Unit :- II

PROPOSITIONAL LOGIC

propositional logic, Basic Logic. Logical Connectives, truth tables, tautologies, Contradiction normal forms [Conjuctive & Disjunctive] universal and existential Quantification. Notion of proof: proof by Implication, Converse, Inverse, Contrapositive, negation, and Contradiction, proof by using truth table.

Proposition OR Statement :-

An statement is a declarative sentence which is true or false but not both; or in other words an statement is a declarative sentence which has a definite truth value.

OR

An expression consisting of some symbols, letters and words is called a sentence, if it is true or false.

Ex :- Declarative sentence	Truth Value.
1) $\{x : x^2 = 36\} = \{6, -6\}$	True
2) $2 + 3 = 5$	True
3) The capital of M.P is Bhopal	True
4) $9 < 6$	False
5) The colour of Human Blood is Blue.	False
6) $5 + 6 = 10$	False.

※ Truth value :-

If any proposition is true then its truth value is denoted by "T" and if the proposition is false then truth value is denoted by "F".

Ex :- 1) 1 is less than 3 — T

2) 14 is odd No. — F

Types of Sentence :-

There are usually 2 type of Sentence / proposition :-

1) Simple Proposition / Atomic sentence.
2) compound Proposition / Molecular Sentence.

(1) SIMPLE PROPOSITION :-

A simple proposition or Atomic Sentence has no Connectives.

OR

The proposition having one subject and one predicate is called a Simple proposition.

Ex :- 1) This flower is pink
2) Every Even No. is divisible by 2.

(2) Compound Proposition :-

Two or more Simple proposition when combined by various connectivities into a Single Composite sentence is called Compound Proposition.

Ex :-
1) Earth is Round and Revolves around the Sun.
2) A triangle is Equilateral iff its 3 Sides are equal.

Logical Connectives :-

Logical connectives are the words or Symbol used to combine 2 or more proposition to form a compound Sentence or compound proposition are called logical Connectives.

	Logical Connectives words	Symbols	Uses.
1)	Denial / NOT / Negation	- or ~	$\sim p$
2)	Join / AND / Conjuction	$\wedge$	$p \wedge q$
3)	meet / OR / Disjuction	$\vee$	$p \vee q$
4)	Equivalent	$\leftrightarrow$	$p \leftrightarrow q$

5)	Conditional "if .... then...."	$\Rightarrow$	$p \Rightarrow q$
6)	Bi-conditional "if and only if" (iff)	$\Leftrightarrow$	$p \Leftrightarrow q$
7)	NAND (NOT + AND)	$\uparrow$	$p \uparrow q$
8)	NOR (NOT + OR)	$\downarrow$	$p \downarrow q$
9)	XOR	$\oplus$	$p \oplus q$

* Basic Logical operations :-

(1) Conjuction ($\wedge$) :-

Any 2 proposition p, q are combined by using the word "and", then we get a new statement represented by $p \wedge q$. The resulting statement $p \wedge q$ is read as "p and q", The statement $p \wedge q$ so obtained is called the Conjunction of p & q. (*)

Truth Table :-

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Ex :-

(1) Delhi is in India and 2+2 = 4 T

(2) Delhi is in India and 2+2 = 5 F

(3) Delhi is in America and 2+2 = 4 F

(2) Disjunction ($\vee$)

When 2 statement p, q are joined by using the word "or", then we get a new statement represented by $p \vee q$.
The resulting statement $p \vee q$ is read as "p or q".
The statement $p \vee q$ so obtained is called the Disjunction of p and q. (+)

Truth Table :-

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Ex :-

(1) Delhi is in India or 2+2 = 4 T

(2) Delhi is in Russia or 2+2 = 6 F

(3) Delhi is in India or 2+2 = 4 T

(3) Negation ($\sim$) :-

The negation of the statement is obtained by putting the word "not" and is represented by $\sim p$ or $\bar{p}$ and is read as "not p".

Truth Table :-

p	$\sim p$
T	F
F	T

Ex :- 1) $p \equiv$ This flower is pink.
$\sim p \equiv$ This flower is not pink.

2) $p \equiv$ Ramesh is good player of Hockey.
$\sim p \equiv$ Ramesh is not good player of Hockey.

(4) Conditional ($\Rightarrow$ or $\rightarrow$) :-

The statement of the type "if p then q" is called Conditional Statement and is represented by "$p \Rightarrow q$", and read by as "p implies q". p is called Antecedent and q is called Consequent. $\{p \Rightarrow q$ or $p \rightarrow q\}$

{The conditional statement is false only in the condition when p is True and q is false.}

Truth table :-

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Ex :- 1) A father said to his son, "If you will stood first in the examination, then I shall purchase a scooter to you.

(5) Biconditional ($\Leftrightarrow$ or $\leftrightarrow$) :-

A statement "p if and only if q" such statement are said to be Bi-conditional statement and denoted by $p \Leftrightarrow q$ or $p \leftrightarrow q$

This statement $p \Leftrightarrow q$ is also read as follows:

a) "p if and only if q"

b) "p than and only than when q"

c) "p iff q"

Truth table :-

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

{Bi-conditional statement is true only in those case when its both component have the same truth values}.

Ex:- The father had promised the son that; "I shall purchase a scooter ~~if~~ for you if and only if you will stand first in the Examination.

⑥ NOR or Joint Denial ($\downarrow$).

Let p and q be 2 statement. The statement of the type $p \downarrow q$, read as "Neither p nor q" is called joint denial or NOR state.

In other words NOR is the "negation of OR of 2 statement". Thus

$$(p \downarrow q) = \sim(p \vee q).$$

{It means that $p \downarrow q$ is true only when p and q both are false}

Truth Table :-

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

⑦ NAND ($\uparrow$) :-

The connective NAND is denoted by the symbol $\uparrow$ and is defined as "negation of AND of 2 statement. and it is denoted $p \uparrow q$,

Thus,

$$p \uparrow q = \sim(p \wedge q)$$

{ $p \uparrow q$ is false only in the case when both p and q are true}

Truth Table :-

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

(8) XOR (⊕) :-

The XOR of 2 statement is True if and only if one of the 2 statement is true. and it is denoted by $p \oplus q$.

Truth Table :-

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

* Tautology :-

A tautology is a proposition which is true for all truth values of its sub proposition or components.

A tautology is also called Logically Valid or Logically true.

* Contradiction :-

A contradiction is a proposition which is always false $\forall$ truth table values of its proposition or components.

A contradiction is also called Logically false.

NOTE :-

A proposition is said to be an Tautologies if it contain only T in last column of truth table and if it contain F in last column of truth table then its is called Contradiction.

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Illustrative Examples :-

Ex:-1 which of the following statement are true and which are false?

(i) $16 + 9 = 4^2 + 3^2 \rightarrow T$

(ii) $9 \times 7 = 21 \times 3 \rightarrow T$

(iii) $\{2, 3\} \subset \{2, 4, 6\} \rightarrow F$

(iv) $5 \in \{1, 3, 5\}. \rightarrow T$

Ex:-4 If p = he is poor, q = he is Laborious, then write down the following statement in Symbol (i.e, in language of logic) :-

(i) He is poor and laborious $\rightarrow p \wedge q$.

(ii) He is poor but is not laborious $\rightarrow p \wedge \sim q$

(iii) It is false that he is poor or is laborious $\rightarrow \sim(p \vee q)$

(iv) It is false that he is not poor ~~and~~ or is laborious $\rightarrow \sim(\sim p \vee q)$

(v) Neither he is poor nor he is laborious $\rightarrow \sim p \wedge \sim q$

(vi) He is poor or is not poor and is laborious $\rightarrow p \vee (\sim p \wedge q)$

(vii) It is not true that he is poor or is not laborious
$\sim(\sim p \vee \sim q)$

Ex:-8
Ex:-9] → Pg - 284 {add in pdf}

Questions Based on Tautology and Contradiction
Pg - 290 to 294

Questions Based on Equivalence : Pg - 295 - 296

Statement is : $p \Rightarrow q, \sim q \Rightarrow \sim p$.

Example 8. *If $p \equiv$ It is 4 o'clock, $q \equiv$ the train is late, then state in words the following results :*

(*i*) $p \vee q$,	(*ii*) $p \wedge q$,
(*iii*) $p \wedge (\sim q)$,	(*iv*) $q \vee \sim p$,
(*v*) $(\sim p) \wedge q$,	(*vi*) $(\sim p) \wedge (\sim q)$,
(*vii*) $(\sim p) \vee (\sim q)$,	(*viii*) $\sim (p \wedge q)$,
(*ix*) $\sim p \Rightarrow p$.	

Solution. (i) It is 4 o'clock or the train is late.

(ii) It is 4 o'clock and the train is late.

(iii) It is 4 o'clock and the train is not late.

Or

It is 4 o'clock but the train is not late.

(iv) The train is late or it is not 4 o'clock.

(v) It is not 4 o'clock and the train is late.

(vi) It is not 4 o'clock and the train is not late.

Or

Neither it is 4 o'clock nor the train is late.

(vii) It is not 4 o'clock or the train is not late.

Or

Either it is not 4 o'clock and the train is not late.

(viii) It is not true that it is 4 o'clock and the train is late.

(ix) If it is not 4 o'clock, then the train is late.

Example 9. *Let p = It is cold, q = It is raining. Give a simple verbal sentence which describes each of the following statements :*

(*i*) $\sim p$,	(*ii*) $p \wedge q$,
(*iii*) $p \vee q$,	(*iv*) $q \Leftrightarrow p$,
(*v*) $p \Rightarrow \sim q$,	(*vi*) $q \vee \sim p$,
(*vii*) $\sim p \wedge \sim q$,	(*viii*) $p \Leftrightarrow \sim q$,
(*ix*) $\sim \sim q$,	(*x*) $(p \wedge \sim q) \Rightarrow p$,
(*xi*) $\sim \sim p$,	(*xii*) $(p \wedge \sim q) \Rightarrow q$.

Solution. (i) It is not cold.

(ii) It is cold and raining.

(iii) It is cold or it is raining.

(iv) It is raining if and only if it is cold.

(v) If it is cold, then it is not raining.

(vi) It is raining or it is not cold.

(vii) It is not cold and it is not raining.

(viii) It is cold if and only if it is not raining.

(ix) It is not true that it is not raining.

(x) If it is cold and not raining, then it is cold.

(xi) It is not true that it is not cold.

(xii) If it is ~~raining~~ and not ~~cold~~, then it is raining.

Cold Raining

Example. 2. (*a*) *If p and q are two statements, prove that* $(p \wedge q) \Rightarrow p$ *is a tautology.*

Solution. Truth table for $p \wedge q \Rightarrow p$ is as follows :

p	q	$p \wedge q$	$p \wedge q \Rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Since all entries in the columns of $p \wedge q \Rightarrow p$ are T, it is a tautology.

Example 2. (*b*) *Prove that* $p \wedge q \Rightarrow q$ *is a tautology.*

Solution. Truth table for $p \wedge q \Rightarrow q$.

p	q	$p \wedge q$	$p \wedge q \Rightarrow q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Note. Above 2(a) and 2(b) are called rules for simplification.

Example 3. *Prove that* $p \vee (-p)$ *is a tautology.*

Solution. Truth table for $p \vee (-p)$.

p	$-p$	$p \vee -p$
T	F	T
F	T	T

Note. $p \vee -p$ is a tautology. It shows any statements p is either true or false.

Example 4. *Laws of addition. Prove that* :

(*i*) $p \Rightarrow p \vee q$, (*ii*) $q \Rightarrow (p \vee q)$,

are tautologies, when p and q are any two statements.

Solution. Truth tables for $p \Rightarrow p \vee q$ and $q \Rightarrow p \vee q$ are shown by a single truth table as given below :

p	q	$p \vee q$	$p \Rightarrow p \vee q$	$q \Rightarrow p \vee q$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	T	T

Since the columns under $p \Rightarrow p \vee q$ and $q \Rightarrow p \vee q$ contain only T's. Hence given propositions are tautologies.

Example 5. *Prove that* $p \wedge q \Rightarrow p \vee q$ *is a tautology.*

Solution. Truth table for $p \wedge q \Rightarrow p \vee q$:

p	q	$p \wedge q$	$p \vee q$	$p \wedge q \Rightarrow p \vee q$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

$\therefore$ $p \wedge q \Rightarrow p \vee q$ is tautology.

Example 6. *Is the statement* $(\sim p) \vee q$ *is a tautology ?*

Solution. Clearly $(\sim p) \vee q$ is not a tautology from the following truth table :

p	$\sim p$	q	$\sim p \vee q$
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T

✓ **Example 7.** *Prove that* $\sim \{p \wedge (\sim p)\}$ *is a tautology.*

Solution. Clearly the statement $\sim \{p \wedge (\sim p)\}$ is a tautology from the following truth table :

p	$\sim p$	$p \wedge (\sim p)$	$\sim \{p \wedge (\sim p)\}$
T	F	F	T
F	T	F	T

Example 8. *Prove that* $p \Rightarrow (\sim p) \Leftrightarrow \sim p$ *is a tautology.*

Solution. Truth table for $p \Rightarrow (\sim p) \Leftrightarrow \sim p$.

p	$\sim p$	$p \Rightarrow \sim p$	$p \Rightarrow (\sim p) \Leftrightarrow (\sim p)$
T	F	F	T
F	T	T	T

Hence the proposition $p \Rightarrow (\sim p) \Leftrightarrow \sim p$ is a tautology.

✓ **Example 9.** *Prove that* $(p \Rightarrow q) \Leftrightarrow (\sim p \vee q)$ *is a tautology.*

Solution.

p	q	$\sim p$	$p \Rightarrow q$	$\sim p \vee q$	$p \Rightarrow q \Leftrightarrow (\sim p \vee q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

✓ **Example 10.** *Prove that* $(p \Leftrightarrow q) \Leftrightarrow (p \Rightarrow q) \wedge (q \Rightarrow p)$ *is a tautology.*

Solution.

p	q	$p \Leftrightarrow q$	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$	$(p \Leftrightarrow q) \Leftrightarrow$ $(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T	T	T
T	F	F	F	T	F	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T

Example 11. *Show that* $(p \Rightarrow q) \wedge (q \Rightarrow p) \Leftrightarrow (p \Leftrightarrow q)$ *is a tautology.*

Solution.

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p) = P$	$p \Leftrightarrow q = Q$	$P \Leftrightarrow Q$
T	T	T	T	T	T	T
T	F	F	T	F	F	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T

Example 12. *Prove that* $(p \Leftrightarrow q) \wedge (q \Leftrightarrow r) \Rightarrow (p \Leftrightarrow r)$ *is a tautology.*

[R.G.P.V. Dec. 2006]

Solution. For convenience, let

$$p \Leftrightarrow r = A \text{ and } (p \Leftrightarrow q) \wedge (q \Leftrightarrow r) = B$$

p	q	r	$p \Leftrightarrow q$	$q \Leftrightarrow r$	$p \Leftrightarrow r = A$ (suppose)	$(p \Leftrightarrow q) \wedge (q \Leftrightarrow r) = B$ (suppose)	$B \Rightarrow A$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	T	F	T
T	F	F	F	T	F	F	T
F	T	T	F	T	F	F	T
F	T	F	F	F	T	F	T
F	F	T	T	F	F	F	T
F	F	F	T	T	T	T	T

Last column shows that $B \Rightarrow A$ *i.e.*, $(p \Leftrightarrow q) \wedge (q \Leftrightarrow r) \Rightarrow (p \Leftrightarrow r)$ is a tautology.

§ 3.15. CONTRADICTION

Definition. *A contradiction is a proposition which is always false for all truth values of its propositions or components.*

A contradiction is also called **logically false.**

Clearly in truth tables, all entries in the column of contradiction are of '*F*' only.

Contingency : **[R.G.P.V. June 2002]**

A contingency is a proposition which is either true or false depending on the truth values of its components or propositions.

ILLUSTRATIVE EXAMPLES

Example 1. *Prove that* $p \wedge (\sim p)$ *is a contradiction.*

Solution. Let p be a statement. Truth table for $p \wedge (\sim p)$ is :

p	$\sim p$	$p \wedge (\sim p)$
T	F	F
F	T	F

Since all entries in the column of $p \wedge (\sim p)$ are of F only, it is a contradiction.

Example 2. *Prove that $p \Leftrightarrow (\sim p)$ is a contradiction.*

Solution. Truth table for $p \Leftrightarrow (\sim p)$ is :

p	$\sim p$	$p \Leftrightarrow (\sim p)$
T	F	F
F	T	F

Example 3. *Prove that $(p \vee q) \wedge (\sim p) \wedge (\sim q)$ is a contradiction.*

Solution. Truth table for given proposition is :

p	q	$p \vee q$	$\sim p$	$\sim q$	$(p \vee q) \wedge (\sim p)$	$(p \vee q) \wedge (\sim p) \wedge (\sim q)$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	T	F
F	F	F	T	T	F	F

Since all entries in the last column are of *F*'s and so it is a contradiction.

Example 4. *Show that the proposition $(\sim p \wedge q) \vee (p \wedge \sim q)$ is a contingency.*

Solution. Truth table for the given proposition is :

p	q	$\sim p$	$\sim q$	$\sim p \wedge q$	$p \wedge \sim q$	$(\sim p \wedge q) \vee (p \wedge \sim q)$
T	T	F	F	F	F	F
T	F	F	T	F	T	T
F	T	T	F	T	F	T
F	F	T	T	F	F	F

The entries in the last column depends upon the truth values of its components. Hence the given proposition (statement) is a contingency.

Example 5. *Show that $p \Rightarrow (p \Rightarrow q)$ is a contingency.*

Solution. Proceed as Ex. 4 above.

§ 3.16. LOGICAL EQUIVALENCE

Definition. *Two statements (or propositions) are called logically equivalent if the truth values of both the statements (or propositions) are always identical.*

In other words, two statements are called logically equivalent if when either is true the other is true and when either is false the other is false.

If two statements *P* and *Q* are logically equivalent, then these are represented by '$P \equiv Q$'. If $P \equiv Q$, then $P \Rightarrow Q$ is a tautology.

Consider two open statements $x = 4$ and $3x = 12$ (when $x \in N$). Both have the same truth value *T* for the value 4 of *x*, therefore, these both open statements are logically equivalent.

Que Prove that the proposition $p \vee \sim(q \wedge r)$ and $(p \vee (\sim q) \vee (\sim r))$ are Equivalent.

Sol:- * Truth Table for $p \vee [\sim(q \wedge r)]$:-

p	q	r	$q \wedge r$	$\sim(q \wedge r)$	$[p \vee \sim(q \wedge r)]$
T	T	T	T	F	T
T	T	F	F	T	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	F	F
F	T	F	F	T	T
F	F	T	F	T	T
F	F	F	F	T	T

* Truth Table for $[p \vee (\sim q)] \vee (\sim r)$:-

p	q	r	$(\sim q)$	$(\sim r)$	$(p \vee (\sim q))$	$[p \vee (\sim q) \vee (\sim r)]$
T	T	T	F	F	T	T
T	T	F	F	T	T	T
T	F	T	T	F	T	T
T	F	F	T	T	T	T
F	T	T	F	F	F	F
F	T	F	F	T	T	T
F	F	T	T	F	T	T
F	F	F	T	T	F	T

fuel page

ILLUSTRATIVE EXAMPLES

Example 1. *If p and q are two statements, show that the implication $p \Rightarrow q$ and its contrapositive $(\sim q) \Rightarrow (\sim p)$ are logically equivalent.*

Solution. Truth table :

p	q	$p \Rightarrow q$	$\sim p$	$\sim q$	$(\sim q) \Rightarrow (\sim p)$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

The entries of third and sixth columns are identical. Hence $p \Rightarrow q$ and $(\sim q) \Rightarrow (\sim p)$ are logically equivalent.

Example 2. *Show that $q \Rightarrow p$ converse of $p \Rightarrow q$ and its inverse $(\sim p) \Rightarrow (\sim q)$ are logically equivalent.*

Solution.

p	q	$q \Rightarrow p$	$\sim p$	$\sim q$	$(\sim p) \Rightarrow (\sim q)$
T	T	T	F	F	T
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	T	T

Since entries of 3rd and 6th columns are identical.

$\therefore \quad (q \Rightarrow p) \equiv \{(\sim p) \Rightarrow (\sim q)\}.$

Example 3. *Show that $\sim (p \Rightarrow q) \equiv \{p \wedge (\sim q)\}$.*

Solution. If $\sim (p \Rightarrow q)$ and $p \wedge (\sim q)$ are equivalent, then

$$\sim (p \Rightarrow q) \Leftrightarrow \{p \wedge (\sim q)\}$$

will be tautology (See Theorem § 2.16), which is clear from the following truth table :

p	q	$\sim q$	$p \Rightarrow q$	$\sim (p \Rightarrow q)$	$p \wedge (\sim q)$	$\sim (p \Rightarrow q) \Leftrightarrow \{p \wedge (\sim q)\}$
T	T	F	T	F	F	T
T	F	T	F	T	T	T
F	T	F	T	F	F	T
F	F	T	T	F	F	T

Example 4. *Show that $p \Rightarrow (q \Rightarrow r) \equiv (p \wedge q) \Rightarrow r$.*

Solution. As in Ex. 3, we shall prove that $p \Rightarrow (q \Rightarrow r) \Leftrightarrow (p \wedge q) \Rightarrow r$ is a tautology.

p	q	r	$q \Rightarrow r$	$p \Rightarrow (q \Rightarrow r)$	$p \wedge q$	$(p \wedge q) \Rightarrow r$	$p \Rightarrow (q \Rightarrow r) \Leftrightarrow (p \wedge q) \Rightarrow r$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	F	T	T
F	T	F	F	T	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

Example 5. *Show that* $p \oplus q \equiv (p \wedge \sim q) \vee (\sim p \wedge q)$.

Solution. As in Ex. 3 above, we shall prove that

$$p \oplus q \Rightarrow (p \wedge \sim q) \vee (\sim p \wedge q) \text{ is a tautology.}$$

p	q	$p \oplus q$	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge q$	$(p \wedge \sim q) \vee (\sim p \wedge q)$	$p \oplus q \Leftrightarrow (p \wedge \sim q) \vee (\sim p \wedge q)$
T	T	F	F	F	F	F	F	T
T	F	T	F	T	T	F	T	T
F	T	T	T	F	F	T	T	T
F	F	F	T	T	F	F	F	T

Example 6. *Show that* $(p \uparrow q) \oplus (p \uparrow q) \equiv (p \vee q) \wedge (p \downarrow q)$.

Solution. As in Ex. 3 above, we shall prove that

$$(p \uparrow q) \oplus (p \uparrow q) \Rightarrow (p \vee q) \wedge (p \downarrow q) \text{ is a tautology.}$$

p	q	$p \uparrow q$	$(p \uparrow q) \oplus (p \uparrow q)$	$p \wedge q$	$p \downarrow q$	$(p \vee q) \wedge (p \downarrow q)$	$(p \uparrow q) \oplus (p \uparrow q) \Leftrightarrow (p \vee q) \wedge (p \downarrow q)$
T	T	F	F	T	F	F	T
T	F	T	F	T	F	F	T
F	T	T	F	T	F	F	T
F	F	T	F	F	T	F	T

Example 7. *Show that* $(p \uparrow q) \oplus (p \uparrow q)$ *is logically false (or is a contradiction).*

Solution. In truth table of Ex. 6 above all entries in column four are of '*F*' only. Hence $(p \uparrow q) \oplus (p \uparrow q)$ is logically false.

Example 8. *Prove that following statements are logically equivalent :*

(*i*) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.

(*ii*) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

(*iii*) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

(*iv*) $p \Rightarrow (q \wedge r) \equiv (p \Rightarrow q) \wedge (p \Rightarrow r)$.

(*v*) $p \Rightarrow (q \vee r) \equiv (p \Rightarrow q) \vee (p \Rightarrow r)$.

(*vi*) $(p \Rightarrow q) \vee r \equiv (p \vee r) \Rightarrow (q \vee r)$. **[R.G.P.V. June 2007]**

(*vii*) $p \vee (q \Leftrightarrow r) \equiv (p \vee q) \Leftrightarrow (p \vee r)$.

Solution. These are easily proved by truth tables.

Kinds of Conditional :-

we can form some new conditional statement related to conditional $p \to q$, i.e, If p then q :-

There are 3 related conditional statement that occurs so often that they have special Names :-

1) Converse :-
2) Inverse.
3 Contrapositive.

* Converse :-

If the statement is "If p, then q", then converse will be "If q, then p".

The converse of $\boxed{p \to q \text{ is } q \to p}$

Ex :- 1) If today is Sunday, then tomorrow is Monday.
If tomorrow is Monday, then today is Sunday.

2) If a triangle is not Isosceles, then it is not Equilateral.
If Δ is not Equilateral, then it is not Isosceles.
*

* Inverse :-

If the statement is "If p, then q" then the Inverse will be "If not p, then not q".

The Inverse of $\boxed{p \to q \text{ is } \sim p \to \sim q}$.

Ex :- 1) If today is Sunday, then tomorrow is Monday.
If today is not Sunday, then tomorrow is not Monday.

2) If p is a Square, then P is a Rectangle.
If p is not a Square, then p is not a Rectangle.

*

* **Contrapositive :-**
If the statement is "If p, then q", then the contra+ will be "If not q, then not p".

The contrapositive of $\boxed{p \to q \text{ is } \sim q \to \sim p.}$

Ex:- 1) If p is a Square, then p is Rectangle.
If p is not Rectangle, then p is not Square.

2) If you do your Homework, you will not be punished.
If you are not punished, then you do not do your H.W.

3) If it rains, then the crop will grow.
If the crop will not grow, then it does not Rain.

*

Question Based on Converse, Inverse, Contrapositive :

Pg - 300 - 301
↳ Ex: 2, 3, 4.

Some more Examples :-

Pg - 303 - 305
↳ Ex: 10, 11, 12, 13, 14

Example 2. (*a*) *Obtain the converse, inverse and contrapositive of the conditional statement* $p \to q$. **[R.G.P.V. Dec. 2002]**

Solution. If p and q are two proposotions then the converse, inverse and contrapositive of $p \to q$ are as follows :

(i) Converse implication is $q \to p$.

(ii) Inverse implication is $\sim p \to \sim q$.

(ii) Contrapositive implication is $\sim q \to \sim p$.

Example 2. (*b*) *Construct converse, inverse and contrapositive of the direct statement 'if* $4x - 2 = 10$ *then* $x = 3$'.

Solution. Let $p \equiv 4x - 2 = 10, q \equiv x = 3$. Thus the statement 'if $4x - 2 = 10$ then $x = 3$' is the direct implication $p \Rightarrow q$.

(i) **Converse.** Converse of '$p \Rightarrow q$' is '$q \Rightarrow p$', hence the converse of given statement is as follows :

"If $x = 3$ then $4x - 2 = 10$".

(ii) Inverse. The inverse of '$p \Rightarrow q$' is '$(\sim p) \Rightarrow (\sim q)$', hence the inverse of the given statement is as follows :

"If $4x - 2 \neq 10$ then $x \neq 3$".

(iii) Contrapositive. The contrapositive of '$p \Rightarrow q$' is '$(\sim q) \Rightarrow (\sim p)$', hence the contrapositive of the given statement is as follows :

"If $x \neq 3$ then $4x - 2 \neq 10$".

Example 3. *Write the converse, inverse and contrapositive of the following direct statements :*

(*a*) *If ABCD is a square then ABCD is a rectangle.*

(*b*) *When sun rises then it is morning.*

Solution (a) Let $p \equiv ABCD$ is a square,

$q \equiv ABCD$ is a rectangle.

$\therefore$ The given statement is the direct implication '$p \Rightarrow q$'.

(i) Converse. Since the converse of '$p \Rightarrow q$' is '$q \Rightarrow p$' so the converse of given statement is :

'If *ABCD* is a rectangle then *ABCD* is a square'.

(ii) Inverse. Since inverse of '$p \Rightarrow q$' is '$(\sim p) \Rightarrow (\sim q)$' so the inverse of given statement is :

'If *ABCD* is not a square then *ABCD* is not a rectangle'.

(iii) Contrapositive. Since contrapositive of '$p \Rightarrow q$' is '$(\sim q) \Rightarrow (\sim p)$', so the contrapositive of the given statement is :

'If *ABCD* is not a rectangle then *ABCD* is not a square'.

(b) Assuming p and q as above in (a), the given statement has :

(i) Converse. 'When it is morning then sun rises'.

(ii) Inverse. 'When sun does not rise then it is not morning'.

(iii) Contrapositive. 'When it is not morning then sun does not rise'.

Example 4. *Let $p \equiv$ it is rainy season, $q \equiv$ the mango is delicious, construct the following statements :*

(*a*) $p \Rightarrow q$, (*b*) *its converse,*

(*c*) *its inverse,* (*d*) *its contrapositive,*

(*e*) *its negation.*

Solution. (a) If it is rainy season, then the mango is delicious.

(b) If the mango is delicious, then it is rainy season.

(c) If it is not rainy season, then the mango is not delicious.

(d) If the mango is not delicious, then it is not rainy season.

(e) It is rainy season and mango is not delicious.

Example 5. *If the truth value of $p \Leftrightarrow q$ is (i) F, (ii) T, then find the truth value of $p \vee q$.*

Solution. We know that the truth value of $p \Leftrightarrow q$ is F, when either one of p and q has truth value F and other T.

Truth table

$p \Leftrightarrow q$	p	q	$p \vee q$
F	T	F	T
	F	T	T

Example 10. *Prepare the truth table of the statement*

$$(P \Rightarrow Q \wedge R) \vee (\sim P \wedge Q).$$

Solution. Truth table for $(P \Rightarrow Q \wedge R) \vee (\sim P \wedge Q)$:

P	Q	R	$Q \wedge R$	$P \Rightarrow (Q \wedge R)$	$\sim P$	$\sim P \wedge Q$	$(P \Rightarrow Q \wedge R) \vee (\sim P \wedge Q)$
T	T	T	T	T	F	F	T
T	T	F	F	F	F	F	F
T	F	T	F	F	F	F	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

Example 11. *Prove that the following statement is a tautology.*

$$(\sim B) \wedge (A \Rightarrow B) \Rightarrow (\sim A).$$

Solution. Truth table for given statement :

A	B	$\sim A$	$\sim B$	$A \Rightarrow B$	$\sim B \wedge (A \Rightarrow B)$	$(\sim B) \wedge (A \Rightarrow B) \Rightarrow (\sim A)$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Hence $(\sim B) \wedge (A \Rightarrow B) \Rightarrow (\sim A)$ is a tautology.

Example 12. *If* $p \equiv$ *Ram is beautiful,*
$q \equiv$ *Ram is mixable,*
$r \equiv$ *His friends like Ram,*

then write the following statements in language :

(*a*) $(p \Rightarrow q) \vee (p \Rightarrow r)$,

(*b*) $p \Rightarrow (q \vee r)$.

Examine, are the above statement equivalent ?

Solution. (a) If Ram is beautiful then either Ram is mixable or his friends like Ram.

(b) If Ram is beautiful then he is mixable or his friends like him.

Combined truth table for (a) and (b) :

p	q	r	$p \Rightarrow q$	$p \Rightarrow r$	$(p \Rightarrow q) \vee (p \Rightarrow r)$	$q \vee r$	$p \Rightarrow (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	F	T

From above truth table, we see that the entries in sixth and eigth column are identical. Hence the statements given by (a) and (b) above are logically equivalent.

Example 13. *Prepare the tables of the following statements :*

(*i*) $(p \Leftrightarrow q) \wedge (r \vee q)$.

(*ii*) $(p \vee q) \wedge (\sim r) \Rightarrow q$.

Solution. (i) Truth table of $p \Leftrightarrow q \wedge (r \vee q)$:

p	q	r	$p \Leftrightarrow q$	$r \vee q$	$(p \Leftrightarrow q) \wedge (r \vee q)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	F	F	F
F	T	T	F	T	F
F	T	F	F	T	F
F	F	T	T	T	T
F	F	F	T	F	F

(ii) Truth table of $(p \vee q) \wedge (\sim r) \Rightarrow q$:

p	q	$\sim r$	$p \vee q$	$(p \vee q) \wedge (\sim r)$	$(p \vee q) \wedge (\sim r) \Rightarrow q$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	T	T	F
T	F	F	T	F	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	F	F	T
F	F	F	F	F	T

Example 14. *Prove by truth table that the following formulae are tautologies* :

(*i*) $(\sim q \Rightarrow \sim p) \wedge (q \Rightarrow p) \Rightarrow (p \Leftrightarrow q)$, (*ii*) $(p \Rightarrow q) \vee (r \Rightarrow p)$.

(*iii*) $(p \Leftrightarrow q \wedge r) \Rightarrow (\sim r \Rightarrow \sim p)$, (*iv*) $(p \Rightarrow q \wedge r) \Rightarrow (\sim r \Rightarrow \sim q)$.

Solution. (i) Let $\sim q \Rightarrow \sim p, q \Rightarrow p, p \Leftrightarrow q$ be respectively denoted by A, B, C. The truth table of given statement is as follows :

p	q	$\sim p$	$\sim q$	A $\sim q \Rightarrow \sim p$	B $q \Rightarrow p$	$A \wedge B$	C $p \Leftrightarrow q$	$(A \wedge B) \Rightarrow C$
T	T	F	F	T	T	T	T	T
T	F	F	T	F	T	F	F	T
F	T	T	F	T	F	F	F	T
F	F	T	T	T	T	T	T	T

Since all entries of last column are T. Hence the given formula is a tautology.

(ii)

p	q	r	$p \Rightarrow q$	$r \Rightarrow p$	$(p \Rightarrow q) \vee (r \Rightarrow p)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	F	T
F	T	F	T	T	T
F	F	T	T	F	T
F	F	F	T	T	T

Since all entries of the last column are T. Hence the given formula is a tautology.

(iii)

p	q	r	$q \wedge r$	A $p \Leftrightarrow q \wedge r$	$\sim r$	$\sim p$	B $\sim r \Rightarrow \sim p$	$A \Rightarrow B$
T	T	T	T	T	F	F	T	T
T	T	F	F	F	T	F	F	T
T	F	T	F	F	F	F	T	T
T	F	F	F	F	T	F	F	T
F	T	T	T	F	F	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	F	T	F	T	T	T
F	F	F	F	T	T	T	T	T

Since all entries of the last column are T. Hence the given formula is a tautology.

(iv) Proceed as above in (iii).

Remark. The connection $\downarrow$, $\uparrow$ and $\oplus$ are called **derived connections.**

Example. *Construct the truth table for the propositions* $p \uparrow q \uparrow r$ *and* $p \oplus q \oplus r$.

Solution.

p	q	r	$p \uparrow q$	$p \uparrow q \uparrow r$	$p \oplus q$	$p \oplus q \oplus r$
T	T	T	F	T	F	T
T	T	F	F	T	F	F
T	F	T	T	F	T	F
T	F	F	T	T	T	T
F	T	T	T	F	T	F
F	T	F	T	T	T	T
F	F	T	T	F	F	T
F	F	F	T	T	F	F

** Algebra of Proposition :-

1) Idempotent Law :- * $p \wedge p \Leftrightarrow p$

* $p \vee p \Leftrightarrow p$

2) Commutative Law :- * $p \vee q = q \vee p$

* $p \wedge q = q \wedge p$

3) Associative Law :- * $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

* $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

4) Distributive Law :- * $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

* $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

5). Demorgan's Law :- * $\sim(p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$

* $\sim(p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$.

Normal form :-

Let $A\{P_1, P_2, P_3 \ldots\ldots P_n\}$ be a statement formula then the construction of truth table may not be practical always. So, we consider alternate procedure known as Reduction to Normal form.

There are 2 types of Normal form :-

1) Disjunctive Normal form [DNF]
2) Conjunctive Normal form [CNF]

* Disjunctive Normal form :-

1) A formula, (or statement) which consist a sum of elementary Product, is called Disjunctive Normal form.

2) A statement form which consist of Disjunction between Conjuction is called DNF.

3) It is Sum of Elementary product.

$\{(\wedge) \vee (\wedge)\}$
Disjunction b/w conjuction

Ex :- 1) $(p \wedge q) \vee (\sim p \wedge q)$

2) $p \vee (q \wedge r)$

3) $p \vee (\sim q \wedge r)$

* Conjunctive Normal form :-

1) A formula which consist the product of elementary sums, is called CNF.

2) A statement form which consist of conjunction b/w disjunction is called CNF.

3) It is product of Elementary Sum.

$\{(\vee) \wedge (\vee)\}$
conjuction b/w Disjunction

Ex :- 1) $(p \vee q) \wedge (\sim p \vee \sim q)$

2) $p \wedge (q \vee r)$

3) $\sim p \wedge (q \vee r)$

* Method to Solve CNF or DNF :-

1) Remove $\rightarrow$ and $\longleftrightarrow$ by using logical Equivalence.
2) Eliminate $\sim$ before sum and product by De-Morgan's law
3) Apply Distributive law.

Note some logical Equivalence :

1) $p \rightarrow q \equiv \sim p \vee q$

2) $p \longleftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ (for CNF)

$p \longleftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$ (for (DNF)

3) Demorgan's law :-

$\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

$\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$

4) Distributive law :-

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

** Examples :-

Statement	DNF	CNF
1) $p \wedge (p \rightarrow q)$	$p \wedge (\sim p \vee q)$ $[p \wedge (\sim p)] \vee (p \wedge q)$	$p \wedge (\sim p \vee q)$
2) $(p \rightarrow q) \wedge (\sim q)$	$(\sim p \vee q) \wedge (\sim q)$ $(\sim p \wedge \sim q) \vee (q \wedge \sim q)$	$(\sim p \vee q) \wedge (\sim q)$

Ex:1 Obtain the DNF of the form $(p \to q) \wedge (\sim p \wedge q)$.

Sol:- we know that;

$$(p \to q) \equiv (\sim p \vee q)$$

then,

$$= (p \to q) \wedge (\sim p \wedge q)$$

$$= (\sim p \vee q) \wedge (\sim p \wedge q)$$

Apply Distributive Law

$$[(\sim p \wedge \sim p) \wedge (\sim p \wedge q)] \vee [(q \wedge \sim p) \wedge (q \wedge q)]$$

$$[\sim p \wedge q] \vee [q \wedge (\sim p)] \quad \text{— Ans}$$

By Idempotent Law
$$\left\{ \begin{array}{l} \because \sim p \wedge \sim p = \sim p \\ \because q \wedge q = q \end{array} \right.$$

Ex:2:- obtain the CNF of the form $(p \wedge q) \vee (\sim p \wedge q \wedge r)$:

Sol:- Given in Que: $(p \wedge q) \vee (\sim p \wedge q \wedge r)$

By using Distributive Law:-

$$[p \vee (\sim p \wedge q \wedge r)] \wedge [q \vee (\sim p \wedge q \wedge r)]$$

$$\{(p \vee \sim p) \wedge (p \vee q) \wedge (p \vee r)\} \wedge \{(q \vee (\sim p) \wedge (q \vee q) \wedge (q \wedge r)\}$$

$$\{(p \vee q) \wedge (p \vee r)\} \wedge \{[q \vee (\sim p)] \wedge q \wedge (q \wedge r)\} \quad \text{— Ans}$$

*

$$\left\{ \begin{array}{l} \because q \vee q = q \text{ ; Idempotent Law} \\ \because p \vee \sim p = 1 \text{ ; Negation} \end{array} \right.$$

Que: Obtain the DNF of : $p \vee (\sim p \to (q \vee (q \to \sim r)))$.

Sol:- Given :- $p \vee (\sim p \to (q \vee (q \to \sim r)))$

$\because$ we know that $p \to q = \sim p \vee q$

So, $q \to -r = \sim q \vee \sim r$

$p \vee (\sim p \to [q \vee (\sim q \vee \sim r)])$

apply Distributive Law,

$p \vee (\sim p \to [(q \vee \sim q) \vee (q \vee \sim r)])$

$\{\because q \vee \sim q = 1$ negation $\}$

$p \vee (\sim p \to (q \vee \sim r))$ $\left\{\begin{array}{l} \because \sim p \to q = \sim p \vee q \\ \sim(\sim p) = p \end{array}\right.$

$p \vee (p \vee (q \vee \sim r))$

$p \vee ((p \vee q) \vee (p \vee \sim r))$

$p \vee \{p \vee q \vee \sim r\}$

$(p \vee p) \vee (p \vee q) \vee (p \vee \sim r)$ $\{\because p \vee p = p\}$

$\{p \vee q \vee \sim r\}$

So, the DNF of $p \vee \{\sim p \to (q \vee (q \to \sim r))\}$ is

$\{p \vee q \vee \sim r\}$.

*

Que: Obtain CNF of $(p \to q) \wedge (q \vee (p \wedge r))$ and determine whether or not it is tautology.

Sol:- we know that:

$$= (p \to q) \wedge (q \vee (p \wedge r))$$

$$= (\sim p \vee q) \wedge (q \vee p) \wedge (q \vee r) = \text{CNF}.$$

Truth Table :-

p	q	r	$\sim p$	$\sim p \vee q$	$q \vee p$	$q \vee r$	(C.N.F) $(\sim p \vee q) \wedge (q \vee p)$	$(\sim p \vee q) \wedge (q \vee p) \wedge (q \vee r)$
T	T	T	F	T	T	T	T	T
T	T	F	F	T	T	T	T	T
T	F	T	F	F	T	T	F	F
T	F	F	F	F	T	F	F	F
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T	T
F	F	T	T	T	F	T	F	F
F	F	F	T	T	F	F	F	F

Hence, the given statement is not tautology.

*.

Que :- Find CNF of $p \wedge (p \rightarrow q)$

Que :- Find CNF of $(\sim p \rightarrow r) \wedge (q \leftrightarrow p)$

Que :- Express the following formula into DNF & CNF:

$$\sim(p \vee q) \leftrightarrow (p \wedge q).$$

*

Quantifiers :-

Quantifiers are the words that refer to Quantities such as "Some" or "all". It tells for how many elements a given predicate is True.

Quantifiers are used to Express the Quantities without giving an exact No..

The word or a Symbol which convey the idea of Quantity or No. are called Quantifers.

Ex:- all, some, many, none, few, etc.

Sentence like: "Can I have some water?"
"Jack has many friends here."

There are 2 most Important type of Quantifiers :-

1) Universal Quantifier.
2) Existential Quantifier.

* **Universal Quantifiers** :-

It states that the statement within its Scope are true "for every value" of the Specific Variable. It is denoted by symbol. $\forall$.

we write it as "$\forall x\, P(x)$" which means that;
$P(x)$ is true for every x in the Domain.
$\forall x$ represent "for all x", "for every x", "for each x".

Ex:- All Human Beings are mortal - ①

$P(x) : x$ is mortal.

Eqⁿ ① can be written as,

$(\forall x \in U)\, P(x)$ OR $\forall x . P(x)$

where, U - domain or universe of Discourse.

Ex:- Let $P(n)$ be a statement $x+1 > x$, where x is +ve Z.

$P(1) : 1+1 > 1 \quad = 2 > 1$ (True)

$P(2) : 2+1 > 2 \quad = 3 > 2$ (True)

$P(3) : 3+1 > 3 \quad = 4 > 2$ (True).

⋮

$P(x) : x+1 > x$ is true for all +ve Integer

OR

$\forall x (Px)$ is true.

*

* Existential Quantifiers :-

It State that "the Statement within its scope are true for some values of the specific Variable. It is denoted by $\exists$.

we can write it as "$\exists x\, P(x)$" which means that, $P(x)$ is true for some x in the Domain.

$\exists x$ represents "there exists an x"

"there is an x"

"for some x"

"there is at least one x"

Ex:- Some Human Beings are mortal

$P(x) : x$ is mortal. $\Rightarrow \exists x\, P(x)$

Ex:- Let $Q(x)$ be the statement $x < 2$. where x is +ve Integer.

$Q(x) : x < 2$

$Q(1) : 1 < 2$ (True)

$Q(2) : 2 < 2$ (False)

$Q(3) : 3 < 2$ (False)

⋮

Question: Is there some value of x for which $Q(x)$ is True? (If domain is set of +ve Integer).

Answer: Yes, $Q(x)$ is true for $x = 1$

∴ there exist an x for which $Q(x)$ is True

OR

$\exists x\, Q(x)$ is true.

*

* Negation Quantifiers :-

To construct the Negation of a Quantified Statement, change the Quantifier from Universal to Existential and Existential to Universal and also replace the Statement :-

Statement	Negation
1) $\forall x\, P(x)$ {all true}	1) $\exists x\, [\sim P(x)]$ {at least one false}
2) $\exists x\, [\sim P(x)]$ {at least one false}	2) $\forall x\, P(x)$ {all true}
3) $\forall x\, [\sim P(x)]$ {all false}	3) $\exists x\, P(x)$ {at least one true}
4) $\exists x\, P(x)$ {at least one false}	4) $\forall x\, [\sim (P(x))]$ {all false}

Que: Negate the statement :-

* All Integers are greater than 8.

Sol:- $P(x)$: x is an Integer.

$Q(x)$: x is greater than 8.

$$\forall x\,[P(x) \rightarrow Q(x)] \equiv \forall x\,[\sim P(x) \vee Q(x)]$$

$$\{\because p \rightarrow q = (\sim p \vee q)\}$$

Negation :- $\exists x \sim[\sim(P(x)) \vee Q(x)]$ $\{$De-Morgans law $\sim(\sim p) = p\}$

$= \exists x\,(P(x) \wedge (\sim Q(x))]$ $\{\sim(p \vee q) = \sim p \wedge \sim q\}$

* Not all students are genius.

$S(x)$: x is student , $G(x)$: x is genius.

$$\sim[\forall x(S(x) \rightarrow G(x))] \equiv \forall x \sim(S(x) \rightarrow G(x))$$

negation : $\exists x\,(S(x) \rightarrow G(x))$

* $(\forall x\, P(x)) \vee (\exists y\, P(y))$

negation : $\sim[\forall x\, P(x) \vee \exists y\,(P(y)]$

$\{\sim(p \vee q) = \sim p \wedge \sim q\}$

$= \sim\forall(x)P(x) \wedge \sim\exists y\, P(y)$

$= \exists x(\sim P(x)) \wedge \forall y\,[\sim P(y))]$

Example 4. *Negate the statement 'He is poor and laborious'.*

Solution. We know that $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

Hence the negation of the given statement is :

'It is false that he is poor and laborious'.

$\equiv$ 'He is not poor or he is not laborious'.

Example 5. *Negate the statement 'It is daylight and all the people have arisen'.*

Soltion. We know that $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

Hence the negation of the given statement is :

'It is false that it is daylight and all the people have arisen'.

$\equiv$ 'It not daylight or it is false that all the people have arisen'.

$\equiv$ 'It is not daylight or some one has not arisen'.

Example 6. *Negate the statements :*

(*i*) $\exists x\, P(x) \vee \forall y\, Q(y)$.

(*ii*) $\forall x\, P(x) \wedge \exists y\, Q(y)$.

Example 10. *Write the following statement into symbols (using quantifier and the symbol '<' for 'less than') :*

(i) A number x, is less than 7 and greater than 5.

(ii) For a given number x there is a greater number y.

(iii) For two given numbers x and y, there is a number z such that the difference of x and y is less than the product of x^2 and z.

(iv) The numbers x, y, z are such that x + y is greater than xz.

Solution. (i) $\exists x\,[x < 7 \wedge 5 < x]$.

(ii) $\forall x \exists y (x < y)$.

(iii) $\forall x \forall y \exists z\,(|x - y| < x^2 z)$.

(iv) $(\exists x)(\exists y)(\exists z)(xz < x + y)$.

Example 11. *Write the following sentences into symbols :*

(i) The square of any rational number is not 2.

(ii) Two non-parallel coplanor straight lines have a common point.

(iii) If there is no prize, then a person does not purchase a ticket.

Solution. (i) Let $Q(x) = x$ is a rational number

$$E(x, y) \equiv \text{'}x = y\text{'}.$$

The given sentence in symbols, using quantifiers, is

$$\exists x\,[Q(x) \wedge E(x^2, 2)].$$

(ii) Let $P(x) = x$ is a point

$L(x) = x$ is a straight line

$I(x, y) =$ the point x is on y

$D(x, y) = x, y$ are non-parallel and coplanar.

Example 15. *Write the following predicate into symbolic form :*

(*i*) *There are real numbers which are greater than all real numbers.*

(*ii*) *Some ladies lawyer are such that they are also homely ladies.*

(*iii*) *Some ladies are lawyer and also homely ladies.*

(*iv*) *Every student is a member of N.C.C. or N.S.S., but some students are not players.*

(*v*) *If a and b are non-zero integers and p is a prime number such that* $p \mid ab$ *then* $p \mid a$ *or* $p \mid b$.

Solution. (i) Let $R(x) \equiv x$ is real number

$$G(x, y) \equiv x > y.$$

Then the given predicate in symbolic form is

$$\therefore \qquad \forall\, x\, [R(x) \Rightarrow \exists\, y\, \{R(y) \wedge G(x, y)\}].$$

(ii) $W(x) \equiv x$ is lady

$L(x) \equiv x$ is lawyer

$H(x) \equiv x$ is homely lady.

Then $\exists\, x\, [W(x) \wedge L(x) \wedge H(x)]$.

(iii) Same as (ii).

(iv) $S(x) \equiv x$ is student.

$N(x) \equiv x$ is a member of N.C.C.

$M(x) \equiv x$ is a member of N.S.S.

$P(x) \equiv x$ is a player.

Then $\forall\, x\, [S(x) \Rightarrow \{N(x) \wedge M(x)\}] \wedge \exists\, y\, [S(y) \wedge \sim P(y)]$.

(v) $I(x) \equiv x$ is a non-zero integer

$P(x) \equiv x$ is prime number

$D(x, y) \equiv x$ divides y.

Then $\qquad \forall\, a, b, p\, [I(a) \wedge I(b) \wedge P(p) \wedge D(p, ab)] \Rightarrow [D(p, a) \vee D(p, b)]$.

each.

5. Explain the following terms and also give examples to explain them :

   (a) Quantifier.

   (b) Universal quantifier.

   (c) Existential quantifier.

   (d) Negation of a quantifier.

6. Define Predicate