and the L-shaped method (see [2]) on two simple two-stage quadratic stochastic programs which satisfy the assumptions of Theorems 4.2 and 4.4.

The first two-stage program is

$$\left\{\begin{array}{l}\min\ c^T x_1 + \mathbb{E}[\mathfrak{Q}(x_1, \xi_2)]\\ x_1 \in \{x_1 \in \mathbb{R}^n : x_1 \geq 0, \sum_{i=1}^n x_1(i) = 1\}\end{array}\right. \tag{5.96}$$

where the second stage recourse function is given by

$$\mathfrak{Q}(x_1, \xi_2) = \left\{\begin{array}{l}\min\limits_{x_2 \in \mathbb{R}^n} \frac{1}{2}\begin{pmatrix} x_1 \\ x_2\end{pmatrix}^T \Big(\xi_2 \xi_2^T + \lambda I_{2n}\Big)\begin{pmatrix} x_1 \\ x_2\end{pmatrix} + \xi_2^T \begin{pmatrix} x_1 \\ x_2\end{pmatrix}\\ x_2 \geq 0, \sum\limits_{i=1}^n x_2(i) = 1.\end{array}\right. \tag{5.97}$$

The second two-stage program is

$$\left\{\begin{array}{l}\min\ c^T x_1 + \mathbb{E}[\mathfrak{Q}(x_1, \xi_2)]\\ x_1 \in \{x_1 \in \mathbb{R}^n : \|x_1 - x_0\|_2 \leq 1\}\end{array}\right. \tag{5.98}$$

where cost-to-go function $\mathfrak{Q}(x_1, \xi_2)$ has nonlinear objective and constraint coupling functions and is given by

$$\mathfrak{Q}(x_1, \xi_2) = \left\{\begin{array}{l}\min\limits_{x_2 \in \mathbb{R}^n} \frac{1}{2}\begin{pmatrix} x_1 \\ x_2\end{pmatrix}^T \Big(\xi_2 \xi_2^T + \lambda I_{2n}\Big)\begin{pmatrix} x_1 \\ x_2\end{pmatrix} + \xi_2^T \begin{pmatrix} x_1 \\ x_2\end{pmatrix}\\ \frac{1}{2}\|x_2 - y_0\|_2^2 + \frac{1}{2}\|x_1 - x_0\|_2^2 - \frac{R^2}{2} \leq 0.\end{array}\right. \tag{5.99}$$

For both problems, ξ_2 is a Gaussian random vector in $\mathbb{R}^{2n}$ and $\lambda > 0$. We consider several instances of these problem with $n = 5$, 10, 200, 400, and $n = 600$. For each instance, the components of ξ_2 are independent with means and standard deviations randomly generated in respectively intervals $[5, 25]$ and $[5, 15]$. We fix $\lambda = 2$ while the components of c are generated randomly in interval $[1, 3]$. For problem (5.98)-(5.99) we also take $R = 5$ and $x_0(i) = y_0(i) = 10, i = 1, \ldots, n$.

In SMD and ISMD, we take $\omega(x) = \sum_{i=1}^n x_i \ln(x_i)$ for problem (5.96)-(5.97). For this distance generating function, $x_+ = \text{Prox}_x(\zeta)$ can be computed analytically for $x \in \mathbb{R}^n$ with $x > 0$ (see [13, 5] for details): defining $z \in \mathbb{R}^n$ by $z(i) = \ln(x(i))$ we have $x_+(i) = \exp(z_+(i))$ where

$$z_+ = w - \ln\left(\sum_{i=1}^n e^{w(i)}\right)\mathbf{1} \text{ with } w = z - \zeta - \max_i[z(i) - \zeta(i)],$$

n	N	Problem	`L-shaped`	`SAA`	`SMD`
5	20 000	(5.96)	57.3	3 698.7	18.5
5	20 000	(5.98)	53.1	3 943.8	22.7
10	20 000	(5.96)	278.1	3.32×10^5	28.2
10	20 000	(5.98)	70.5	4 126.5	33.4

Table 2. CPU time in seconds required to solve instances of problems (5.96)-(5.97) and (5.98)-(5.99) (for $n = 5, 10$ and $N = 20\ 000$) obtained with the L-shaped method, SAA, and SMD.

and with $\mathbf{1}$ a vector in $\mathbb{R}^n$ of ones.

For problem (5.98)-(5.99), SMD and ISMD are run taking distance generating function $\omega(x) = \frac{1}{2}\|x\|_2^2$ (in this case, SMD is just the Robust Stochastic Approximation). For this choice of ω, if $x_+ = \text{Prox}_x(\zeta)$ we have

$$x_+ = \left\{ \begin{array}{ll} x - \zeta & \text{if } \|x - \zeta - x_0\|_2 \leq 1, \\ x_0 + \frac{x-\zeta-x_0}{\|x-\zeta-x_0\|_2} & \text{otherwise.} \end{array} \right.$$

In SMD and ISMD, the interior point solver of the Mosek Optimization Toolbox [1] is used at each iteration to solve the quadratic second stage problem (given first stage decision x_1^t and realization ξ_2^t of ξ_2 at iteration t) and constant steps are used: if there are N iterations, the step γ_t for iteration t is $\gamma_t = \frac{1}{\sqrt{N}}$. For ISMD, we limit the number of iterations of Mosek solver used to solve subproblems.[6] More precisely, we consider four strategies for the limitation of these numbers of iterations given in Table 5 in the Appendix, which define four variants of ISMD denoted by ISMD 1, ISMD 2, ISMD 3, and ISMD 4. The variants that most limit the number of iterations are ISMD 1 and ISMD 2. All methods were implemented in Matlab and run on an Intel Core i7, 1.8GHz, processor with 12,0 Go of RAM.

To check the implementations and compare the accuracy and CPU time of all methods, we first consider problems (5.96)-(5.97) and (5.98)-(5.99) with $n = 5, 10$, and a large sample of size $N = 20\ 000$ of ξ_2.[7] In these experiments, the L-shaped method terminates when the relative error is at most 5%. The CPU time needed to solve these instances with the L-shaped method, SAA, and SMD are given in Table 2. For these instances, we also report in Table 3 the approximate optimal values given by all methods knowing that for the L-shaped method we report the value of the last upper bound computed. For SMD, the approximate optimal value after N iterations is given by $\hat{f}_N$. On the four experiments, all methods give very close approximations of the optimal value, which is a good indication that the methods were well implemented. SMD is by far the quickest and SAA by far the slowest. For the instance of Problem (5.96)-(5.97) with $n = 10$, we report in the left plot of Figure 1 the evolution of the approximate optimal value along the iterations of SMD.[8] We also report on the right plot of this figure the evolution of the upper and lower bounds computed along the iterations of the L-shaped method for the instance of Problem (5.96)-(5.97) with $n = 10$. For problem (5.98)-(5.99), the evolution of the approximate optimal value along the iterations of SMD is represented in Figure 2. Observe that with SMD the approximate optimal value is not the value of the objective function at a feasible point and therefore some of these approximations can be below the optimal value of the problem.

We now consider larger instances taking $n = 200$, 400, and 600. For these simulations we do not use SAA and L-shaped method anymore which were not as efficient as SMD on previous simulations and require prohibitive computational time for $n = 200, 400, 600$, and we compare the performance of SMD and the four variants ISMD 1, ISMD 2, ISMD 3, and ISMD 4 of ISMD defined above.

[6] According to current Mosek documentation, it is not possible to use absolute errors. Therefore, early termination of the solver can either be obtained limiting the number of iterations or defining relative errors.

[7] The deterministic equivalents of these instances are already large size quadratic programs. For instance, for $n = 10$, the deterministic equivalent of Problem (5.98)-(5.99) is a quadratically constrained quadratic program with 200 010 variables and 20 0001 quadratic constraints.

[8] Naturally, after running $t-1$ of the $N-1$ total iterations, the approximate optimal value computed by SMD is $\frac{1}{\sum_{\tau=1}^t \gamma_\tau(N)} \sum_{\tau=1}^t \gamma_\tau(N)\Big(f_1(x_1^{N,\tau}) + f_2(x_2^{N,\tau}, x_1^{N,\tau}, \xi_2^{N,\tau})\Big)$ obtained on the basis of sample $\xi_2^{N,1}, \ldots, \xi_2^{N,t}$ of ξ_2.

n	N	Problem	`L-shaped`	`SAA`	`SMD`
5	20 000	(5.96)	210.9	210.7	210.6
5	20 000	(5.98)	$1.122{\times}10^6$	$1.121{\times}10^6$	$1.120{\times}10^6$
10	20 000	(5.96)	78.8	78.9	78.6
10	20 000	(5.98)	$3.020{\times}10^6$	$3.016{\times}10^6$	$3.015{\times}10^6$

TABLE 3. Approximate optimal value of instances of problems (5.96)-(5.97) and (5.98)-(5.99) (for $n = 5, 10$ and $N = 20\ 000$) obtained with the L-shaped method, SAA, and SMD.

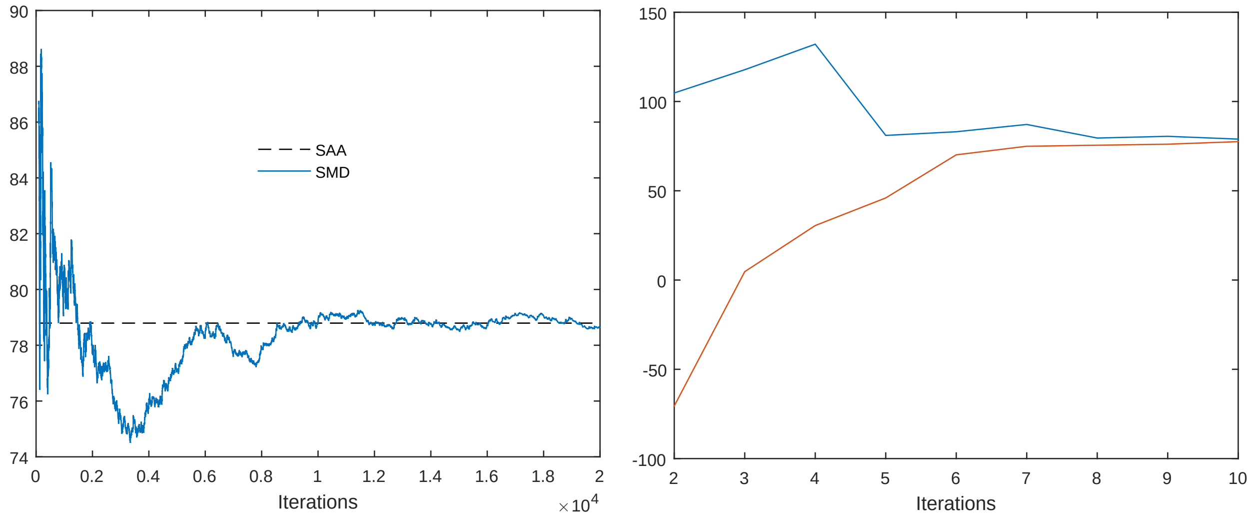


FIGURE 1. Left plot: optimal value of our instance of Problem (5.96)-(5.97) with $n = 10$ estimated using SAA as well as evolution of the approximate optimal value computed along the iterations of SMD. Right plot: for the same instance, evolution of the lower and upper bounds computed along the iterations of the L-shaped method.

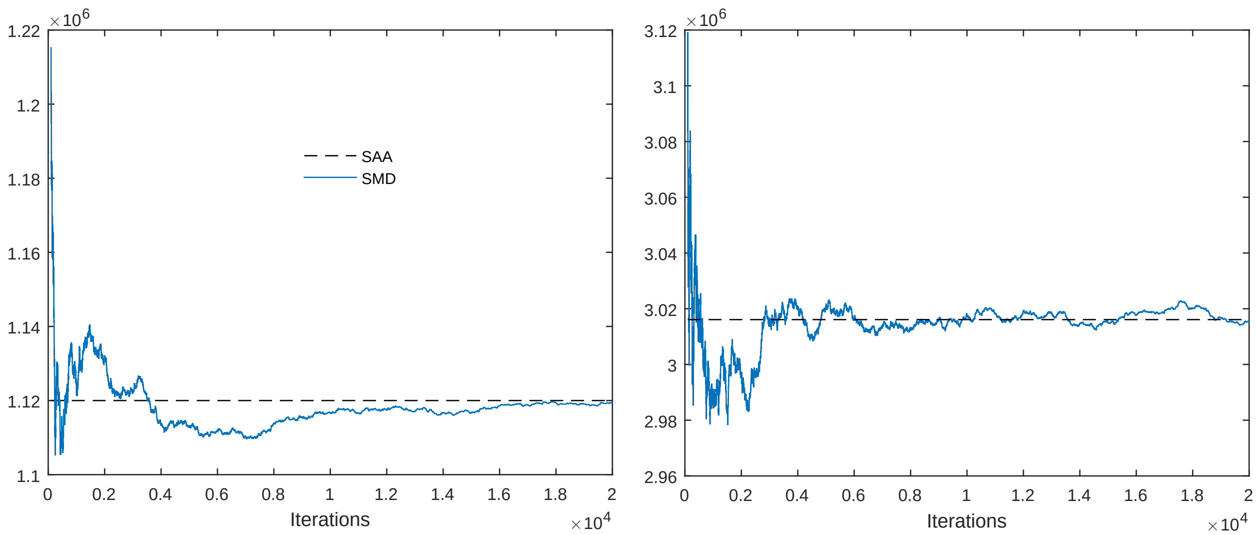


FIGURE 2. Left plot: optimal value of our instance of Problem (5.98)-(5.99) with $n = 5$ estimated using SAA as well as evolution of the approximate optimal value computed along the iterations of SMD. Right plot: same outputs for Problem (5.98)-(5.99) and $n = 10$.

Instance	SMD	ISMD 1	ISMD 2	ISMD 3	ISMD 4
$n = 200$, Problem (5.96)	1.2	3.2	1.7	1.2	1.2
$n = 400$, Problem (5.96)	0.86	3.14	1.27	0.86	0.86
$n = 600$, Problem (5.96)	0.81	6.59	3,33	0.81	0.81
$n = 200$, Problem (5.98)	1.7523×10^9	1.3335×10^9	1.5762×10^9	1.7472×10^9	1.7508×10^9
$n = 400$, Problem (5.98)	6.9978×10^9	6.2402×10^9	6.7624×10^9	6.9943×10^9	6.9972×10^9
$n = 600$, Problem (5.98)	1.5524×10^{10}	1.1339×10^{10}	1.3838×10^{10}	1.5481×10^{10}	1.5512×10^{10}

Table 4. Approximate optimal values of instances of Problems (5.96) and (5.98) estimated with SMD, ISMD 1, ISMD 2, ISMD 3, and ISMD 4.

For $n = 200$ and $n = 400$, we run all methods 10 times taking samples of ξ_2 of size $N = 2000$ for $n = 200$, of size $N = 1000$ for Problem (5.96)-(5.97) and $n = 400$, and of size $N = 500$ for Problem (5.98)-(5.99) and $n = 400$. For $n = 600$, it takes much more time to load and solve subproblems and we only run SMD and ISMD once taking a sample of size $N = 500$ for Problem (5.96)-(5.97) and of size $N = 300$ for Problem (5.98)-(5.99).[9]

In Figure 3, we report for our instances of Problem (5.96)-(5.97) the mean (computed over the 10 runs of the methods for $n = 200, 400$) approximate optimal values along the iterations of SMD and our variants of ISMD.[10] We also report on this figure the empirical distribution (over the 10 runs of the methods for $n = 200, 400$) of the total time required to solve the problem instances with SMD and our variants of ISMD.

As expected, ISMD 1 and ISMD 2 complete the N iterations quicker (since they run Mosek for less iterations) but start with worse approximations of the optimal values. ISMD 3 and ISMD 4 also complete the N iterations quicker than SMD but provide approximations of the optimal values very close to SMD along the iterations of the method and in particular at termination, see also Table 4 which gives the mean approximate optimal value at the last iteration N for all methods. We should also note that most of the computational time for these methods is spent in loading the data for Mosek solver through a series of loops and this step requires the same computational time for all methods. Therefore, the difference in computational time only comes from the time spent by Mosek to solve subproblems. With a C++ or Fortran implementation, this time would remain similar but the loops for loading the data would be much quicker and the total solution time would decrease by a much more important factor. However, even with our Matlab implementation, the total time decreases significantly.

For our instances of Problem (5.97)-(5.99), we report in Figure 4 the mean (over the 10 runs for $n = 200$ and $n = 400$) approximate optimal values computed along the iterations of SMD and our variants of ISMD. For the instances $n = 200$ and $n = 400$, we also report in Figure 5 the empirical distribution of the total solution time and of the time required for Mosek to solve subproblems for SMD and all variants of ISMD. The remarks made for Problem (5.96) still apply for these simulations performed on Problem (5.98). We also refer to Table 4 which provides the mean approximate optimal value at the last iteration N for all methods. As for Problem (5.96), ISMD 3 and ISMD 4 provide after our $N-1$ iterations a good approximation of the optimal value, very close to the approximation obtained with SMD but require less computational time.

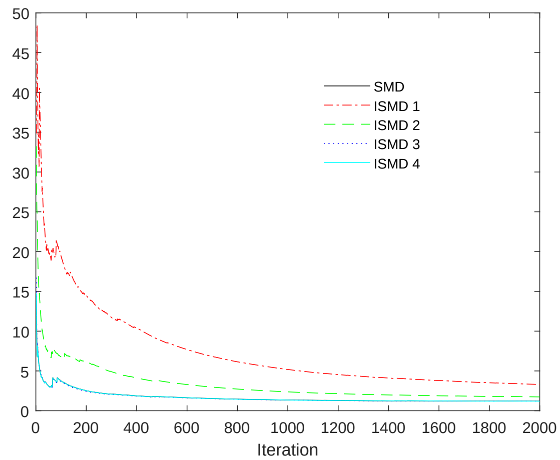
6. Conclusion

We introduced an inexact variant of SMD called ISMD to solve (general) nonlinear two-stage stochastic programs. We have shown on two examples of two-stage nonlinear problems that ISMD can allow us to obtain quicker than SMD a good solution and a good approximation of the optimal value.

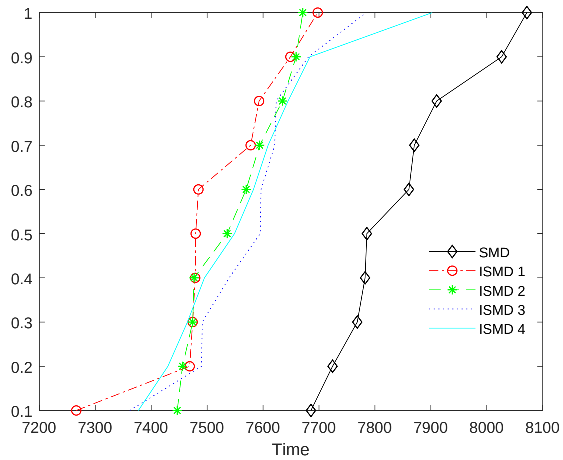
The method and convergence analysis was based on two studies of convex analysis:

[9]Due to the increase in computational time when N increases, we do not take the largest sample size $N = 2000$ for all instances. However, for all instances and values of N chosen, we observe a stabilization of the approximate optimal value before stopping the algorithm, which indicates a good solution has been found at termination.

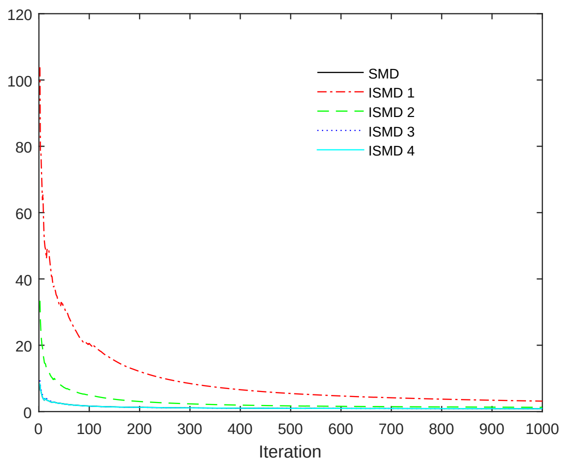
[10]When SMD (and similarly for ISMD) is run on samples of ξ_2 of size N, we have seen how to compute at iteration $t-1$ an estimation $\frac{1}{\sum_{\tau=1}^{t}\gamma_\tau(N)}\sum_{\tau=1}^{t}\gamma_\tau(N)\Big(f_1(x_1^{N,\tau})+f_2(x_2^{N,\tau},x_1^{N,\tau},\xi_2^{N,\tau})\Big)$ of the optimal value on the basis of sample $\xi_2^{N,1},\ldots,\xi_2^{N,t}$ of ξ_2. The mean approximate optimal value after $t-1$ iterations is obtained running SMD on 10 independent samples of ξ_2 of size N and computing the mean of these values on these samples.



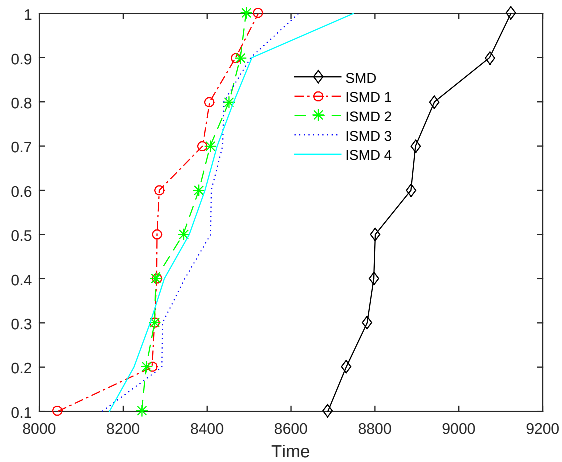
$n = 200$



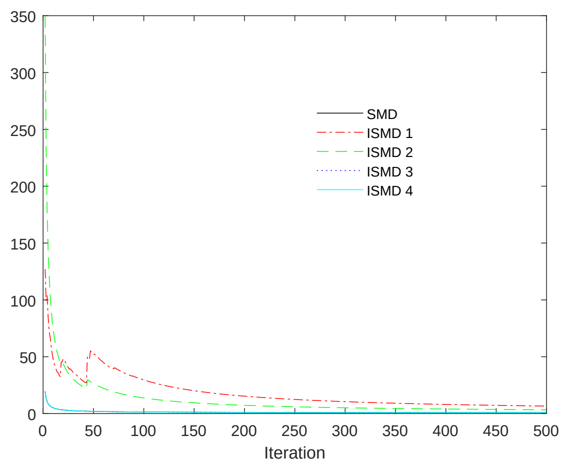
$n = 200$



$n = 400$



$n = 400$



$n = 600$

FIGURE 3. Top left plot: approximate optimal values of our instance of Problem (5.96) with $n = 200$ along the iterations of SMD and our variants of ISMD. Top right plot: empirical distribution of the solution time in seconds on this instance and for these methods. Middle plots: same as the top plot replacing $n = 200$ by $n = 400$. Bottom plot: same as the top left plot replacing $n = 200$ by $n = 600$.

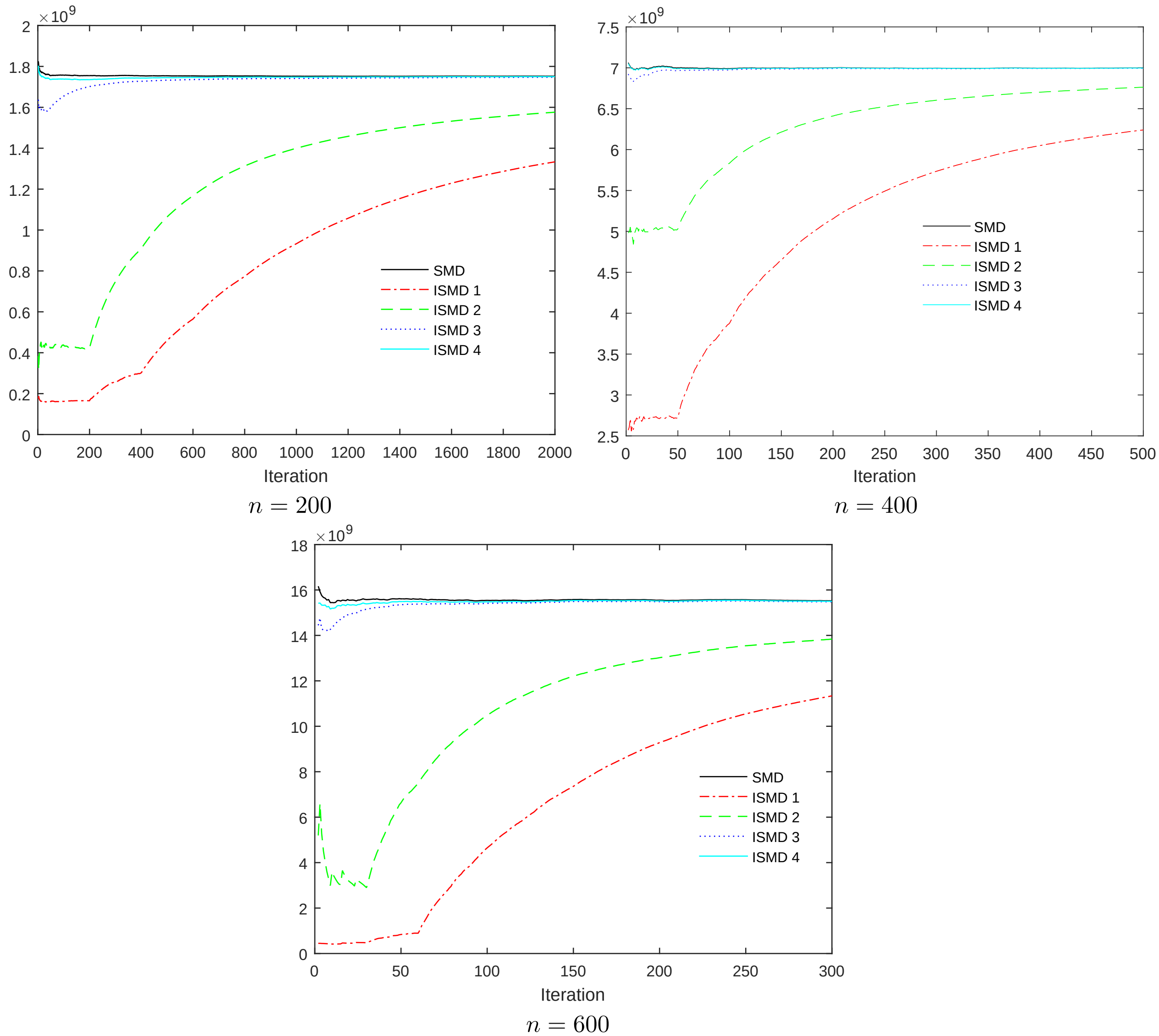


FIGURE 4. Top left plot: approximate optimal values of our instance of Problem (5.98) with $n = 200$ along the iterations of SMD and our variants of ISMD. Top right and bottom plots provide the same graphs for respectively $n = 400$ and $n = 600$.

(a) the computation of inexact cuts for value functions of a large class of convex optimization problems having nonlinear objective and constraints which couple the argument of the value function and the decision variable;
(b) the study of the strong concavity of the dual function of an optimization problem (used to derive one of our formulas for inexact cuts).

It is worth mentioning that the formulas we derived for inexact cuts could also be used to propose inexact level methods [12] to solve nonlinear two-stage stochastic programs (4.62)-(4.63), when primal and dual second stage problems are solved approximately (inexactly).

It would also be interesting to test ISMD and the aforementioned inexact level methods on several relevant instances of nonlinear two-stage stochastic programs.

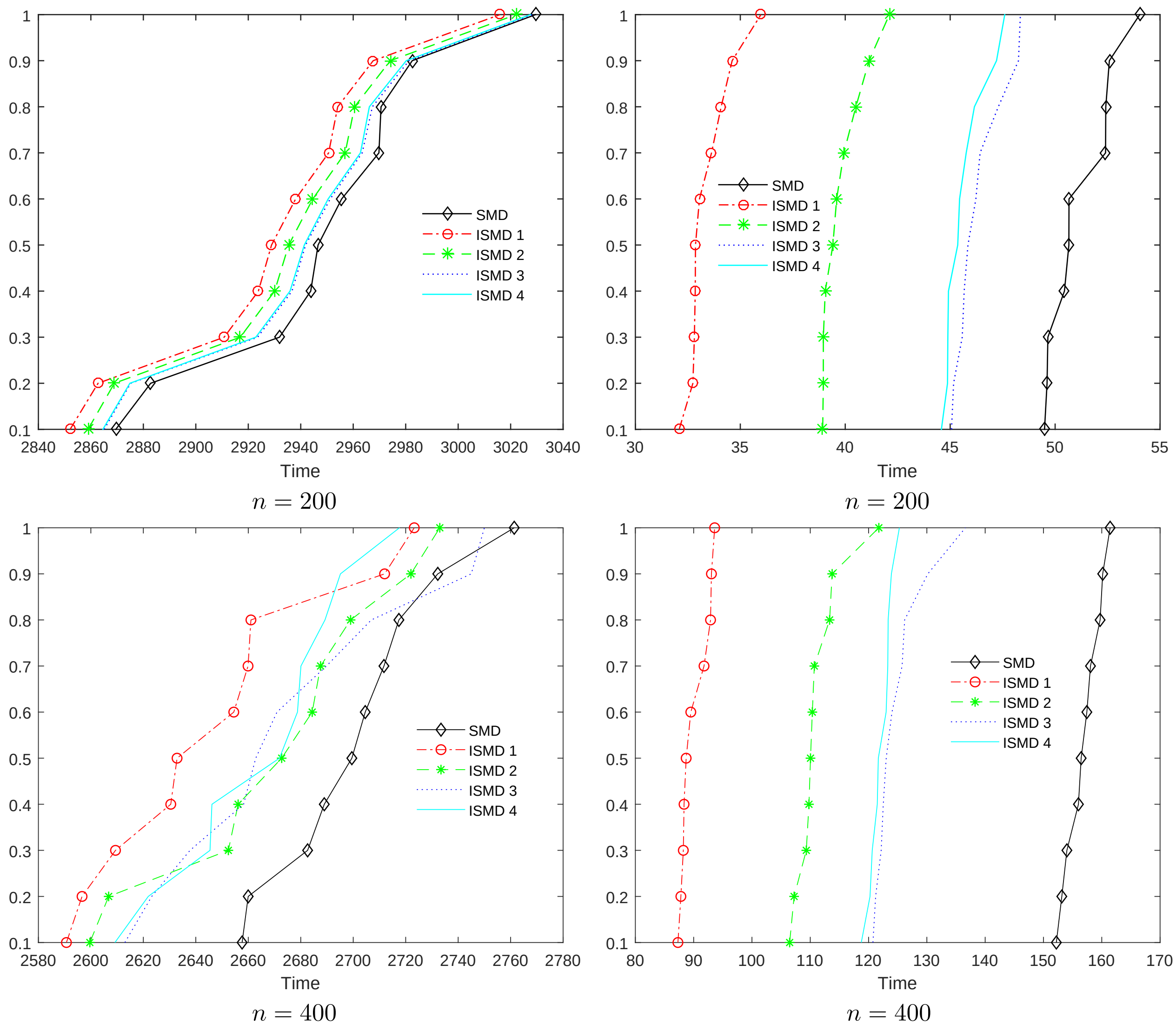


FIGURE 5. Top left: empirical distribution of the total solution time to solve Problem (5.98) for SMD and and our four variants of ISMD for the instance with $n = 200$. Right plot: empirical distribution of the time required for Mosek to solve all subproblems for that instance. Bottom plots: same outputs for $n = 400$.

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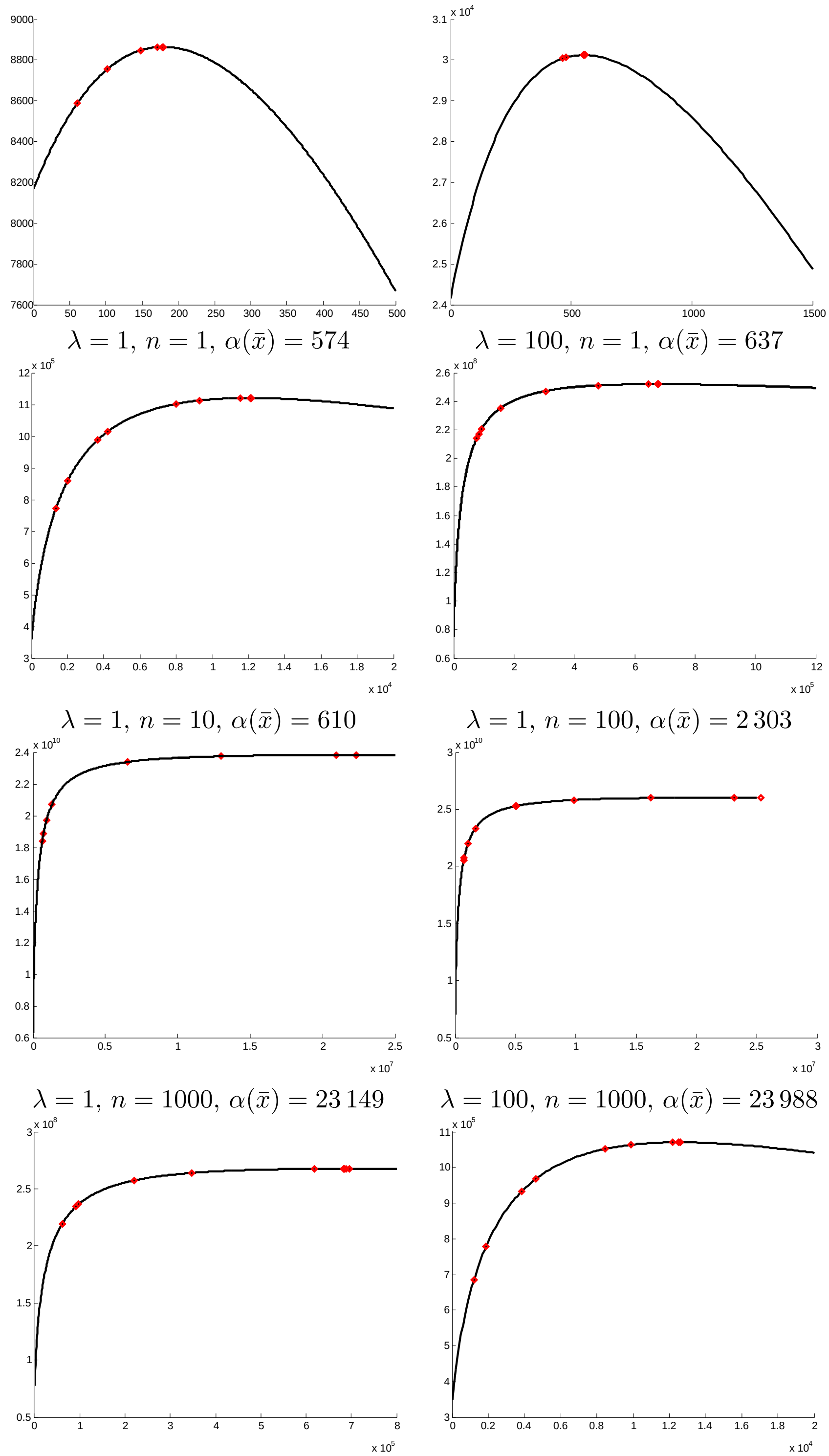


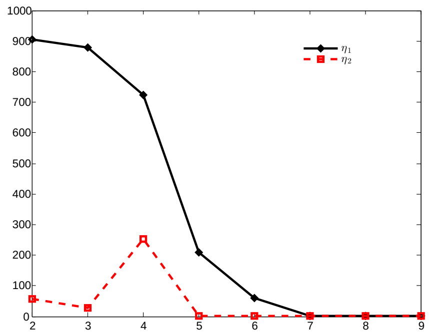
$\lambda = 1,\ n = 1,\ \alpha(\bar{x}) = 574$ $\lambda = 100,\ n = 1,\ \alpha(\bar{x}) = 637$

$\lambda = 1,\ n = 10,\ \alpha(\bar{x}) = 610$ $\lambda = 1,\ n = 100,\ \alpha(\bar{x}) = 2\,303$

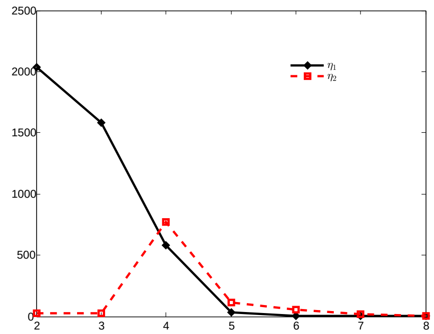
$\lambda = 1,\ n = 1000,\ \alpha(\bar{x}) = 23\,149$ $\lambda = 100,\ n = 1000,\ \alpha(\bar{x}) = 23\,988$

$\lambda = 100,\ n = 100,\ \alpha(\bar{x}) = 2\,839$ $\lambda = 100,\ n = 10,\ \alpha(\bar{x}) = 526$

FIGURE 6. Dual function $\theta_{\bar{x}}$ of problem (3.54) for some $\bar{x}$ randomly drawn in ball $\{x \in \mathbb{R}^n : \|x - x_0\|_2 \leq 1\}$, $S = AA^T + \lambda I_{2n}$ for some random matrix A with random entries in $[-20, 20]$, and several values of the pair (n, λ). The dual iterates are represented by red diamonds.



$\lambda = 1,\ n = 1,\ \alpha(\bar{x}) = 574$



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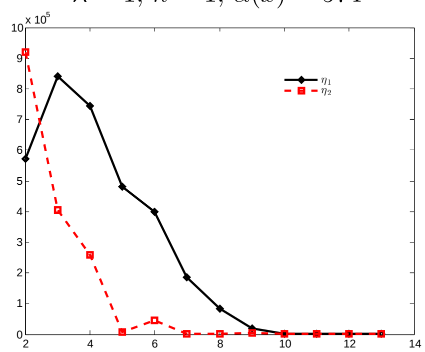
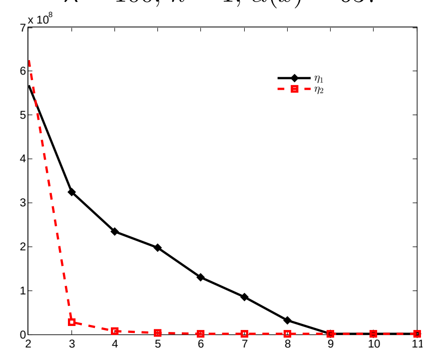
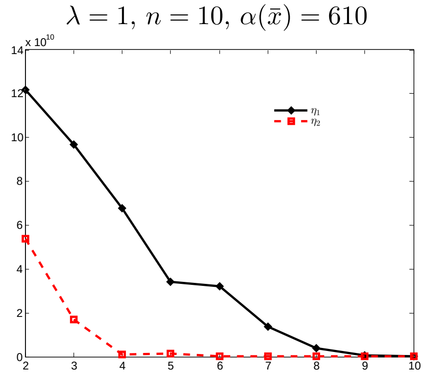


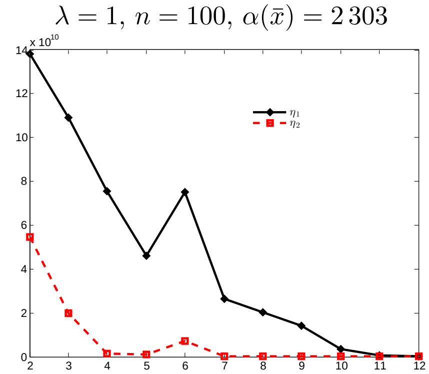
$\lambda = 1,\ n = 10,\ \alpha(\bar{x}) = 610$



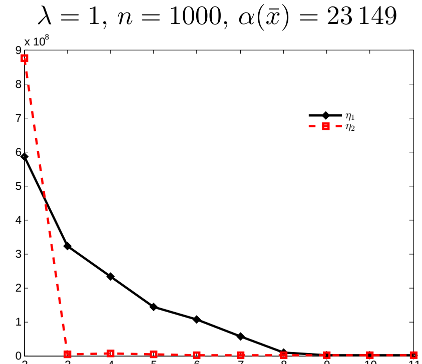
$\lambda = 1,\ n = 100,\ \alpha(\bar{x}) = 2\,303$



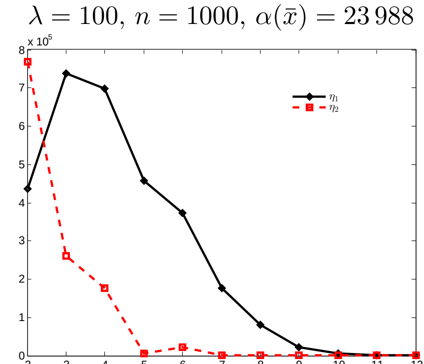
$\lambda = 1,\ n = 1000,\ \alpha(\bar{x}) = 23\,149$



$\lambda = 100,\ n = 1000,\ \alpha(\bar{x}) = 23\,988$



$\lambda = 100,\ n = 100,\ \alpha(\bar{x}) = 2\,839$



$\lambda = 100,\ n = 10,\ \alpha(\bar{x}) = 526$

FIGURE 7. Plots of $\eta_1(\varepsilon_k, \bar{x})$ and $\eta_2(\varepsilon_k, \bar{x})$ as a function of iteration k where ε_k is the duality gap at iteration k for problem (3.54) for some $\bar{x}$ randomly drawn in ball $\{x \in \mathbb{R}^n : \|x - x_0\|_2 \leq 1\}$, $S = AA^T + \lambda I_{2n}$ for some random matrix A with random entries in $[-20, 20]$, and several values of the pair (n, λ).

ISMD 1

Iteration number	$[1, \lceil 0.1N \rceil]$	$[\lceil 0.1N \rceil + 1, \lceil 0.2N \rceil]$	$[\lceil 0.2N \rceil + 1, \lceil 0.3N \rceil]$	$[\lceil 0.3N \rceil + 1, \lceil 0.4N \rceil]$
IP solver maximal number of iterations	$\lceil 0.1 I_{\max} \rceil$	$\lceil 0.2 I_{\max} \rceil$	$\lceil 0.3 I_{\max} \rceil$	$\lceil 0.4 I_{\max} \rceil$

Iteration number	$[\lceil 0.4N \rceil + 1, \lceil 0.5N \rceil]$	$[\lceil 0.5N \rceil + 1, \lceil 0.6N \rceil]$	$[\lceil 0.6N \rceil + 1, \lceil 0.7N \rceil]$
IP solver maximal number of iterations	$\lceil 0.5 I_{\max} \rceil$	$\lceil 0.6 I_{\max} \rceil$	$\lceil 0.7 I_{\max} \rceil$

Iteration number	$[\lceil 0.7N \rceil + 1, \lceil 0.8N \rceil]$	$[\lceil 0.8N \rceil + 1, \lceil 0.9N \rceil]$	$[\lceil 0.9N \rceil + 1, N]$
IP solver maximal number of iterations	$\lceil 0.8 I_{\max} \rceil$	$\lceil 0.9 I_{\max} \rceil$	$I_{\max}$

ISMD 2

Iteration number	$[1, \lceil 0.1N \rceil]$	$[\lceil 0.1N \rceil + 1, \lceil 0.2N \rceil]$	$[\lceil 0.2N \rceil + 1, \lceil 0.3N \rceil]$
IP solver maximal number of iterations	$\lceil 0.2 I_{\max} \rceil$	$\lceil 0.4 I_{\max} \rceil$	$\lceil 0.6 I_{\max} \rceil$

Iteration number	$[\lceil 0.3N \rceil + 1, \lceil 0.4N \rceil]$	$[\lceil 0.4N \rceil + 1, \lceil 0.5N \rceil]$	$[\lceil 0.5N \rceil + 1, N]$
IP solver maximal number of iterations	$\lceil 0.8 I_{\max} \rceil$	$\lceil 0.9 I_{\max} \rceil$	$I_{\max}$

ISMD 3

Iteration number	$[1, \lceil 0.02N \rceil]$	$[\lceil 0.02N \rceil + 1, \lceil 0.04N \rceil]$	$[\lceil 0.04N \rceil + 1, \lceil 0.06N \rceil]$
IP solver maximal number of iterations	$\lceil 0.5 I_{\max} \rceil$	$\lceil 0.6 I_{\max} \rceil$	$\lceil 0.7 I_{\max} \rceil$

Iteration number	$[\lceil 0.06N \rceil + 1, \lceil 0.08N \rceil]$	$[\lceil 0.08N \rceil + 1, \lceil 0.1N \rceil]$	$[\lceil 0.1N \rceil + 1, N]$
IP solver maximal number of iterations	$\lceil 0.8 I_{\max} \rceil$	$\lceil 0.9 I_{\max} \rceil$	$I_{\max}$

ISMD 4

Iteration number	$[1, \lceil 0.1N \rceil]$	$[\lceil 0.1N \rceil + 1, \lceil 0.2N \rceil]$	$[\lceil 0.2N \rceil + 1, \lceil 0.3N \rceil]$	$[\lceil 0.3N \rceil + 1, N]$
IP solver maximal number of iterations	$\lceil 0.7 I_{\max} \rceil$	$\lceil 0.8 I_{\max} \rceil$	$\lceil 0.9 I_{\max} \rceil$	$I_{\max}$

TABLE 5. Maximal number of iterations for Mosek interior point solver used to solve second stage problems as a function of the iteration number $i = 1, \ldots, N$, of ISMD and the maximal number of iterations $I_{\max}$ allowed for Mosek solver to solve subproblems with SMD. In this table, $\lceil x \rceil$ is the smallest integer larger than or equal to x. For problem (5.96)-(5.97) and $n = 200, 400, 600$ and problem (5.98)-(5.99) and $n = 200$, we take $I_{\max} = 15$, for problem (5.98)-(5.99) and $n = 400$ we take $I_{\max} = 25$, and for problem (5.98)-(5.99) and $n = 600$ we take $I_{\max} = 28$. For instance for ISMD 1, $N = 2000$, and problem (5.96)-(5.97), for iterations $[\lceil 0.4N \rceil + 1, \lceil 0.5N \rceil]$, i.e., for iterations $0.4 \times 2000 + 1, \ldots, 0.5 \times 2000 = 801, \ldots, 1000$, Mosek interior point solver is run to solve second stage problems limiting the maximal number of iterations to $\lceil 0.5 I_{\max} \rceil = 8$.