Mathematics I

B.Tech. I Semester, all Branches

Shukla B.B.

Nature is always subject to Change,
Change defines Rate,
Rate gives rise to Differentials,
Hence, Derivatives and Differentals are omnipresent

— Satish Shukla

To my teachers and family,
with whose love and support
nothing is unattainable.

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Unit-I

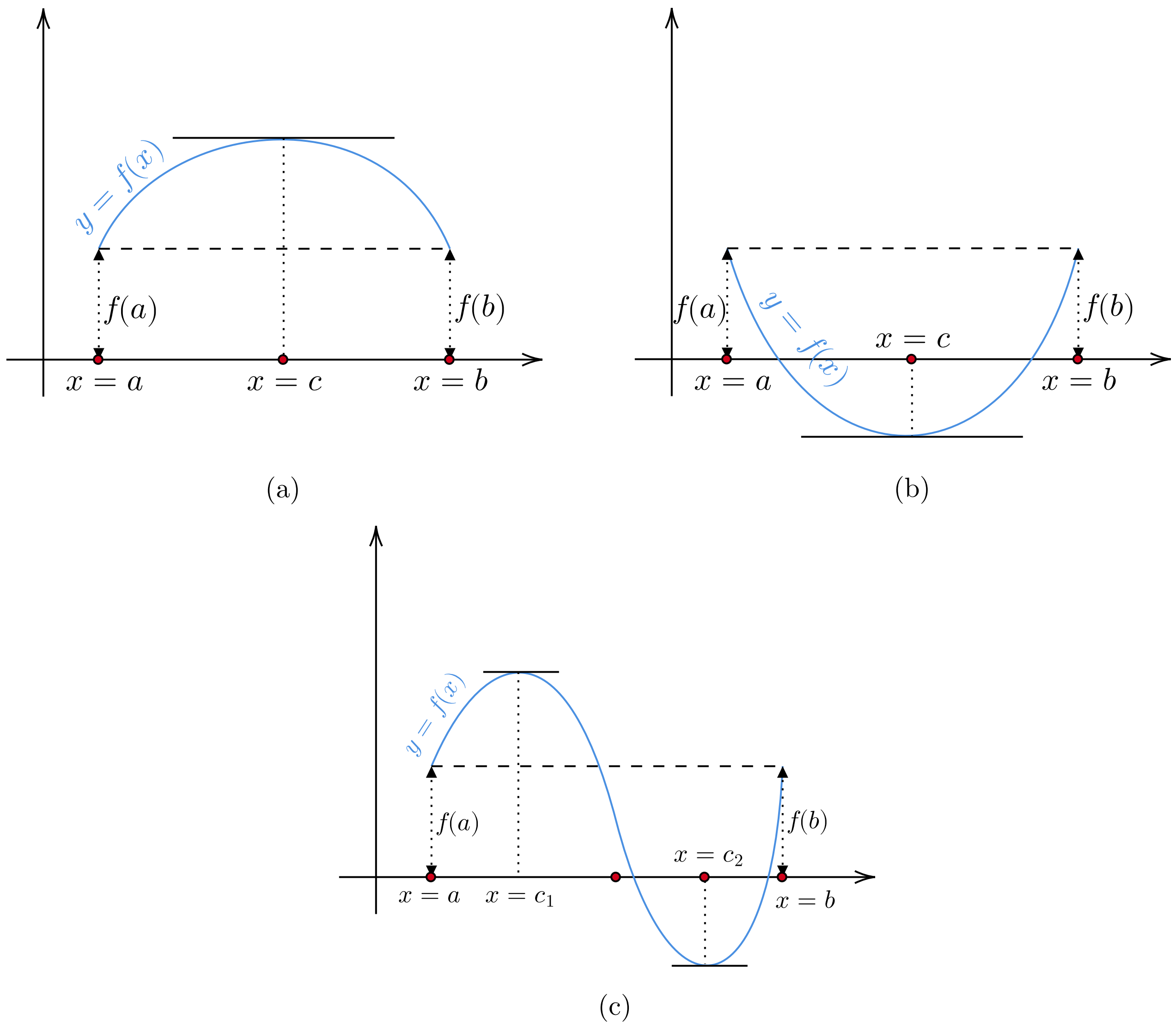
Differential Calculus: Rolle's theorem, mean value theorem, expansion of functions of one variable, Taylors series Maclaurin series.

The history of the mean value theorem and its variants are studied by several mathematicians. Vatasseri Parameshvara Nambudiri (1380-1460) was a Hindu from the Bhrgugotra, adhering to the Ashvalayanasutra of the Rigveda. His family, known by the surname Vatasseri, lived in the village of Alathiyur (referred to as Asvatthagrama in Sanskrit) in Tirur, Kerala. Alathiyur is located on the northern bank of the river Nila (Bharathappuzha) at its mouth in Kerala. He was the grandson of a disciple of Govinda Bhattathiri (1237-1295 CE), a renowned figure in Kerala's astrological traditions.
Parameshvara wrote commentaries on many mathematical and astronomical works, including those by Bhaskara I and Aryabhata. Over 55 years, he made a series of eclipse observations, constantly comparing these with the theoretically computed positions of the planets. He revised planetary parameters based on his observations. One of Parameshvara's most significant contributions was his mean value type formula for the inverse interpolation of the sine (see, [1], [2]). In 1691, Michel Rolle proved a particular case of the generalized mean value theorem. In the modern form, the mean value theorem was proved by Augustin-Louis Cauchy in 1823. A more formal statement of this theorem was known as Lagrange's mean value theorem, named after Joseph-Louis Lagrange.

1.1 Rolle's theorem

Theorem 1. Let f be a function which is continuous everywhere on the interval $[a, b]$ and has a derivative at each point of the open interval (a, b). Also, assume that $f(a) = f(b)$. Then there is at least one point c in the interval (a, b) such that $f'(c) = 0$.

Proof. We prove the Rolle's theorem geometrically.
Since $f(a) = f(b)$ and function f is continuous in $[a, b]$ we have the following three cases:



Geometric Interpretation of Roll's Theorem

Case (a): Suppose that the function increases after point $x = a$. Since $f(a) = f(b)$ and function f is continuous, there must exist a point c such that $a < c < b$ and f has its maximum value at c. Therefore, we have $f'(c) = 0$.

Case (b): Suppose that the function decreases after point $x = a$. Since $f(a) = f(b)$ and function f is continuous, there must exist a point c such that $a < c < b$ and f has its minimum value at c. Therefore, we have $f'(c) = 0$.

Case (c): Suppose that the function increases after point $x = a$ and then attains its maximum values and then decreases and attains its minimum value, i.e., function oscillates. Since $f(a) = f(b)$ and function f is continuous, it finally returns to its initial value. Thus, we have more than one point $c_1, c_2, \ldots$ such that $a < c_1 < c_2 < \cdots < b$ and f has its maximum and minimum values at $c_1, c_2, \ldots$. Therefore, we have $f'(c_1) = f'(c_2) = \cdots = 0$.

Thus, in each case, we obtain the desired point c. □

Example 1.1. Verify the Rolle's theorem for $f(x) = |x|$ in $[-1, 1]$.

Solution. Here $a = -1, b = 1$. Given function $f(x)$ is continuous in $[-1, 1]$ and $f(a) = f(1) = |1| = 1$, $f(b) = f(-1) = |-1| = 1$, but we know that the function $f(x) = |x|$ is not differentiable at point $x = 0$, and $0 \in [-1, 1]$, therefore the Rolle's theorem cannot be verified. □

Example 1.2. Verify the Rolle's theorem for $f(x) = e^x \sin x$ in $[0, \pi]$.

Solution. Here $a = 0, b = \pi$. Given function $f(x)$ is continuous in $[0, \pi]$ and $f(a) = f(0) = e^0 \sin 0 = 0$, $f(b) = f(\pi) = e^\pi \sin \pi = 0$, so $f(a) = f(b)$. Also, the function $f(x) = e^x \sin x$ is differentiable at every point of the interval $(0, \pi)$. Therefore, all the conditions of Rolle's theorem are satisfied and by Rolle's theorem, there exists $0 < c < \pi$ such that $f'(c) = 0$. Then

$$f'(x) = \frac{d}{dx}(e^x \sin x) = e^x \sin x + e^x \cos x.$$

Therefore,

$$\begin{aligned} f'(c) = 0 &\implies e^c \sin c + e^c \cos c = 0 \implies e^c[\sin c + \cos c] = 0 \\ &\implies \sin c + \cos c = 0 \implies (\sin c + \cos c)^2 = 0 \\ &\implies \sin 2c = -1 \implies 2c = \frac{3\pi}{2} \\ &\implies c = \frac{3\pi}{4}. \end{aligned}$$

Since $c = \dfrac{3\pi}{4} \in (0, \pi)$ the Rolle's theorem is verified. □

Example 1.3. Verify the Rolle's theorem for $f(x) = \sin 3x$ in $\left[0, \dfrac{\pi}{3}\right]$.

Solution. Here $a = 0, b = \dfrac{\pi}{3}$. Given function $f(x)$ is continuous in $\left[0, \dfrac{\pi}{3}\right]$ and $f(a) = f(0) = \sin 0 = 0$, $f(b) = f\left(\dfrac{\pi}{3}\right) = \sin\left(\dfrac{3\pi}{3}\right) = 0$, so $f(a) = f(b)$.

Also, the function $f(x) = \sin 3x$ is differentiable at every point of the interval $\left(0, \frac{\pi}{3}\right)$. Therefore, all the conditions of Rolle's theorem are satisfied and by Rolle's theorem, there exists $c \in \left(0, \frac{\pi}{3}\right)$ such that $f'(c) = 0$. Then

$$f'(x) = \frac{d}{dx}(\sin 3x) = 3\cos 3x.$$

Therefore,

$$\begin{aligned} f'(c) = 0 \quad &\implies \quad 3\cos 3c = 0 \implies \cos 3c = 0 \\ &\implies \quad 3c = \frac{\pi}{2} \\ &\implies \quad c = \frac{\pi}{6}. \end{aligned}$$

Since $c = \frac{\pi}{6} \in \left(0, \frac{\pi}{3}\right)$ the Rolle's theorem is verified. □

Example 1.4. Verify the Rolle's theorem for $f(x) = \cos 2x$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

Solution. Here $a = -\frac{\pi}{4}, b = \frac{\pi}{4}$. Given function $f(x)$ is continuous in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ and $f(a) = f\left(-\frac{\pi}{4}\right) = \cos\left(-\frac{2\pi}{4}\right) = 0$, $f(b) = f\left(\frac{\pi}{4}\right) = \cos\left(\frac{2\pi}{4}\right) = 0$, so $f(a) = f(b)$. Also, the function $f(x) = \cos 2x$ is differentiable at every point of the interval $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$. Therefore, all the conditions of Rolle's theorem are satisfied and by Rolle's theorem, there exists $c \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ such that $f'(c) = 0$. Then

$$f'(x) = \frac{d}{dx}(\cos 3x) = -2\sin 2x.$$

Therefore,

$$\begin{aligned} f'(c) = 0 \quad &\implies \quad -2\sin 2c = 0 \implies \sin 2c = 0 \\ &\implies \quad 2c = 0 \\ &\implies \quad c = 0. \end{aligned}$$

Since $c = 0 \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ the Rolle's theorem is verified. □

Example 1.5. Verify the Rolle's theorem for $f(x) = 2 + (x-1)^{2/3}$ in $[0, 2]$.

Solution. Here $a = 0, b = 2$. Given function $f(x)$ is continuous in $[0, 2]$ and $f(a) = f(0) = 2 + (0-1)^{2/3} = 3$, $f(b) = f(2) = 2 + (2-1)^{2/3} = 3$, so $f(a) = f(b)$. Note that f is not differentiable in the interval $(0, 2)$. Indeed:

$$f'(x) = \frac{d}{dx}\left(2 + (x-1)^{2/3}\right) = \frac{2}{3}(x-1)^{-1/3}.$$

Therefore, $f'(1)$ does not exist and since $1 \in (0, 2)$, therefore all the conditions of Rolle's theorem are not satisfied, and so, it cannot be verified. □

Example 1.6. Verify the Rolle's theorem for $f(x) = x^3 - 4x$.

Solution. Here the interval where the theorem is to be verified is not given. To find the interval put $f(x) = 0$, i.e.,

$$x^3 - 4x \implies x(x^2 - 4) = 0 \implies x = 0, \pm 2.$$

So we obtain the intervals $[-2, 0], [0, 2]$ and $[-2, 2]$. Given function $f(x)$ is a polynomial in x, so, continuous and differentiable everywhere and $f(-2) = f(0) = f(2) = 0$. Therefore, all the conditions of Rolle's theorem are satisfied and by Rolle's theorem, there exists $c \in (0, 2)$ such that $f'(c) = 0$. Then

$$f'(x) = \frac{d}{dx}\left(x^3 - 4x\right) = 3x^2 - 4.$$

Therefore,

$$f'(c) = 0 \implies 3c^2 - 4 = 0 \implies c = \pm\frac{2}{\sqrt{3}}.$$

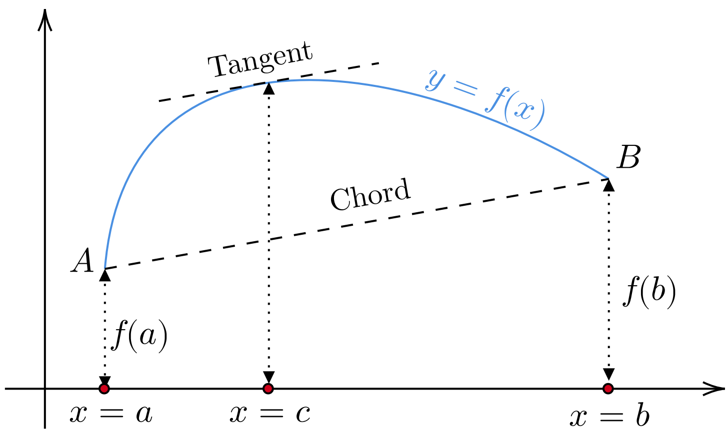
Since $c = -\frac{2}{\sqrt{3}} \in (-2, 0)$ and $c = \frac{2}{\sqrt{3}} \in (0, 2)$ the Rolle's theorem is verified.

□

1.2 Mean value theorem or Lagrange's mean value theorem

Theorem 2. Let f be a function which is continuous everywhere on the interval $[a, b]$ and has a derivative at each point of the open interval (a, b). Then there is at least one point c in the interval (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Proof. We prove this theorem with the help of Rolle's theorem.



Geometric Interpretation of Mean Value Theorem

Define a function $F(x)$ by

$$F(x) = f(x) + \alpha x \tag{1.1}$$

where α is an arbitrary constant. Then, we shall show that F satisfies all the conditions of Rolle's theorem. Then:

(I) Since f is continuous in $[a, b]$ and αx is a polynomial, it is continuous everywhere, and so, their sum $F(x) = f(x) + \alpha x$ is also continuous in $[a, b]$.

(II) Since f is differentiable in (a, b) and αx is a polynomial, it is differentiable everywhere, and so, their sum $F(x) = f(x) + \alpha x$ is also differentiable in (a, b).

(III) Finally, since α was an arbitrary constant, choose α such that:

$$\begin{aligned} F(a) = F(b) \quad &\Longrightarrow \quad f(a) + \alpha a = f(b) + \alpha b \\ &\Longrightarrow \quad \alpha = -\frac{f(b) - f(a)}{b - a)}. \end{aligned}$$

Thus, F satisfies all the conditions of Rolle's theorem. Therefore, by Rolle's theorem there exists $c \in (a, b)$ such that

$$\begin{aligned} F'(c) = 0 \quad &\Longrightarrow \quad f'(c) + \alpha = 0 \\ &\Longrightarrow \quad f'(c) = -\alpha \\ &\Longrightarrow \quad f'(c) = \frac{f(b) - f(a)}{b - a)}. \end{aligned}$$

Hence the proof is complete. □

Example 1.7. Find the c of mean value theorem for the function $f(x) = (x-1)(x-2)(x-3)$ in the interval $[0, 4]$.

Solution. Here $a = 0, b = 4$ and the function f is a polynomial, so, it is continuous and differentiable everywhere. Therefore, all the conditions of the mean value theorem are satisfied. By mean value theorem there exists a point $c \in (0, 4)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. Now

$$f'(x) = (x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2) = 3x^2 - 12x + 11.$$

Therefore,

$$\begin{aligned} f'(c) = \frac{f(b) - f(a)}{b - a} \quad &\Longrightarrow \quad 3c^2 - 12c + 11 = \frac{f(4) - f(0)}{4 - 0} \\ &\Longrightarrow \quad 3c^2 - 12c + 11 = \frac{6 - (-6)}{4} \\ &\Longrightarrow \quad 3c^2 - 12c + 11 = 3 \\ &\Longrightarrow \quad 3c^2 - 12c + 8 = 0 \\ &\Longrightarrow \quad c = 2 \pm \frac{2\sqrt{3}}{3}. \end{aligned}$$

Since $c = \dfrac{2\sqrt{3}}{3} \in (0, 4)$, hence the mean value theorem is verified. □

Example 1.8. Verify the mean value theorem for the function $f(x) = \ln x$ in the interval $\dfrac{1}{e} \leq x \leq e$.

Solution. Here $a = \dfrac{1}{e}, b = e$ and the function f is logarithmic, and so, it is continuous in the interval $\left[\frac{1}{e}, e\right]$ and differentiable in the interval $\left(\frac{1}{e}, e\right)$. Therefore, all the conditions of the mean value theorem are satisfied. By mean value theorem there exists a point $c \in \left(\frac{1}{e}, e\right)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. Now

$$f'(x) = \frac{1}{x}.$$

Therefore,

$$\begin{aligned} f'(c) = \frac{f(b)-f(a)}{b-a} \quad &\Longrightarrow \quad \frac{1}{c} = \frac{f(e) - f\left(\frac{1}{e}\right)}{e - 1/e} \\ &\Longrightarrow \quad \frac{1}{c} = \frac{\ln(e) - \ln\left(\frac{1}{e}\right)}{e - 1/e} \\ &\Longrightarrow \quad \frac{1}{c} = \frac{e(1-(-1))}{e^2-1} \\ &\Longrightarrow \quad c = \frac{e^2-1}{2e}. \end{aligned}$$

Since $c = \dfrac{e^2-1}{2e} \in \left(\frac{1}{e}, e\right)$, hence the mean value theorem is verified. □

Example 1.9. Verify the mean value theorem for the function $f(x) = \ln x$ in the interval $[1, e]$.

Solution. Here $a = 1, b = e$ and the function f is logarithmic, and so, it is continuous in the interval $[1, e]$ and differentiable in the interval $(1, e)$. Therefore, all the conditions of the mean value theorem are satisfied. By mean value theorem there exists a point $c \in (1, e)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$. Now

$$f'(x) = \frac{1}{x}.$$

Therefore,

$$\begin{aligned} f'(c) = \frac{f(b)-f(a)}{b-a} \quad &\Longrightarrow \quad \frac{1}{c} = \frac{f(e) - f(1)}{e-1} \\ &\Longrightarrow \quad \frac{1}{c} = \frac{\ln(e) - \ln(1)}{e-1} \\ &\Longrightarrow \quad \frac{1}{c} = \frac{1}{e-1} \\ &\Longrightarrow \quad c = e - 1. \end{aligned}$$

Since $c = e - 1 \in (1, e)$, hence the mean value theorem is verified. □

Example 1.10. Show that on the graph of any quadratic polynomial the chord joining the points for which $x = a, x = b$ is parallel to the tangent line at the midpoint $x = \dfrac{a+b}{2}$.

OR

If $f(x) = \alpha x^2 + \beta x + \gamma$, where α, β, γ are constants and $\alpha \neq 0$, then find the value of c in Lagrange's mean value theorem in the interval $[a, b]$.

Solution. The function f is polynomial, and so, it is continuous in the interval $[a, b]$ and differentiable in the interval (a, b). Therefore, all the conditions of the mean value theorem are satisfied. By mean value theorem there exists a point $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. Now

$$f'(x) = 2\alpha x + \beta.$$

Therefore,

$$\begin{aligned} f'(c) = \frac{f(b) - f(a)}{b - a} \quad &\Longrightarrow \quad 2\alpha c + \beta = \frac{f(b) - f(a)}{b - a} \\ &\Longrightarrow \quad 2\alpha c + \beta = \frac{\alpha b^2 + \beta b + \gamma - \left(\alpha a^2 + \beta a + \gamma\right)}{b - a} \\ &\Longrightarrow \quad 2\alpha c + \beta = \frac{\alpha(b^2 - a^2) + \beta(b - a)}{b - a} \\ &\Longrightarrow \quad 2\alpha c + \beta = \alpha(b + a) + \beta \\ &\Longrightarrow \quad c = \frac{b + a}{2} \text{ midpoint of } a, b. \end{aligned}$$

Since $f'(c) = \dfrac{f(b) - f(a)}{b - a}$, hence the slope of the tangent at midpoint $c = \dfrac{b + a}{2}$ (i.e., $f'(c)$) is equal to the slope of a chord at the endpoints a, b. Therefore, the tangent and cord are parallel. □

1.3 Exercise

(Q.1) Discuss the conditions of Rolle's theorem for the function $f(x) = \tan x$ in the interval $0 \leq x \leq \pi$.

Ans. Since $\tan x$ is not continuous at $x = \frac{\pi}{2}$, the Rolle's theorem is not applicable.

(Q.2) Verify the Rolle's theorem for the function $f(x) = x^2$ in the interval $[-1, 1]$.

Ans. $c = 0$.

(Q.3) Can Rolle's theorem be applied for the function $f(x) = 1 - (x-3)^{2/3}$.

Hint. For the interval, put $f(x) = 0$, it gives the interval $[2, 4]$. Then, since f is not differentiable at $x = 3 \in (2, 4)$, so, Rolle's theorem cannot be verified.

(Q.4) Explain Rolle's theorem for the function $f(x) = (x-a)^m(x-b)^n$ in the interval $[a, b]$.

Ans. $c = \frac{mb+na}{m+n} \in (a, b)$.

(Q.5) Find the c of mean value theorem for the function $f(x) = x^3$ in the interval $[-2, 2]$.

Ans. $c = \pm\frac{2}{\sqrt{3}}$.

(Q.6) Verifiy mean value theorem for the function $f(x) = x^3 - 3x - 1$ in the interval $[0, 1]$.

Ans. $c = \frac{1}{\sqrt{3}}$.

1.4 Taylor's theorem

Theorem 3. Suppose that the $(n-1)$th derivative $f^{(n-1)}$ of f is continuous on the interval $[a, b]$ and the nth derivative $f^{(n)}$ of f exists in the open interval (a, b). Then for each $x \neq a$ in I there is a value c such that $a < c < x$ and

$$\begin{aligned} f(x) &= f(a) + \frac{x-a}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) \\ &\quad + \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(c). \end{aligned}$$

The last term $R_n = \dfrac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(c)$ is called the remainder therm (Lagrange's form) after n terms.

1.4.1 Taylor's series

Theorem 4. Suppose $R_n \to 0$ as $n \to \infty$, then the expression for $f(x)$ in the Taylor's theorem reduces into an infinite series and this series is called the Taylor's series or Taylor's series expansion of $f(x)$ about the point $x = a$; and it is given by:

$$f(x) = f(a) + \frac{x-a}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots.$$

1.4.2 Various forms of Taylor's series

Maclaurin's series Put $a = 0$ in Taylor's series obtain:

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots .$$

Expansion of $f(x+h)$ in powers of x Replace x by $x+h$ and a by h in Taylor's series obtain:

$$f(x+h) = f(h) + \frac{x}{1!}f'(h) + \frac{x^2}{2!}f''(h) + \cdots + \frac{x^n}{n!}f^{(n)}(h) + \cdots .$$

Expansion of $f(x+h)$ in powers of h Replace x, h by h, x respectively, in the previous series:

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + \cdots .$$

Example 1.11. Expand $\ln\left(\frac{1+x}{1-x}\right)$ using Maclaurin's theorem.

Solution. Here $f(x) = \ln\left(\frac{1+x}{1-x}\right)$. By Maclaurin's theorem we know that

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots .$$

Putting $y = f(x)$, $(y)_0 = f(0), (y_1)_0 = f'(0), (y_2)_0 = f''(0)$ etc., in the above we obtain:

$$y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \cdots + \frac{x^n}{n!}(y_n)_0 + \cdots . \tag{1.2}$$

Now differentiating successively and putting $x = 0$ we obtain:

$$\begin{aligned} y = f(x) = \ln(1+x) - \ln(1-x) &\implies (y)_0 = 0 \\ y_1 = \frac{1}{1+x} + \frac{1}{1-x} &\implies (y_1)_0 = 2 \\ y_2 = -\frac{1}{(1+x)^2} + \frac{1}{(1-x)^2} &\implies (y_2)_0 = 0 \\ y_3 = \frac{2}{(1+x)^3} + \frac{2}{(1-x)^3} &\implies (y_3)_0 = 4 \end{aligned}$$

$$y_4 = -\frac{6}{(1+x)^4} + \frac{6}{(1-x)^4} \implies (y_4)_0 = 0$$
$$y_5 = \frac{24}{(1+x)^5} + \frac{24}{(1-x)^5} \implies (y_5)_0 = 48 \text{ and so on.}$$

Putting these values in (1.2) we obtain:

$$\begin{aligned}\ln\left(\frac{1+x}{1-x}\right) &= 0 + \frac{x}{1!}(2) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(4) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(48) + \cdots \\ &= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots .\end{aligned}$$

It is the required series. □

Example 1.12. If $\ln \sec x = \frac{1}{2}x^2 + Ax^4 + Bx^6 + \cdots$, then find the values of A and B.

Solution. Since the given value of $\ln \sec x$ is a series in powers of x, we will expand $\ln \sec x$ by Maclaurin's series. Then, here $f(x) = \ln \sec x$ and by Maclaurin's theorem we know that

$$y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \cdots + \frac{x^n}{n!}(y_n)_0 + \cdots . \tag{1.3}$$

Now differentiating successively and putting $x = 0$ we obtain:

$$y = f(x) = \ln \sec x \implies (y)_0 = 0$$
$$y_1 = \frac{\sec x \tan x}{\sec x} = \tan x \implies (y_1)_0 = 0$$
$$y_2 = \sec^2 x = 1 + \tan^2 x = 1 + y_1^2 \implies (y_2)_0 = 1$$
$$y_3 = 2y_1y_2 \implies (y_3)_0 = 2(y_1)_0(y_2)_0 = 0$$
$$y_4 = 2y_1y_3 + 2y_2y_2 = 2y_1y_3 + 2y_2^2 \implies (y_4)_0 = 2(y_1)_0(y_3)_0 + 2(y_2)_0^2 = 2$$

$$y_5 = 2y_1y_4 + 2y_2y_3 + 4y_2y_3 = 2y_1y_4 + 6y_2y_3$$
$$\implies (y_5)_0 = 2(y_1)_0(y_4)_0 + 6(y_2)_0(y_3)_0 = 0$$
$$y_6 = 2y_1y_5 + 2y_2y_4 + 6y_2y_4 + 6y_3y_3 = 2y_1y_5 + 8y_2y_4 + 6y_3^2$$
$$\implies (y_6)_0 = 2(y_1)_0(y_5)_0 + 8(y_2)_0(y_4)_0 + 6y_3^2(0) = 16 \text{ and so on.}$$

Putting these values in (1.3) we obtain:

$$\begin{aligned}\ln\sec x &= 0+\frac{x}{1!}(0)+\frac{x^2}{2!}(1)+\frac{x^3}{3!}(0)+\frac{x^4}{4!}(2)+\frac{x^5}{5!}(0)+\frac{x^6}{6!}(16)+\cdots\\ &= \frac{1}{2}x^2+\frac{1}{12}x^4+\frac{1}{45}x^6+\cdots.\end{aligned}$$

On comparing the coefficients of various powers of x in the above and given series we obtain

$$A=\frac{1}{12},\ B=\frac{1}{45}.$$

□

Example 1.13. Find the first five terms in the expansion of $e^{\sin x}$ by Maclaurin's series.

Solution. Here $f(x)=e^{\sin x}$. By Maclaurin's theorem we know that

$$y=(y)_0+\frac{x}{1!}(y_1)_0+\frac{x^2}{2!}(y_2)_0+\cdots+\frac{x^n}{n!}(y_n)_0+\cdots. \tag{1.4}$$

Now differentiating successively and putting $x=0$ we obtain:

$$\begin{aligned}y=f(x)=e^{\sin x} &\implies (y)_0=1\\ y_1=\cos x e^{\sin x}=y\cos x &\implies (y_1)_0=1\\ y_2=y_1\cos x-y\sin x &\implies (y_2)_0=1\\ y_3=y_2\cos x-2y_1\sin x-y\cos x=y_2\cos x-2y_1\sin x-y_1 &\implies (y_3)_0=0\\ y_4=y_3\cos x-3y_2\sin x-2y_1\cos x-y_2 &\implies (y_4)_0=-3\\ y_5=y_4\cos x-4y_3\sin x-5y_2\cos x+2y_1\sin x-y_3 &\implies (y_5)_0=-8\end{aligned}$$

and so on. Putting these values in (1.4) we obtain:

$$\begin{aligned}e^{\sin x} &= 1+\frac{x}{1!}(1)+\frac{x^2}{2!}(1)+\frac{x^3}{3!}(0)+\frac{x^4}{4!}(-3)+\frac{x^5}{5!}(-8)+\cdots\\ &= 1+x+\frac{x^3}{2}-\frac{x^4}{8}-\frac{x^5}{15}+\cdots.\end{aligned}$$

It is the required series. □

Example 1.14. Expand $e^{ax}\cos(bx)$ by Maclaurin's theorem.

Solution. Here $f(x) = e^{ax}\cos(bx)$. By Maclaurin's theorem we know that

$$y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \cdots + \frac{x^n}{n!}(y_n)_0 + \cdots . \tag{1.5}$$

Now differentiating successively and putting $x = 0$ we obtain:

$$\begin{aligned} y = f(x) = e^{ax}\cos(bx) &\implies (y)_0 = 1 \\ y_1 = ae^{ax}\cos(bx) - be^{ax}\sin(bx) = ay - be^{ax}\sin(bx) &\implies (y_1)_0 = a \\ y_2 = ay_1 - b^2e^{ax}\cos(bx) - abe^{ax}\sin(bx) = ay_1 - b^2y + a(y_1 - ay) &= 2ay_1 - (a^2+b^2)y \\ &\implies (y_2)_0 = a^2 - b^2 \\ y_3 = 2ay_2 - (a^2+b^2)y_1 &\implies (y_3)_0 = a(a^2 - 3 \end{aligned}$$

and so on. Putting these values in (1.5) we obtain:

$$\begin{aligned} e^{\sin x} &= 1 + \frac{x}{1!}(a) + \frac{x^2}{2!}(a^2 - b^2) + \frac{x^3}{3!}a(a^2 - 3b^2) + \cdots \\ &= 1 + ax + (a^2 - b^2)\frac{x^2}{2!} + a(a^2 - 3b^2)\frac{x^3}{3!} + \cdots . \end{aligned}$$

It is the required series. □

Example 1.15. Expand $e^{a\sin^{-1}x}$ by Maclaurin's theorem. Hence show that

$$e^{\theta} = 1 + \sin\theta + \frac{1}{2!}\sin^2\theta + \frac{2}{3!}\sin^3\theta + \cdots$$

where $\theta = \sin^{-1}x$.

Solution. Here $f(x) = e^{a\sin^{-1}x}$. By Maclaurin's theorem we know that

$$y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \cdots + \frac{x^n}{n!}(y_n)_0 + \cdots . \tag{1.6}$$

Since $y = f(x) = e^{a\sin^{-1}x}$ we have $\boxed{(y)_0 = 1}$.
Differentiating we get

$$\begin{aligned} y_1 &= e^{a\sin^{-1}x} \times \frac{a}{\sqrt{1-x^2}} \\ \implies \quad y_1 &= \frac{ay}{\sqrt{1-x^2}} \\ \implies (1-x^2)y_1^2 &= a^2y^2. \end{aligned} \tag{1.7}$$

Therefore, $\boxed{(y_1)_0 = a}$. Again differentiating (1.7) we get:

$$\begin{aligned}(1-x^2)y_2 - 2xy_1 &= 2a^2yy_1 \\ \implies (1-x^2)y_2 - xy_1 - a^2y &= 0. \end{aligned} \tag{1.8}$$

Therefore, $(y_2)_0 - a^2(y)_0 = 0$, i.e., $\boxed{(y_2)_0 = a^2}$. Again differentiating (1.8) we get:

$$(1-x^2)y_3 - 3xy_2 - (1+a^2)y_1 = 0.$$

Therefore, $(y_3)_0 - (1+a^2)(y_1)_0 = 0$, i.e., $\boxed{(y_3)_0 = a(1+a^2)}$ and so on. Putting these values in (1.6) we obtain:

$$\begin{aligned} e^{a\sin^{-1}x} &= 1 + \frac{x}{1!}(a) + \frac{x^2}{2!}(a^2) + \frac{x^3}{3!}(1+a^2) + \cdots \\ &= 1 + ax + \frac{a^2x^2}{2!} + \frac{a(1+a^2)x^3}{3!} + \cdots . \end{aligned}$$

Putting $a = 1$ and $\sin^{-1} x = \theta$ we get

$$e^\theta = 1 + \sin\theta + \frac{\sin^2\theta}{2!} + \frac{2\sin^3\theta}{3!} + \cdots .$$

It is the required series. □

Example 1.16. Find the first five terms in the expansion of $\ln(1 + \sin x)$ by Maclaurin's series.

Solution. Here $f(x) = \ln(1 + \sin x)$. By Maclaurin's series we know that

$$y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \cdots + \frac{x^n}{n!}(y_n)_0 + \cdots . \tag{1.9}$$

Since $y = f(x) = \ln(1 + \sin x)$ we have $\boxed{(y)_0 = 0}$.
Differentiating we get

$$\begin{aligned} y_1 &= \frac{\cos x}{1+\sin x} \\ \implies (1+\sin x)y_1 &= \cos x. \end{aligned} \tag{1.10}$$

Therefore, $\boxed{(y_1)_0 = 1}$. Again differentiating (1.10) we get:

$$(1+\sin x)y_2 + y_1\cos x = -\sin x. \tag{1.11}$$

Therefore, $(1+0)(y_2)_0 + (y_1)_0 = 0$, i.e., $\boxed{(y_2)_0 = -1}$. Again differentiating (1.11) we get:

$$(1+\sin x)y_3 + 2y_2\cos x - y_1 \sin x = -\cos x. \quad (1.12)$$

Therefore, $(1+0)(y_3)_0 + 2(y_2)_0 - 0 = -1$, i.e., $\boxed{(y_3)_0 = 1}$. Again differentiating (1.12) we get:

$$(1+\sin x)y_4 + 3y_3\cos x - 3y_2 \sin x - y_1\cos x = \sin x. \quad (1.13)$$

Therefore, $(1+0)(y_4)_0 + 3(y_3)_0 - 0 - (y_1)_0 = 0$, i.e., $\boxed{(y_4)_0 = -2}$. Again differentiating (1.13) we get:

$$(1+\sin x)y_5 + 4y_4\cos x - 6y_3\sin x - 4y_2\cos x + y_1\sin x = \cos x. (1.14)$$

Therefore, $(1+0)(y_5)_0 + 4(y_4)_0 - 0 - 4(y_2)_0 + 0 = 1$, i.e., $\boxed{(y_5)_0 = 5}$. Putting these values in (1.9) we obtain:

$$\begin{aligned}\ln(1+\sin x) &= 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(-2) + \frac{x^4}{5!}(5) + \cdots \\ &= x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{2x^4}{4!} + \frac{5x^5}{5!} + \cdots .\end{aligned}$$

It is the required series. □

Example 1.17. Expand $\tan^{-1} x$ in the ascending powers of $x-1$.

Solution. By Taylor's series, we know that

$$f(x) = f(a) + \frac{x-a}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots .$$

Here $f(x) = \tan^{-1} x$ and $a = 1$, therefore:

$$f(x) = f(1) + \frac{x-1}{1!}f'(1) + \frac{(x-1)^2}{2!}f''(1) + \cdots . \quad (1.15)$$

Since $f(x) = \tan^{-1} x$ we have $\boxed{f(1) = \frac{\pi}{4}}$. Differentiating we get:

$$f'(x) = \frac{1}{1+x^2} \implies \boxed{f'(1) = \frac{1}{2}}.$$

Rearranging the terms in the above we get:

$$(1+x^2)f'(x) = 1.$$

Again differentiating we get:

$$(1+x^2)f''(x) + 2xf'(x) = 0 \implies \boxed{f''(1) = -1}.$$

On putting these values in (1.15) we get

$$\tan^{-1} x = \frac{\pi}{4} + \frac{x-1}{2\cdot 1!} - \frac{(x-1)^2}{2!} + \cdots.$$

□

Example 1.18. Expand $\sin x$ in powers of $x - \frac{\pi}{2}$ and hence evaluate $\sin 91°$ correct to four places of decimals.

Solution. By Taylor's series, we know that

$$f(x) = f(a) + \frac{x-a}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots.$$

Here $f(x) = \sin x$ and $a = \frac{\pi}{2}$, therefore:

$$f(x) = f\left(\frac{\pi}{2}\right) + \frac{x-\frac{\pi}{2}}{1!}f'\left(\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^2}{2!}f''\left(\frac{\pi}{2}\right) + \cdots. \tag{1.16}$$

Since $f(x) = \sin x$ we have $\boxed{f\left(\frac{\pi}{2}\right) = 1}$. Differentiating we get:

$$f'(x) = \cos x \implies \boxed{f'\left(\frac{\pi}{2}\right) = 0}.$$

Again differentiating we get:

$$f''(x) = -\sin x \implies \boxed{f''\left(\frac{\pi}{2}\right) = -1}.$$

Again differentiating we get:

$$f'''(x) = -\cos x \implies \boxed{f'''\left(\frac{\pi}{2}\right) = 0}.$$

Again differentiating we get:

$$f^{(\mathrm{iv})}(x) = \sin x \implies \boxed{f^{(\mathrm{iv})}\left(\frac{\pi}{2}\right) = 1}.$$

On putting these values in (1.16) we get

$$\sin x = 1 - \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} + \frac{\left(x - \frac{\pi}{2}\right)^4}{4!} - \cdots . + \cdots .$$

Let $x = 91^\circ$, so that,

$$x - \frac{\pi}{2} = 91^\circ - 90^\circ = 1^\circ = \frac{\pi}{180} \text{ radians} = 0.0174 \text{ radians}.$$

Putting the value of $x - \dfrac{\pi}{2}$ in the above series we obtain:

$$\begin{aligned} \sin 91^\circ &= 1 - \frac{(0.0174)^2}{2!} + \frac{(0.0174)^4}{4!} \\ &= 0.9999 \end{aligned}$$

correct up to four places of decimals. □

Example 1.19. Expand $\ln x$ in powers of $x - 1$ and hence evaluate $\ln(1.1)$ correct to four decimal places.

Solution. By Taylor's series, we know that

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^4}{4!} f^{(\mathrm{iv})}(a) + \cdots .$$

Here $f(x) = \ln x$ and $a = 1$, therefore:

$$f(x) = f(1) + \frac{x-1}{1!} f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \frac{(x-1)^4}{4!} f^{(\mathrm{iv})}(1) + \cdots . \tag{1.17}$$

Since $f(x) = \ln x$ we have $\boxed{f(1) = 0}$. Differentiating we get:

$$f'(x) = \frac{1}{x} \implies \boxed{f'(1) = 1}.$$

Again differentiating we get:

$$f''(x) = -\frac{1}{x^2} \implies \boxed{f''(1) = -1}.$$

Again differentiating we get:

$$f'''(x) = \frac{2}{x^3} \implies \boxed{f'''(1) = 2}.$$

Again differentiating we get:

$$f^{(\text{iv})}(x) = -\frac{6}{x^4} \implies \boxed{f^{(\text{iv})}\left(\frac{\pi}{2}\right) = -6}.$$

On putting these values in (1.17) we get

$$\ln x = x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots .$$

Putting $x = 1.1$ in the above we get:

$$\begin{aligned} \ln(1.1) &= 1.1 - 1 - \frac{1}{2}(1.1-1)^2 + \frac{1}{3}(1.1-1)^3 - \frac{1}{4}(1.1-1)^4 + \cdots \\ &= 0.1 - 0.005 + 0.0003 - 0.00002 \\ &= 0.0953 \end{aligned}$$

correct up to four places of decimals. □

Example 1.20. Expand $2x^3 + 7x^2 + x - 1$ in powers of $x - 2$.

Solution. By Taylor's series, we know that

$$f(x) = f(a) + \frac{x-a}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{(\text{iv})}(a) + \cdots .$$

Here $f(x) = 2x^3 + 7x^2 + x - 1$ and $a = 2$, therefore:

$$f(x) = f(2) + \frac{x-2}{1!}f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \frac{(x-1)^4}{4!}f^{(\text{iv})}(1) + \cdots . \quad (1.18)$$

Since $f(x) = 2x^3 + 7x^2 + x - 1$ we have $\boxed{f(2) = 45}$. Differentiating we get:

$$f'(x) = 6x^2 + 14x + 1 \implies \boxed{f'(2) = 53}.$$

Again differentiating we get:

$$f''(x) = 12x + 14 \implies \boxed{f''(2) = 38}.$$

Again differentiating we get:

$$f'''(x) = 12 \implies \boxed{f'''(2) = 12}.$$

All other higher-order derivatives are zero. On putting these values in (1.18) we get

$$2x^3 + 7x^2 + x - 1 = 45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3.$$

It is the required expansion. □

Example 1.21. Use Taylor's theorem to prove that

$$\begin{aligned}\tan^{-1}(x+h) &= \tan^{-1}x + h\sin\theta\cdot\frac{\sin\theta}{1} - (h\sin\theta)^2\cdot\frac{\sin 2\theta}{2} + (h\sin\theta)^3\cdot\frac{\sin 3\theta}{3}\\ &\quad - \cdots + (-1)^{n-1}(h\sin\theta)^n\cdot\frac{\sin n\theta}{n} + \cdots\end{aligned}$$

where $\theta = \cot^{-1}x$.

Solution. By Taylor's series, we know that

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots. \tag{1.19}$$

Here $f(x+h) = \tan^{-1}(x+h)$, and so, $f(x) = \tan^{-1}x$ therefore differentiating we get:

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1+\cot^2\theta} = \sin^2\theta.$$

Again differentiating (w.r.t. x) we get:

$$\begin{aligned}f''(x) &= 2\sin\theta\cos\theta\cdot\frac{d\theta}{dx} = \sin 2\theta\cdot\frac{d}{dx}(\cot^{-1}x)\\ &= -\sin 2\theta\cdot\frac{1}{1+x^2}\\ &= -\sin 2\theta\sin^2\theta \qquad \text{(since } x = \cot\theta).\end{aligned}$$

Again differentiating we get:

$$\begin{aligned}f'''(x) &= \left(-2\cos 2\theta\sin^2\theta - 2\sin\theta\sin 2\theta\cos\theta\right)\cdot\frac{d\theta}{dx}\\ &= 2\sin\theta\,(\cos 2\theta\sin\theta + \sin 2\theta\cos\theta)\cdot\frac{1}{1+x^2}\\ &= 2\sin^3\theta\sin 3\theta \qquad \text{(since } x = \cot\theta).\end{aligned}$$

On putting these values in (1.19) we get

$$\begin{aligned}\tan^{-1}(x+h) &= \tan^{-1}x + \frac{h}{1!}(\sin^2\theta) + \frac{h^2}{2!}(-\sin 2\theta \sin^2\theta) + \frac{h^3}{3!}(2\sin^3\theta\sin 3\theta) + \cdots \\ &= \tan^{-1}x + h\sin\theta\cdot\frac{\sin\theta}{1} - (h\sin\theta)^2\cdot\frac{\sin 2\theta}{2} + (h\sin\theta)^3\cdot\frac{\sin 3\theta}{3} \\ & - \cdots + (-1)^{n-1}(h\sin\theta)^n\cdot\frac{\sin n\theta}{n} + \cdots\end{aligned}$$

□

Example 1.22. Expand $\tan\left(x+\frac{\pi}{4}\right)$ as far as the term x^4 and evaluate $\tan 46.5°$ to four places of decimals.

OR

Approximate the value of $\tan(46°30')$ using Taylor's theorem.($1° = 60'$)

Solution. By Taylor's series we know that the expansion of $f(x+h)$ in powers of x is:

$$f(x+h) = f(h) + \frac{x}{1!}f'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \cdots. \tag{1.20}$$

Here $f(x+h) = \tan\left(x+\frac{\pi}{4}\right)$, $f(x) = \tan x$, $h = \frac{\pi}{4}$, and so, $f\left(\frac{\pi}{4}\right) = 1$. Differentiating $f(x)$ we get:

$$f'(x) = \sec^2 x = 1 + \tan^2 x = 1 + [f(x)]^2 \implies f'\left(\frac{\pi}{4}\right) = 2.$$

Again differentiating we get:

$$f''(x) = 2f(x)f'(x) \implies f''\left(\frac{\pi}{4}\right) = 4.$$

Again differentiating we get:

$$\begin{aligned}f'''(x) &= 2f(x)f''(x) + 2f'(x)f'(x) = 2f(x)f''(x) + 2[f'(x)]^2 \\ &\implies f'''\left(\frac{\pi}{4}\right) = 16.\end{aligned}$$

Again differentiating we get:

$$\begin{aligned}f^{(iv)}(x) &= 2f(x)f'''(x) + 2f'(x)f''(x) + 4f'(x)f''(x) = 2f(x)f'''(x) + 6f'(x)f''(x) \\ &\implies f^{(iv)}\left(\frac{\pi}{4}\right) = 80.\end{aligned}$$

On putting these values in (1.20) we get

$$\begin{aligned}\tan\left(x+\frac{\pi}{4}\right) &= 1+\frac{x}{1!}(2)+\frac{x^2}{2!}(4)+\frac{x^3}{3!}(16)+\frac{x^4}{4!}(80)+\cdots\\ &= 1+2x+2x^2+\frac{8x^3}{3}+\frac{10x^4}{3}+\cdots.\end{aligned}$$

On putting $x = 1.5^\circ = 1.5\times\frac{\pi}{180}$radians $= 0.0262$ (approximately) in the above equation we get:

$$\begin{aligned}\tan(46.5^\circ) &= 1+2(0.0262)+2(0.0262)^2+\frac{8(0.0262)^3}{3}+\frac{10(0.0262)^4}{3}+\cdots\\ &= 1.0538.\end{aligned}$$

Thus, $\tan(46^\circ 30') = \tan(46.5^\circ) = 1.0538$ (correct to four places of decimals).

□

Example 1.23. Find the value of $\sqrt{10}$.

Solution. Let $f(x+h) = \sqrt{x+h}$. By Taylor's series we know that the expansion of $f(x+h)$ in powers of h is:

$$f(x+h) = f(x)+\frac{h}{1!}f'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f''(x)+\cdots. \quad (1.21)$$

Here $f(x+h) = \sqrt{x+h}$, and so, $f(x) = \sqrt{x}$. Differentiating $f(x)$ we get:

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

Again differentiating we get:

$$f''(x) = -\frac{1}{4x^{3/2}}.$$

Again differentiating we get:

$$f'''(x) = \frac{3}{8x^{5/2}}.$$

On putting these values in (1.21) we get

$$\sqrt{x+h} = \sqrt{x}+\frac{h}{2\sqrt{x}}-\frac{h^2}{8x^{3/2}}+\frac{h^3}{16x^{5/2}}+\cdots.$$

On putting $x = 9, h = 1$ in the above equation we get:

$$\begin{aligned}\sqrt{10} &= \sqrt{9} + \frac{1}{2\sqrt{9}} - \frac{1}{8 \cdot 9^{3/2}} + \frac{1}{16 \cdot 9^{5/2}} + \cdots \\ &= 3 + 0.16667 - 0.00463 + 0.00025 \\ &= 3.16229.\end{aligned}$$

Thus, $\sqrt{10} = 3.1623$ (correct to four places of decimals). □

1.5 Exercise

(Q.1) Expand $\dfrac{e^x}{1+e^x}$ in Maclaurin's series as far as the terms x^3.

Ans. $\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{8x^3}{3!} + \cdots$.

(Q.2) Expand $e^{x\cos x}$ in Maclaurin's series.

Ans. $e^{x\cos x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \cdots$.

(Q.3) Prove that: $\left(\sin^{-1} x\right)^2 = \dfrac{2}{2!}x^2 + \dfrac{2 \cdot 2^2}{4!}x^4 + \cdots$.

Hint. Use Maclaurin's series for $y = \left(\sin^{-1} x\right)^2$.

(Q.4) Prove that: $\ln\left(1 + e^x\right) = \ln(2) + \dfrac{1}{2}x + \dfrac{1}{8}x^2 - \dfrac{1}{192}x^4 + \cdots$.

Hint. Use Maclaurin's series for $y = \ln\left(1 + e^x\right)$.

(Q.5) Prove that: $e^x \sin x = x + x^2 + \dfrac{2}{3!}x^3 - \dfrac{2^2}{5!}x^5 + \cdots$.

Hint. Use Maclaurin's series for $y = e^x \sin x$.

(Q.6) Find the Maclaurin's series for $y = \sin\left(m \sin^{-1} x\right)$.

Ans. $y = mx + \dfrac{m(m^2-1)}{3!}x^3 + \cdots$.

(Q.7) Expand $\tan x$ in powers of $x - \dfrac{\pi}{4}$.

Ans. $\tan x = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \cdots$.

(Q.8) Expand $7x^6 - 3x^5 + x^2 + 2$ in powers of $x - 1$.

Ans. $7x^6 - 3x^5 + x^2 + 2 = 7 + 29(x-1) + 76(x-1)^2 + 110(x-1)^3 + 90(x-1)^4 + 39(x-1)^5 + 7(x-1)^6$.

(Q.9) Find the Taylor's series expansion of $\ln(\cos x)$ about the point $\dfrac{\pi}{3}$.

Ans. $\ln(\cos x) = \ln\frac{1}{2} - \sqrt{3}\left(x - \frac{\pi}{3}\right) - \frac{4}{2!}\left(x - \frac{\pi}{3}\right)^2 - \cdots$.

(Q.10) Prove that $\ln(x+h) = \ln x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \cdots$.

Hint. Use Taylor's series and expand $f(x+h)$ in powers of h.

(Q.11) Calculate the value of $\sqrt{5}$ correct to four places of decimals by taking the first four terms in Taylor's series.

Hint. Use Taylor's series and expand $f(x+h) = \sqrt{x+h}$ in powers of h, and put $x = 4, h = 1$.

(Q.12) Approximate the value of $\sin(61°30')$ using Taylor's theorem.

Ans. $\sin(61°30') = 0.87881711$(approximate).

Unit-II

Partial differentiation: Euler's theorem, total differential, maxima and minima of functions of two variables only.

2.1 Functions of several variables

Suppose, a particle is moving parallel to the earth's surface, then at any instant its energy depends only upon its velocity (surely, we neglect the effect of other celestial and terrestrial bodies on the energy of the particle). Precisely, the energy of particle

$$E(v) = \frac{1}{2}mv^2 + K_0$$

where v is the velocity of particle and K_0 is its potential energy (which is constant). Thus, the $E(v)$ depends only on the velocity v. We say that the energy E of the particle is an output, while its velocity is the input for this output function, and for various values of input, we obtain the different outputs.

Now consider the same particle but with a different situation. Suppose, the particle is moving in such a way that its height from the earth's surface changes continuously. Then, at any instant, its energy depends upon its velocity v, as well as, its height h from the earth's surface. Precisely, the energy of particle

$$\mathcal{E}(v, h) = \frac{1}{2}mv^2 + mgh.$$

What do we see? We now see that the output function $\mathcal{E}$ depends on the two inputs, namely, the velocity v and the height h of the particle.

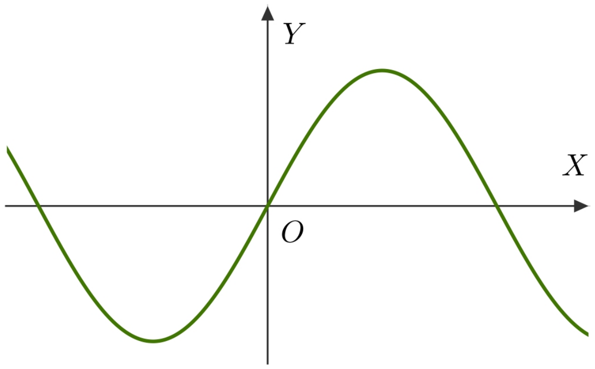
We say that the energy function E is a function of a single variable v, while the energy function $\mathcal{E}$ is a function of two variables v, h.

In general, we say that a quantity y is a function f of n variables if its value depends on n variables $x_1, x_2, \ldots, x_n$. Mathematically, we represent this fact by:

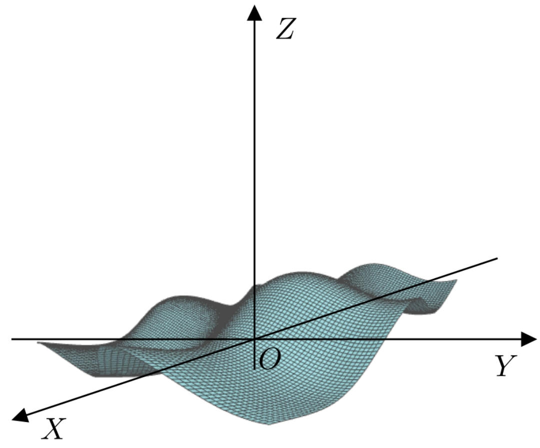
$$y = f(x_1, x_2, \ldots, x_n).$$

2.2 Partial derivatives

Suppose, $y = f(x)$ is a function of a single variable. If we draw a graph of this function by taking the values of x on the X-axis and of y on the Y-axis, then we get a two-dimensional curve. The input x can change only along the X-axis (either towards left or towards right), and so, we can find the rate of change of y only along the X-axis. This rate is called the derivative (total derivative) of y with respect to x and is denoted by $\frac{dy}{dx}$.



Graph of a function of single variable



Graph of a function of two variables

We call the inputs as *independent variable* and the output as *dependent variable*. Now consider a different case, when the dependent variable z is a function of two independent variables (x, y). We write $z = f(x, y)$. Now if we draw the graph of this function by taking the values of x, y and z on three mutually perpendicular axes, we obtain a three-dimensional surface. Then, apart from the previous case the independent variables (x, y) (the inputs) now can change in the XY-plane in any direction (right or left, up or down; or in any direction different from these two), and so, we can find the rate of change of z along any such direction. Such a rate of change is called the directional derivative of f. In particular, we are interested in finding the rate of change (directional derivative) of f in two directions (i) along the X-axis; and (ii) along the Y-axis, and so, we get two directional derivatives along these two axes. The rate of change of f (or z) along the X-axis is called the partial derivative of f (or z) with respect to x and it is denoted by $\frac{\partial f}{\partial x}$. Similarly, the rate of change of f (or z) along the Y-axis is called the partial derivative of f (or z) with respect to y and it is denoted by $\frac{\partial f}{\partial y}$.

Because, in moving along the X-axis, y remains constant, and $\frac{\partial f}{\partial x}$ is the rate of change of f along the X axis, we have:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

Similarly, in moving along the Y-axis x remains constant, and $\frac{\partial f}{\partial y}$ is the rate of change of f along the Y axis, we have:

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{k}.$$

Similarly, we can define the partial derivatives of higher orders.

To find the partial derivative of $z = f(x, y)$ with respect to x we differentiate z by usual rules of differentiation with respect to x but treat the variable y as constant. Similarly, when we find the partial derivative of $z = f(x, y)$ with respect to y we differentiate z by usual rules of differentiation with respect to y but treat the variable x as constant. If $u = f(x, y, z)$ is a function of three variables, then find the partial derivative of $u = f(x, y, z)$ with respect to x we differentiate u by usual rules of differentiation with respect to x, but treat all other variables y and z as constant, and so on.

Example 2.1. Find the first and second partial derivatives of the function $z = x^3 + y^3 - 3axy$.

Solution. Given function is

$$z = x^3 + y^3 - 3axy. \tag{2.1}$$

Differentiating (2.1) partially with respect to x we get:

$$\frac{\partial z}{\partial x} = 3x^2 - 3ay. \tag{2.2}$$

Differentiating (2.1) partially with respect to y we get:

$$\frac{\partial z}{\partial y} = 3y^2 - 3ax. \tag{2.3}$$

Differentiating (2.2) partially with respect to x and y we get:

$$\frac{\partial^2 z}{\partial x^2} = 6x; \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = -3a.$$

Differentiating (2.3) partially with respect to y we get:

$$\frac{\partial^2 z}{\partial y^2} = 6y.$$

□

Example 2.2. If $z(x + y) = x^2 + y^2$, then show that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$

Solution. Given function is

$$z = \frac{x^2 + y^2}{x + y}. \tag{2.4}$$

Differentiating (2.4) partially with respect to x we get:

$$\frac{\partial z}{\partial x} = \frac{(x+y)2x - (x^2+y^2)}{(x+y)^2} = \frac{x^2 + 2xy - y^2}{(x+y)^2}. \tag{2.5}$$

Differentiating (2.4) partially with respect to y we get:

$$\frac{\partial z}{\partial x} = \frac{(x+y)2y - (x^2+y^2)}{(x+y)^2} = \frac{y^2 + 2xy - x^2}{(x+y)^2}. \tag{2.6}$$

From (2.5) and (2.6) we obtain:

$$\begin{aligned}
\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 &= \left[\frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}\right]^2 \\
&= \left[\frac{2x^2 - 2y^2}{(x+y)^2}\right]^2 \\
&= \left[\frac{2(x-y)(x+y)}{(x+y)^2}\right]^2 \\
&= \frac{4(x-y)^2}{(x+y)^2}
\end{aligned}$$

and

$$\begin{aligned}
4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) &= 4\left[1 - \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}\right] \\
&= 4\left[\frac{(x+y)^2 - (x^2+2xy-y^2) - (y^2+2xy-x^2)}{(x+y)^2}\right] \\
&= 4\left[\frac{(x^2+y^2+2xy) - (x^2+2xy-y^2) - (y^2+2xy-x^2)}{(x+y)^2}\right] \\
&= 4\left[\frac{x^2+y^2-2xy}{(x+y)^2}\right] \\
&= \frac{4(x-y)^2}{(x+y)^2}.
\end{aligned}$$

Therefore:

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$

□

Example 2.3. If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, then show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$ and $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Solution. Differentiating the given function partially with respect to y we get:

$$\begin{aligned} \frac{\partial u}{\partial y} &= x^2 \times \frac{1}{1 + (y/x)^2} \times \frac{1}{x} - \left[2y \tan^{-1}\left(\frac{x}{y}\right) + y^2 \times \frac{1}{1 + (x/y)^2} \times \left(-\frac{x}{y^2}\right)\right] \\ &= \frac{x^3}{x^2 + y^2} - 2y \tan^{-1}\left(\frac{x}{y}\right) + \frac{xy^2}{x^2 + y^2} \\ &= x - 2y \tan^{-1}\left(\frac{x}{y}\right). \end{aligned}$$

Differentiating the above equation partially with respect to x we get:

$$\begin{aligned} \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x}\left[x - 2y \tan^{-1}\left(\frac{x}{y}\right)\right] \\ &= 1 - 2y \times \frac{1}{1 + (x/y)^2} \times \frac{1}{y} = \frac{x^2 - y^2}{x^2 + y^2}. \end{aligned}$$

Similarly, $\frac{\partial u}{\partial x} = 2x \tan^{-1}\left(\frac{y}{x}\right) - y$. Differentiating with respect to y we get:

$$\frac{\partial^2 u}{\partial y \partial x} = 2x \times \frac{1}{1 + (y/x)^2} \times \frac{1}{x} - 1 = \frac{x^2 - y^2}{x^2 + y^2}.$$

Therefore:

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}.$$

□

Example 2.4. If $v = \left(x^2 + y^2 + z^2\right)^{-1/2}$, then prove that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$.

Solution. Differentiating the given function partially with respect to x we get:

$$\begin{aligned} \frac{\partial v}{\partial x} &= -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2} \times 2x \\ &= -x\left(x^2 + y^2 + z^2\right)^{-3/2}. \end{aligned}$$

Again differentiating with respect to x we obtain:

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2} &= -\left(x^2+y^2+z^2\right)^{-3/2} - x\left(-\frac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2} \times 2x\\ &= \left(x^2+y^2+z^2\right)^{-5/2}\left[3x^2-\left(x^2+y^2+z^2\right)\right]\\ &= \left(x^2+y^2+z^2\right)^{-5/2}\left(2x^2-y^2-z^2\right).\end{aligned}$$

Using symmetry of v in x, y and z we obtain:

$$\frac{\partial^2 v}{\partial y^2} = \left(x^2+y^2+z^2\right)^{-5/2}\left(2y^2-x^2-z^2\right)$$

and

$$\frac{\partial^2 v}{\partial z^2} = \left(x^2+y^2+z^2\right)^{-5/2}\left(2z^2-x^2-z^2\right).$$

Adding the above three we get:

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2} &= \left(x^2+y^2+z^2\right)^{-5/2}\left(2x^2-y^2-z^2\right)\\ &\quad+\left(x^2+y^2+z^2\right)^{-5/2}\left(2y^2-x^2-z^2\right)\\ &\quad+\left(x^2+y^2+z^2\right)^{-5/2}\left(2z^2-x^2-z^2\right)\\ &= \left(x^2+y^2+z^2\right)^{-5/2}\left(2x^2-y^2-z^2+2y^2-x^2\right.\\ &\quad\left.-z^2+2z^2-x^2-z^2\right)\\ &= \left(x^2+y^2+z^2\right)^{-5/2}\cdot 0\\ &= 0.\end{aligned}$$

□

Example 2.5. If $u = \ln\left(x^3+y^3+z^3-3xyz\right)$, then show that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$

Solution. Differentiating the given function partially with respect to x we get:

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}.$$

Using symmetry of u in x, y and z we obtain:

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

and
$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}.$$

Adding the above three we get:

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} &= \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} + \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz} \\
&\quad + \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz} \\
&= \frac{3x^2 - 3yz + 3y^2 - 3xz + 3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz} \\
&= \frac{3(x^2 + y^2 + z^2 - yz - xz - xy)}{x^3 + y^3 + z^3 - 3xyz} \\
&= \frac{3(x^2 + y^2 + z^2 - yz - xz - xy)}{(x + y + z)(x^2 + y^2 + z^2 - yz - xz - xy)} \\
&= \frac{3}{x + y + z}.
\end{aligned}$$

Thus:
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x + y + z}.$$

Therefore:

$$\begin{aligned}
\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) \\
&= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x + y + z}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{3}{x + y + z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x + y + z}\right) \\
&\quad + \frac{\partial}{\partial z}\left(\frac{3}{x + y + z}\right) \\
&= -\frac{3}{(x + y + z)^2} - \frac{3}{(x + y + z)^2} - \frac{3}{(x + y + z)^2} \\
&= -\frac{9}{(x + y + z)^2}.
\end{aligned}$$

□

Example 2.6. If $u = e^{xyz}$, then show that $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$.

Solution. Differentiating the given function partially with respect to x we get:

$$\frac{\partial u}{\partial z} = e^{xyz} \cdot xy.$$

Again differentiating with respect to y we get:

$$\begin{aligned}\frac{\partial^2 u}{\partial y \partial z} &= (e^{xyz} \cdot xz) \cdot xy + e^{xyz} \cdot x \\ &= e^{xyz} \left(x + x^2 yz\right).\end{aligned}$$

Again differentiating with respect to x we get:

$$\begin{aligned}\frac{\partial^3 u}{\partial x \partial y \partial z} &= e^{xyz} \cdot yz \cdot \left(x + x^2 yz\right) + e^{xyz} \cdot (1 + 2xyz). \\ &= (1 + 3xyz + x^2 y^2 z^2) e^{xyz}.\end{aligned}$$

□

Example 2.7. If $x^x y^y z^z = c$, then show that $\dfrac{\partial^2 z}{\partial x \partial y} = -(x \ln ex)^{-1}$ at point $x = y = z$.

Solution. Given that: $x^x y^y z^z = c$. Taking logarithm we obtain:

$$x \ln x + y \ln y + z \ln z = \ln c.$$

Differentiating the given function partially with respect to x (note that, z is a function of x and y both, so, it will not be treated as constant) we get:

$$\begin{aligned}x \cdot \frac{1}{x} + \ln x + z \cdot \frac{1}{z} \frac{\partial z}{\partial x} + \ln z \frac{\partial z}{\partial x} &= 0 \\ \implies 1 + \ln x + (1 + \ln z) \frac{\partial z}{\partial x} &= 0 \\ \implies \frac{\partial z}{\partial x} &= -\frac{1 + \ln x}{1 + \ln z}.\end{aligned}$$

By symmetry of the function z in the variables x and y we obtain:

$$\frac{\partial z}{\partial y} = -\frac{1 + \ln y}{1 + \ln z}.$$

Differentiating the above equation partially with respect to x we obtain:

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left[-\frac{1+\ln y}{1+\ln z}\right] \\
&= (1+\ln y)\cdot\frac{1}{(1+\ln z)^2}\cdot\frac{1}{z}\cdot\frac{\partial z}{\partial x} \\
&= (1+\ln y)\cdot\frac{1}{(1+\ln z)^2}\cdot\frac{1}{z}\cdot\left[-\frac{1+\ln x}{1+\ln z}\right] \\
&= -\frac{(1+\ln x)(1+\ln y)}{z(1+\ln z)^3}.
\end{aligned}$$

Putting $x = y = z$ in the above equation we get:

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= -\frac{(1+\ln x)(1+\ln x)}{z(1+\ln x)^3} \\
&= -\frac{1}{z(1+\ln x)} \\
&= -\frac{1}{z(\ln e+\ln x)} \\
&= -(x\ln ex)^{-1}.
\end{aligned}$$

□

Example 2.8. If $v = r^n$, where $r^2 = x^2 + y^2 + z^2$, then show that

$$v_{xx} + v_{yy} + v_{zz} = n(n+1)r^{n-2}.$$

Solution. Given that: $r^2 = x^2 + y^2 + z^2$. Differentiating partially with respect to x we get:

$$2r\frac{\partial r}{\partial x} = 2x \implies \frac{\partial r}{\partial x} = \frac{x}{r}.$$

By symmetry of the function r in the variables x and y we obtain:

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

Now, given that $v = r^n$. Differentiating partially with respect to x we obtain:

$$\begin{aligned}
\frac{\partial v}{\partial x} &= nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}\cdot\frac{x}{r} \\
&= nxr^{n-2}.
\end{aligned}$$

Again differentiating with respect to x we get:

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2} &= \frac{\partial}{\partial x}(nxr^{n-2}) = nr^{n-2} + n(n-2)xr^{n-3}\frac{\partial r}{\partial x}\\ &= nr^{n-2} + n(n-2)xr^{n-3}\cdot\frac{x}{r}\\ &= nr^{n-4}\left[r^2+(n-2)x^2\right]\end{aligned}$$

Again by symmetry, we obtain:

$$\frac{\partial^2 v}{\partial y^2} = nr^{n-4}\left[r^2+(n-2)y^2\right]$$

and

$$\frac{\partial^2 v}{\partial z^2} = nr^{n-4}\left[r^2+(n-2)z^2\right].$$

Adding the above three equalities we obtain:

$$\begin{aligned}v_{xx}+v_{yy}+v_{zz} &= \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\\ &= nr^{n-4}\left[r^2+(n-2)x^2\right] + nr^{n-4}\left[r^2+(n-2)y^2\right]\\ &\quad + nr^{n-4}\left[r^2+(n-2)z^2\right]\\ &= nr^{n-4}\left[3r^2+(n-2)\left(x^2+y^2+z^2\right)\right]\\ &= nr^{n-4}\left[3r^2+(n-2)r^2\right]\\ &= nr^{n-4}(n+1)r^2\\ &= n(n+1)r^{n-2}.\end{aligned}$$

□

Example 2.9. If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$, then prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r).$$

Solution. Given that $x = r\cos\theta$ and $y = r\sin\theta$. By squaring and adding these two we obtain

$$r^2 = x^2+y^2.$$

Differentiating the above equation with respect to x partially we obtain: $2r\,\frac{\partial r}{\partial x} = 2x$, i.e.

$$\frac{\partial r}{\partial x} = \frac{x}{r}.$$

Similarly, we obtain:

$$\frac{\partial r}{\partial y} = \frac{y}{r}.$$

Given that $u = f(r)$. Differentiating u with respect to x partially we get:

$$\begin{aligned} \frac{\partial u}{\partial x} &= f'(r)\,\frac{\partial r}{\partial x} \\ \implies \frac{\partial u}{\partial x} &= \frac{x}{r} f'(r). \end{aligned}$$

Again differentiating the above equation with respect to x partially and using the formula $\frac{d}{dx}(f_1 f_2 f_3) = f_1' f_2 f_3 + f_1 f_2' f_3 + f_1 f_2 f_3'$ we get:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{x}{r} f'(r)\right] \\ \implies \frac{\partial^2 u}{\partial x^2} &= \frac{x}{r} f''(r)\,\frac{\partial r}{\partial x} + \frac{1}{r} f'(r) - \frac{x}{r^2}\,\frac{\partial r}{\partial x} f'(r) \\ \implies \frac{\partial^2 u}{\partial x^2} &= \frac{x^2}{r^2} f''(r) + \frac{1}{r} f'(r) - \frac{x^2}{r^3} f'(r). \end{aligned} \tag{2.7}$$

Since the given functions are symmetric in x and y we obtain:

$$\frac{\partial^2 u}{\partial y^2} = \frac{y^2}{r^2} f''(r) + \frac{1}{r} f'(r) - \frac{y^2}{r^3} f'(r). \tag{2.8}$$

Adding equations (2.7) and (2.8) we obtain

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{1}{r^2} f''(r)\left(x^2 + y^2\right) + \frac{2}{r} f'(r) - \frac{1}{r^3} f'(r)\left(x^2 + y^2\right) \\ &= \frac{1}{r^2} f''(r)\cdot r^2 + \frac{2}{r} f'(r) - \frac{1}{r^3} f'(r)\cdot r^2 \\ &= f''(r) + \frac{2}{r} f'(r) - \frac{1}{r} f'(r) \\ &= f''(r) + \frac{1}{r} f'(r). \end{aligned}$$

□

Example 2.10. If $x = r\cos\theta$, $y = r\sin\theta$, then prove that $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$ and $\frac{1}{r}\,\frac{\partial x}{\partial \theta} = r\,\frac{\partial \theta}{\partial x}$.

Solution. Given that

$$x = r\cos\theta \tag{2.9}$$
$$y = r\sin\theta. \tag{2.10}$$

Squaring and adding these two we obtain $r^2 = x^2 + y^2$. So, as we found in the previous example: $\frac{\partial r}{\partial x} = \frac{x}{r}$. Again dividing (2.10) by (2.9) we get

$$\begin{aligned} \frac{y}{x} &= \frac{\sin\theta}{\cos\theta} = \tan\theta \\ \implies \theta &= \tan^{-1}\left(\frac{y}{x}\right). \end{aligned}$$

Differentiating the above equation partially with respect to θ we obtain:

$$\begin{aligned} \frac{\partial\theta}{\partial x} &= \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \left(-\frac{y}{x^2}\right) \\ &= -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} \\ &= -\frac{\sin\theta}{r}. \end{aligned}$$

On differentiating equation (2.9) partially with respect to r we get:

$$\begin{aligned} \frac{\partial x}{\partial r} &= \cos\theta = \frac{x}{r} \\ \implies \frac{\partial x}{\partial r} &= \frac{\partial r}{\partial x}. \end{aligned}$$

Again differentiating equation (2.9) partially with respect to θ we get:

$$\begin{aligned} \frac{\partial x}{\partial \theta} &= -r\sin\theta = -r\left(-r\frac{\partial\theta}{\partial x}\right) \\ \implies \frac{1}{r}\frac{\partial x}{\partial \theta} &= r\frac{\partial\theta}{\partial x}. \end{aligned}$$

□

2.3 Chain Rule for Partial Differentiation

Suppose $z = f(x, y)$ be a function of two variables, where $x = x(t), y = y(t)$ are functions of another variable t. Suppose there is a small change δt in the variable t, due to which there are small changes δx and δy in the variables x and y respectively. Because of these changes in x and y, suppose there is a small change δz in the function $z = f(x, y)$. Then, the rate of change of z in X direction will be $\frac{\partial z}{\partial x}$, and so the change in z along the X direction will be $\frac{\partial z}{\partial x}\delta x$. Similarly, the change in z in

Y direction will be $\dfrac{\partial z}{\partial y}\delta y$. Since the changes are very small the total approximate change in z will be:

$$\delta z \approx \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y.$$

Therefore, the rate of change of z with respect to t:

$$\frac{\delta z}{\delta t} \approx \frac{\partial z}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial z}{\partial y}\frac{\delta y}{\delta t}.$$

For instantaneous rate of change, letting $\delta \to 0$, and so, $\delta x, \delta y \to 0$ in the above inequality we obtain:

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.}$$

In a general case, if $z = f(x, y)$, $x = x(r, s)$ and $y = y(r, s)$, then we have:

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\\ \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}.\end{aligned}$$

The above results can be generalized for a function of n variables.

Example 2.11. If $u = f(x, y)$ and $x = r\cos\theta, y = r\sin\theta$, then prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$

Solution. Since $x = r\cos\theta, y = r\sin\theta$ we have:

$$\frac{\partial x}{\partial r} = \cos\theta,\ \frac{\partial x}{\partial \theta} = -r\sin\theta$$

and

$$\frac{\partial y}{\partial r} = \sin\theta,\ \frac{\partial y}{\partial \theta} = r\cos\theta.$$

Now by chain rule, we have:

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\\ &= \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \\ &= -r\sin\theta\,\frac{\partial u}{\partial x} + r\cos\theta\,\frac{\partial u}{\partial y}.\end{aligned}$$

Therefore:

$$\begin{aligned}\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 &= \left(\cos\theta\,\frac{\partial u}{\partial x} + \sin\theta\,\frac{\partial u}{\partial y}\right)^2 + \frac{1}{r^2}\left(-r\sin\theta\,\frac{\partial u}{\partial x} + r\cos\theta\,\frac{\partial u}{\partial y}\right)^2 \\ &= \left(\cos^2\theta + \sin^2\theta\right)\left(\frac{\partial u}{\partial x}\right)^2 + \left(\sin^2\theta + \cos^2\theta\right)\left(\frac{\partial u}{\partial y}\right)^2 \\ &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.\end{aligned}$$

□

Example 2.12. If $u = x\log(xy)$ where $x^3 + y^3 + 3xy = 1$, find $\frac{du}{dx}$.

Solution. Since

$$u = x\log(xy) \tag{2.11}$$

we have:

$$\frac{\partial u}{\partial x} = x\left\{\frac{1}{xy}.y\right\} + \log(xy) = 1 + \log(xy),$$

and

$$\frac{\partial u}{\partial y} = x\left\{\frac{1}{xy}.x\right\} = \frac{x}{y}.$$

also,

$$x^3 + y^3 + 3xy = 1 \tag{2.12}$$

Differentiating (2.12), w.r.t. x, we get

$$3x^2 + 3y^2\frac{dy}{dx} + 3\left(x\frac{dy}{dx} + y\right) = 0$$

$$\frac{dy}{dx} = -\left(\frac{x^2 + y}{x + y^2}\right) \tag{2.13}$$

we know that

$$\begin{aligned}\frac{du}{dx} &= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} \\ &= 1 + \log(xy) + \frac{x}{y}\left\{-\left(\frac{x^2+y}{x+y^2}\right)\right\} \\ &= 1 + \log(xy) - \frac{x(x^2+y)}{y(x+y^2)}.\end{aligned}$$

□

Example 2.13. If $u = f(x-y, y-z, z-x)$, then prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

Solution. Let

$$X = x - y \tag{2.14}$$
$$Y = y - z \tag{2.15}$$
$$Z = z - x. \tag{2.16}$$

Then we have $u = f(X, Y, Z)$, i.e., u is a function of X, Y, Z. Differentiating (2.14), (2.15) and (2.16) with respect to x, y, z we get:

$$\begin{aligned}&\frac{\partial X}{\partial x} = 1, \ \frac{\partial X}{\partial y} = -1, \ \frac{\partial X}{\partial z} = 0 \\ &\frac{\partial Y}{\partial x} = 0, \ \frac{\partial Y}{\partial y} = 1, \ \frac{\partial Y}{\partial z} = -1 \\ &\frac{\partial Z}{\partial x} = -1, \ \frac{\partial Z}{\partial y} = 0, \ \frac{\partial Z}{\partial z} = 1.\end{aligned}$$

Now by the chain rule of partial differentiation, we get:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial x} \\ &= \frac{\partial u}{\partial X}\cdot 1 + \frac{\partial u}{\partial Y}\cdot 0 + \frac{\partial u}{\partial Z}(-1) \\ &= \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z},\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial y}\\ &= \frac{\partial u}{\partial X}(-1) + \frac{\partial u}{\partial Y}\cdot 1 + \frac{\partial u}{\partial Z}\cdot 0\\ &= -\frac{\partial u}{\partial X} + \frac{\partial u}{\partial Y}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial u}{\partial z} &= \frac{\partial u}{\partial X}\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial z}\\ &= \frac{\partial u}{\partial X}\cdot 0 + \frac{\partial u}{\partial Y}(-1) + \frac{\partial u}{\partial Z}\cdot 1\\ &= -\frac{\partial u}{\partial Y} + \frac{\partial u}{\partial Z}.\end{aligned}$$

Adding the above three equalities we get

$$\begin{aligned}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z} - \frac{\partial u}{\partial X} + \frac{\partial u}{\partial Y} - \frac{\partial u}{\partial Y} + \frac{\partial u}{\partial Z}\\ &= 0.\end{aligned}$$

□

Example 2.14. Transform the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ into the polar coordinates.

Solution. We know that the relation between cartesian coordinates (x, y) and the polar coordinates (r, θ) are given by $x = r\cos\theta, y = r\sin\theta$, i.e., $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. Therefore:

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta,\ \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

and

$$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r},\ \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}.$$

Now by chain rule, we have:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\\ &= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)u.\end{aligned}$$

The above relation is true for all functions u, and so:

$$\frac{\partial}{\partial x} \equiv \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta}.$$

Similarly, we have

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} \\ &= \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right) u.\end{aligned}$$

The above relation is true for all functions u, and so:

$$\frac{\partial}{\partial y} \equiv \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta}.$$

Therefore:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \\ &= \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) \\ &= \cos\theta \frac{\partial}{\partial r}\left(\cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) \\ &= \cos^2\theta \frac{\partial^2 u}{\partial r^2} - \sin\theta\cos\theta\left(-\frac{1}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial^2 u}{\partial r \partial \theta}\right) \\ &\quad - \frac{\sin\theta}{r}\left(-\sin\theta \frac{\partial u}{\partial r} + \cos\theta \frac{\partial^2 u}{\partial r \partial \theta}\right) + \frac{\sin\theta}{r^2}\left(\cos\theta \frac{\partial u}{\partial \theta} + \sin\theta \frac{\partial^2 u}{\partial \theta^2}\right)\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \\ &= \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right) \\ &= \sin^2\theta \frac{\partial^2 u}{\partial r^2} + \sin\theta\cos\theta\left(-\frac{1}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial^2 u}{\partial r \partial \theta}\right) \\ &\quad + \frac{\cos\theta}{r}\left(\cos\theta \frac{\partial u}{\partial r} + \sin\theta \frac{\partial^2 u}{\partial r \partial \theta}\right) + \frac{\cos\theta}{r^2}\left(-\sin\theta \frac{\partial u}{\partial \theta} + \cos\theta \frac{\partial^2 u}{\partial \theta^2}\right).\end{aligned}$$

Therefore the Laplace equation will be:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\implies \left(\cos^2\theta + \sin^2\theta\right)\frac{\partial^2 u}{\partial r^2} + \frac{\sin^2\theta + \cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta + \cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\implies \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

□

2.4 Exercise

(Q.1) If $z = f(x, y)$, $x = e^u + e^{-v}, y = e^{-u} + e^v$, then prove that:

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

Hint: Use the chain rule of partial differentiation.

2.5 Homogeneous function

A function f in two variables x and y is called homogeneous of degree n in x and y if for any variable t we have

$$f(tx, ty) = t^n f(x, y).$$

Example 2.15. The function $f(x, y) = ax^2 + bxy + cz^2$ is a homogeneous function of degree 2 in x and y as

$$\begin{aligned} f(tx, ty) &= a(tx)^2 + b(tx)(ty) + c(tz)^2 \\ &= t^2\left[ax^2 + bxy + cz^2\right] \\ &= x^2 f(x, y). \end{aligned}$$

In general, an expression of the form:

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_ny^n$$

is a homogeneous function of degree n in x and y.

Remark 2.1. If $u = f(x, y)$ is a homogeneous function of degree n in x and y, then for any value t we have:

$$f(tx, ty) = t^n f(x, y).$$

Replacing t by $\frac{1}{x}$ we get:

$$f\left(1, \frac{y}{x}\right) = \frac{1}{x^n} f(x, y) \implies f(x, y) = x^n f\left(1, \frac{y}{x}\right).$$

Since, in $f\left(1, \frac{y}{x}\right)$ first argument is 1 (constant) and the second is $\frac{y}{x}$, hence $f\left(1, \frac{y}{x}\right)$ can be assumed as a function of $\frac{y}{x}$ only, i.e.,

$$f\left(1, \frac{y}{x}\right) = \text{a function of } \frac{y}{x} = \phi\left(\frac{y}{x}\right) \text{ (say)}.$$

Therefore, we conclude that if $u = f(x, y)$ is a homogeneous function of degree n in x and y, then it can be written as:

$$\boxed{u = f(x, y) = x^n \phi\left(\frac{y}{x}\right).}$$

In general, a function $f(x, y, z, \ldots)$ is said to be a homogeneous function of degree n in $x, y, z, \ldots$, if it can be expressed in the form $x^n \phi\left(\frac{y}{x}, \frac{z}{x}, \ldots\right)$.

2.5.1 Euler's theorem on homogeneous functions

Theorem 5. If u be a homogeneous function of degree n in x and y, then:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Proof. Since u is a homogeneous function of degree n in x and y, therefore we can assume that

$$u = x^n \phi\left(\frac{y}{x}\right).$$

On differentiating with respect to x and y partially we obtain:

$$\begin{aligned} \frac{\partial u}{\partial x} &= nx^{n-1}\phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right) \cdot y\left(-\frac{1}{x^2}\right) \\ \implies \frac{\partial u}{\partial x} &= nx^{n-1}\phi\left(\frac{y}{x}\right) - yx^{n-2}\phi'\left(\frac{y}{x}\right) \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \frac{\partial u}{\partial y} &= x^n \phi'\left(\frac{y}{x}\right) \cdot \frac{1}{x} \\ \implies \frac{\partial u}{\partial y} &= x^{n-1}\phi'\left(\frac{y}{x}\right). \end{aligned} \tag{2.18}$$

Multiplying (2.17) by x and (2.18) by y and adding we get:

$$x\,\frac{\partial u}{\partial x}+y\,\frac{\partial u}{\partial y}=nx^n\phi\left(\frac{y}{x}\right)=nu. \qquad \square$$

Remark 2.2. In general, if u be a homogeneous function of degree n in $x, y, z, \dots,$ then:

$$x\,\frac{\partial u}{\partial x}+y\,\frac{\partial u}{\partial y}+z\,\frac{\partial u}{\partial z}+t\,\frac{\partial u}{\partial t}\dots=nu.$$

2.5.2 Relation between second order derivatives of homogeneous functions

Theorem 6. If u is a homogeneous function of degree n, then:

(i) $x\dfrac{\partial^2 u}{\partial x^2}+y\dfrac{\partial^2 u}{\partial x\partial y}=(n-1)\,\dfrac{\partial u}{\partial x};$

(ii) $x\dfrac{\partial^2 u}{\partial x\partial y}+y\dfrac{\partial^2 u}{\partial y^2}=(n-1)\,\dfrac{\partial u}{\partial y};$

(iii) $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=n(n-1)u.$

Proof. Since, u is a homogeneous function of x and y of degree n, therefore by Euler's theorem, we have:

$$x\,\frac{\partial u}{\partial x}+y\,\frac{\partial u}{\partial y}=nu. \tag{2.19}$$

Differentiating (2.19) partially with respect to x, we get:

$$\begin{aligned} x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y} &= n\,\frac{\partial u}{\partial x}\\ \implies x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y} &= (n-1)\,\frac{\partial u}{\partial x}. \end{aligned} \tag{2.20}$$

Again, differentiating (2.19) partially with respect to y, we get:

$$\begin{aligned} x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2} &= n\,\frac{\partial u}{\partial y}\\ \implies x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}+ &= (n-1)\,\frac{\partial u}{\partial y}. \end{aligned} \tag{2.21}$$

Multiplying (2.20) by x and (2.21) by y and adding, we get:

$$\begin{aligned} x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} &= (n-1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) \\ \Longrightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} &= (n-1)nu \\ \Longrightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} &= n(n-1)u. \end{aligned} \tag{2.22}$$

□

Theorem 7. If u is a function of x and y (not necessarily homogeneous in x and y) and F is a function such that $F(u) = f(x, y)$ is homogeneous of degree n in x and y, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{F(u)}{F'(u)}.$$

Proof. Given that $F(u) = f(x, y)$ is a homogeneous function of degree n in x and y, therefore by Euler's theorems, we have

$$\begin{aligned} x\frac{\partial F(u)}{\partial x} + y\frac{\partial F(u)}{\partial y} &= nF(u) \\ \Longrightarrow xF'(u)\frac{\partial u}{\partial x} + yF'(u)\frac{\partial u}{\partial y} &= nF(u) \\ \Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= n\frac{F(u)}{F'(u)}. \end{aligned}$$

This proves the result. □

Example 2.16. If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Solution. Given that

$$\begin{aligned} u &= \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right) \\ &= \sin^{-1}\left\{\left(\frac{y}{x}\right)^{-1}\right\} + \tan^{-1}\left(\frac{y}{x}\right) \\ &= \psi\left(\frac{y}{x}\right). \end{aligned}$$

Hence, $u = x^0\psi\left(\frac{y}{x}\right)$ is a homogeneous function of degree $n = 0$ in x and y. Hence, by Euler's theorem, we have:

$$\begin{aligned} x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= nu \\ &= 0 \cdot u \\ &= 0. \end{aligned}$$

This proves the result. □

Example 2.17. If $u = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$, then prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

Solution. Let $v = x\phi\left(\frac{y}{x}\right)$ and $w = \psi\left(\frac{y}{x}\right)$, then $u = v + w$. Now $v = x\phi\left(\frac{y}{x}\right)$ is a homogeneous function of x and y of degree 1, so from the relation (2.22) we have,

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2} = 1(1-1)v = 0 \tag{2.23}$$

again $w = \psi\left(\frac{y}{x}\right)$ is a homogeneous function of x and y of degree 0, so from the relation (2.22) we have,

$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x \partial y} + y^2\frac{\partial^2 w}{\partial y^2} = 0(0-1)w = 0 \tag{2.24}$$

adding (2.23) and (2.24), we get

$$\begin{aligned} x^2\frac{\partial^2 (v+w)}{\partial x^2} + 2xy\frac{\partial^2 (v+w)}{\partial x \partial y} + y^2\frac{\partial^2 (v+w)}{\partial y^2} &= 0 \\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} &= 0. \end{aligned}$$

This is the required result. □

Example 2.18. If $u = \sin^{-1}\left(\dfrac{x^2+y^2}{x+y}\right)$, then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$$

Solution. Given that:

$$\begin{aligned} u &= \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right) \\ \implies \sin u &= \frac{x^2+y^2}{x+y} = f(x,y) \text{ (say)}. \end{aligned}$$

Again,

$$\begin{aligned} f(tx,ty) &= \frac{(tx)^2+(ty)^2}{tx+ty} = t\cdot\frac{x^2+y^2}{x+y} \\ &= t^1 f(x,y). \end{aligned}$$

Thus, f is a homogeneous function of degree 1. Hence, by Euler's theorem, we have:

$$\begin{aligned} x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= 1\cdot f \\ \implies x\frac{\partial(\sin u)}{\partial x} + y\frac{\partial(\sin u)}{\partial y} &= 1\cdot \sin u \\ \implies x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} &= \sin u \\ \implies x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= \frac{\sin u}{\cos u} \\ \implies x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= \tan u. \end{aligned}$$

This is the required result. □

Example 2.19. Show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u\log u$, where $\log u = \dfrac{x^3+y^3}{3x+4y}$.

Solution. Given that

$$\log u = \frac{x^3+y^3}{3x+4y} = f(x,y) \text{ (say)}.$$

Then:

$$\begin{aligned} f(tx,ty) &= \frac{(tx)^3+(ty)^3}{3(tx)+4(ty)} = t^2\frac{x^3+y^3}{3x+4y} \\ &= t^2 f(x,y). \end{aligned}$$

Hence, f is a homogeneous function of degree 2 in x and y. By Euler's theorem, we get

$$x\,\frac{\partial f}{\partial x} + y\,\frac{\partial f}{\partial y} = 2f \tag{2.25}$$

But

$$\frac{\partial f}{\partial x} = \frac{1}{u}\frac{\partial u}{\partial x} \text{ and } \frac{\partial f}{\partial y} = \frac{1}{u}\frac{\partial u}{\partial y}$$

Hence (2.25) becomes

$$\begin{aligned} & x\left(\frac{1}{u}\frac{\partial u}{\partial x}\right) + y\left(\frac{1}{u}\frac{\partial u}{\partial y}\right) = 2\log u \\ \implies & x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} = 2u\log u. \end{aligned}$$

This gives the required result. □

Example 2.20. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, then show that:

(A) $x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} = \sin 2u.$

(B) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\cos 3u\sin u.$

Solution. Given that:

$$\begin{aligned} & u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right) \\ \implies & \tan u = \frac{x^3+y^3}{x+y} = f(x,y) \text{ (say).} \end{aligned}$$

Again,

$$f(tx,ty) = \frac{(tx)^3+(ty)^3}{tx+ty} = t^2\cdot\frac{x^3+y^3}{x+y} = t^2 f(x,y).$$

Thus, f is a homogeneous function of degree 2. Hence, by Euler's theorem, we have:

$$\begin{aligned} x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= 2\cdot f \\ \Longrightarrow\ x\frac{\partial(\tan u)}{\partial x} + y\frac{\partial(\tan u)}{\partial y} &= 2\tan u \\ \Longrightarrow\ x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} &= 2\tan u \\ \Longrightarrow\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= \frac{2\tan u}{\sec^2 u} \\ \Longrightarrow\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= \sin 2u. \end{aligned}$$

which is the required result (A). Again, differentiating (A), i.e., the above equation partially with respect to x, we get:

$$\begin{aligned} x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} &= 2\cos 2u\frac{\partial u}{\partial x} \\ \Longrightarrow\ x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} &= (2\cos 2u - 1)\frac{\partial u}{\partial x}. \end{aligned} \tag{2.26}$$

Again, differentiating (A) partially with respect to y, we get:

$$\begin{aligned} x\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} &= 2\cos 2u\frac{\partial u}{\partial y} \\ \Longrightarrow\ x\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial^2 u}{\partial y^2} + &= (2\cos 2u - 1)\frac{\partial u}{\partial y}. \end{aligned} \tag{2.27}$$

Multiplying (2.26) by x and (2.27) by y and adding, we get:

$$\begin{aligned} x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} &= (2\cos 2u - 1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) \\ \Longrightarrow\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} &= (2\cos 2u - 1)\sin 2u \\ \Longrightarrow\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} &= 2\cos 2u\sin 2u - \sin 2u \\ \Longrightarrow\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} &= \sin 4u - \sin 2u = 2\cos 3u\sin u. \end{aligned}$$

This proves the required result (B). □

2.6 Exercise

(Q.1) If $u = \sin^{-1}\left(\dfrac{x^2y^2}{x+y}\right)$, then show that: $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3\tan u$.

Hint: $f(x,y) = \sin u = \dfrac{x^2y^2}{x+y}$ is a homogeneous function of degree 3 in x and y.

(Q.2) If $u = \cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then show that: $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{2}\cot u$.

Hint: $f(x,y) = \cos u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ is a homogeneous function of degree $\dfrac{1}{2}$ in x and y.

(Q.3) If $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then show that:

- $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\tan u$.
- $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\sin u \cos 2u}{4\cos^3 u}$.

Hint: $f(x,y) = \sin u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ is a homogeneous function of degree $\dfrac{1}{2}$ in x and y.

2.7 Maxima and minima of function of two variables

2.7.1 Necessary condition for maxima or minima of a function of two variables

Suppose $z = f(x,y)$ is a function of two variables x and y. We say that there is a maxima of the function f at point (a,b) if $f(a+h,b+k) - f(a,b) < 0$ for all h,k (positive or negative). Similarly, say that there is a minima of function f at point (a,b) if $f(a+h,b+k) - f(a,b) > 0$ for all h,k (positive or negative). We discuss the necessary conditions for maxima or minima of f analytically and geometrically. The Taylor's series for the function f about the point (a,b) is given by:

$$f(a+h,b+k) = f(a,b) + \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right] f(a,b) + \frac{1}{2!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f(a,b) + \cdots$$

Neglecting the higher-order terms we get:

$$\begin{aligned} f(a+h,b+k) &= f(a,b) + hf_x(a,b) + kf_y(a,b) \\ &\quad + \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2 f_{yy}(a,b)\right] \end{aligned}$$

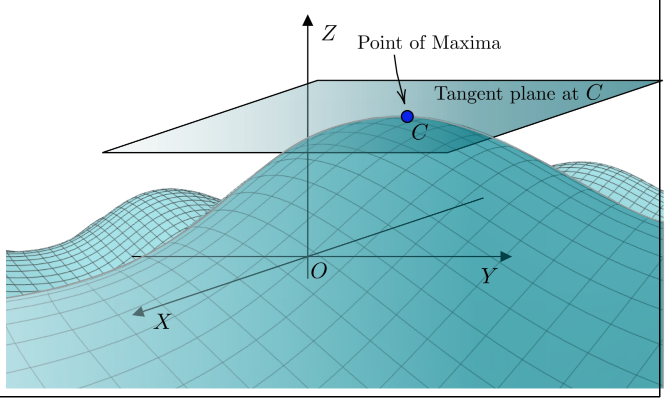
or

$$f(a+h,b+k)-f(a,b)=hf_x(a,b)+kf_y(a,b)+\frac{1}{2!}\left[h^2f_{xx}(a,b)+2hkf_{xy}(a,b)+k^2f_{yy}(a,b)\right]. \quad (2.28)$$

If there is a maxima (or minima) at (a, b) the LHS of the above equation is negative (or positive). Therefore the RHS must be negative (or positive) for all values of h and k. Note that, the first two terms of the RHS change their sign with a change in the signs of h and k (as h and k become positive and negative), and so, LHS will be negative (or positive) for all h and k if the first two terms become zero, i.e.,

$$\boxed{f_x(a,b)=f_y(a,b)=0.}$$

Geometrically, since at maxima or minima, the tangent plane to the surface $z = f(x, y)$ becomes parallel to the XY-plane, its normal at point (a, b) must be in Z-direction. Since the direction ratios of normal are $f_x(a, b), f_y(a, b)$ and $f_z(a, b)$, at maxima or minima we must have $f_x(a, b) = f_y(a, b) = 0$.



2.7.2 Second derivative test

Putting $f_x(a, b) = f_y(a, b) = 0$ in equation (2.28) we obtain:

$$f(a+h,b+k)-f(a,b)=\frac{1}{2!}\left[h^2f_{xx}(a,b)+2hkf_{xy}(a,b)+k^2f_{yy}(a,b)\right]. \quad (2.29)$$

Let $r = f_{xx}(a, b), s = f_{xy}(a, b), t = f_{yy}(a, b)$, then:

$$\begin{aligned} h^2f_{xx}(a,b)+2hkf_{xy}(a,b)+k^2f_{yy}(a,b) &= h^2r+2hks+k^2t \\ &= \frac{1}{r}(hr+ks)^2+k^2\left(t-\frac{s^2}{r}\right). \end{aligned}$$

On putting this value in (2.29) we get:

$$f(a+h,b+k)-f(a,b)=\frac{1}{2!}\left[\frac{1}{r}(hr+ks)^2+k^2\left(t-\frac{s^2}{r}\right)\right].$$

For maxima, the LHS, and so the RHS should be negative and it is possible if $r < 0$ and $t - \frac{s^2}{r} < 0$, i.e., $rt - s^2 > 0$.

For minima the LHS, and so the RHS should be positive and it is possible if $r > 0$ and $t - \frac{s^2}{r} > 0$, i.e., $rt - s^2 > 0$.

For saddle point the LHS, and so the RHS should be positive as well as negative (should change the sign) and it is possible in the following two ways: (i) if $r > 0$ and $t - \frac{s^2}{r} < 0$, i.e., $rt - s^2 < 0$. (ii) if $r < 0$ and $t - \frac{s^2}{r} > 0$, i.e., $rt - s^2 < 0$. Thus, for saddle point, we must have $rt - s^2 < 0$.
Finally, if $rt - s^2 = 0$, then the neglected terms in the series become effective and we need further investigation.

2.7.3 Working rules for finding the maxima and minima.

Let

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - \left[f_{xy}(x, y)\right]^2.$$

We follow the following steps:

(1) Find the first derivatives $f_x(x, y)$ and $f_y(x, y)$ and solve the equations:

$$\begin{aligned} f_x(x, y) &= 0 \\ f_y(x, y) &= 0. \end{aligned}$$

Solution(s) of the above system is (are) the *critical* point(s). Suppose, a critical point is (a, b);

(2) if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(x, y)$ has a local minimum at (a, b);

(3) if $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(x, y)$ has a local maximum at (a, b);

(4) if $D(a, b) < 0$, then $f(x, y)$ has a saddle point at (a, b);

(5) if $D(a, b) = 0$, then we cannot draw any conclusions and further investigations are required.

Example 2.21. Discuss the maxima and minima of $f(x, y) = x^3 + y^3 - 3axy$.

Solution. Given function is $f(x, y) = x^3 + y^3 - 3axy$. Differentiating partially with respect to x and y we get:

$$f_x(x, y) = 3x^2 - 3ay, \quad f_y(x, y) = 3y^2 - 3ax.$$

First, we find the critical point. Then:

$$f_x(x, y) = 0, \quad f_y(x, y) = 0 \implies 3x^2 - 3ay = 0, \quad 3y^2 - 3ax = 0.$$

Since f is symmetric in x and y, a solution of the above system is $x = y$. Putting $x = y$ in the above equation we get:

$$\begin{aligned} 3x^2 - 3ax = 0 &\implies 3x(x - a) = 0 \\ &\implies x = 0, a. \end{aligned}$$

Since $x = y$, we get two critical points $(0,0)$ and (a,a). Now, by differentiating $f_x(x,y)$ and $f_y(x,y)$ again with respect to x and y we get:

$$f_{xx}(x,y) = 6x, \quad f_{xy}(x,y) = -3a, \quad f_{yy}(x,y) = 6y.$$

Now we find D at each critical point. Then:

(i).

$$\begin{aligned} D(0,0) &= f_{xx}(0,0)f_{yy}(0,0) - \big[f_{xy}(0,0)\big]^2 \\ &= 0 \cdot 0 - \big[-3a\big]^2 \\ &= -9a^2 \\ &< 0. \end{aligned}$$

Since $D(0,0) < 0$, the critical point $(0,0)$ is a saddle point.

(ii).

$$\begin{aligned} D(a,a) &= f_{xx}(a,a)f_{yy}(a,a) - \big[f_{xy}(a,a)\big]^2 \\ &= 6a \cdot 6a - \big[-3a\big]^2 \\ &= 36a^2 - 9a^2 = 27a^2 \\ &> 0. \end{aligned}$$

Since $D(0,0) > 0$, there are maxima or minima at the critical point (a,a). We consider two cases:

If $a < 0$, then $f_{xx}(a,a) = 6a < 0$ and so there is a maxima of function f and its maximum value is

$$f_{\max} = f(a,a) = a^3 + a^3 - 3a \cdot a \cdot a = -a^3.$$

If $a > 0$, then $f_{xx}(a,a) = 6a > 0$ and so there is a minima of function f and its minimum value is

$$f_{\min} = f(a,a) = a^3 + a^3 - 3a \cdot a \cdot a = -a^3.$$ □

Example 2.22. Discuss the maxima and minima of $u = xy + \dfrac{a^3}{x} + \dfrac{a^3}{y}$.

Solution. Given function is $u = xy + \dfrac{a^3}{x} + \dfrac{a^3}{y}$. Differentiating partially with respect to x and y we get:

$$u_x(x,y) = y - \frac{a^3}{x^2}, \quad u_y(x,y) = x - \frac{a^3}{y^2}.$$

First, we find the critical point. Then:

$$u_x(x,y)=0,\;\; u_y(x,y)=0 \implies y-\frac{a^3}{x^2}=0,\;\; x-\frac{a^3}{y^2}=0.$$

Since f is symmetric in x and y, a solution of the above system is $x=y$. Putting $x=y$ in the above equation we get:

$$\begin{aligned} x-\frac{a^3}{x^2}=0 &\implies x^3-a^3=0\\ &\implies x=a.\end{aligned}$$

Since $x=y$, we get the critical point (a,a). Now, by differentiating $u_x(x,y)$ and $u_y(x,y)$ again with respect to x and y we get:

$$u_{xx}(x,y)=\frac{2a^3}{x^3},\;\; u_{xy}(x,y)=1,\;\; u_{yy}(x,y)=\frac{2a^3}{y^3}.$$

Now we find D at each critical point. Then:

$$\begin{aligned} D(a,a) &= u_{xx}(a,a)u_{yy}(a,a)-\big[u_{xy}(a,a)\big]^2\\ &= 2\cdot 2-\big[1\big]^2\\ &= 3\\ &> 0.\end{aligned}$$

Since $D(a,a)>0$, there are maxima or minima at critical point (a,a). Then $u_{x,x}(a,a)=2>0$, and so, there is a minima of function u and its minimum value is

$$u_{\min}=u(a,a)=a^2+a^2+a^2=3a^2.$$ □

Example 2.23. Discuss the maxima and minima of $f(x,y)=xy(a-x-y)$.

Solution. Given function is $f(x,y)=xy(a-x-y)=axy-x^2y-xy^2$. Differentiating partially with respect to x and y we get:

$$f_x(x,y)=ay-2xy-y^2,\;\; f_y(x,y)=ax-x^2-2xy.$$

First, we find the critical point. Then:

$$f_x(x,y)=0,\;\; f_y(x,y)=0 \implies ay-2xy-y^2=0,\;\; ax-x^2-2xy=0.$$

Since f is symmetric in x and y, a solution of the above system is $x=y$.

Putting $x = y$ in the above equation we get:

$$ax - x^2 - 2x \cdot x = 0 \implies x(a - 3x) = 0$$
$$\implies x = 0, \frac{a}{3}.$$

Since $x = y$, we get two critical points $\left(\frac{a}{3}, \frac{a}{3}\right)$ and (a, a). Now, by differentiating $f_x(x, y)$ and $f_y(x, y)$ again with respect to x and y we get:

$$f_{xx}(x, y) = -2y, \quad f_{xy}(x, y) = a - 2x - 2y, \quad f_{yy}(x, y) = -2x.$$

Now we find D at each critical point. Then:
(i).

$$\begin{aligned} D(0,0) &= f_{xx}(0,0) f_{yy}(0,0) - \big[f_{xy}(0,0)\big]^2 \\ &= 0 \cdot 0 - \big[a\big]^2 \\ &= -a^2 \\ &< 0. \end{aligned}$$

Since $D(0, 0) < 0$, the critical point $(0, 0)$ is a saddle point.
(ii).

$$\begin{aligned} D\left(\frac{a}{3}, \frac{a}{3}\right) &= f_{xx}\left(\frac{a}{3}, \frac{a}{3}\right) f_{yy}\left(\frac{a}{3}, \frac{a}{3}\right) - \big[f_{xy}\left(\frac{a}{3}, \frac{a}{3}\right)\big]^2 \\ &= -\frac{2a}{3} \cdot \left(-\frac{2a}{3}\right) - \big[-\frac{a}{3}\big]^2 \\ &= \frac{4a^2}{9} - \frac{a^2}{9} \\ &> 0. \end{aligned}$$

Since $D\left(\frac{a}{3}, \frac{a}{3}\right) > 0$, there is a maxima or minima at the critical point $\left(\frac{a}{3}, \frac{a}{3}\right)$.
We consider two cases:
If $a > 0$, then $f_{xx}\left(\frac{a}{3}, \frac{a}{3}\right) = -\frac{2a}{3} < 0$ and so there is a maxima of function f and its maximum value is

$$f_{\max} = f\left(\frac{a}{3}, \frac{a}{3}\right) = \frac{a^2}{9}\left(a - \frac{a}{3} - \frac{a}{3}\right) = \frac{a^3}{27}.$$

If $a < 0$, then $f_{xx}\left(\frac{a}{3}, \frac{a}{3}\right) = -\frac{2a}{3} > 0$ and so there is a minima of function f and its minimum value is

$$f_{\min} = f\left(\frac{a}{3}, \frac{a}{3}\right) = \frac{a^2}{9}\left(a - \frac{a}{3} - \frac{a}{3}\right) = \frac{a^3}{27}.$$

□

Example 2.24. Discuss the maxima and minima of $f(x, y) = x^3y^2(1 - x - y)$.

Solution. Given function is $f(x, y) = x^3y^2(1 - x - y) = x^3y^2 - x^4y^2 - x^3y^3$. Differentiating partially with respect to x and y we get:

$$f_x(x, y) = 3x^2y^2 - 4x^3y^2 - 3x^2y^3, \quad f_y(x, y) = 2x^3y - 2x^4y - 3x^3y^2.$$

First, we find the critical point. Then: $f_x(x, y) = 0, \quad f_y(x, y) = 0$ implies

$$\begin{aligned} & 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0; \\ & 2x^3y - 2x^4y - 3x^3y^2 = 0 \\ \implies & x^2y^2(3 - 4x - 3y) = 0; \\ & x^3y(2 - 2x - 3y) = 0 \\ \implies & 4x + 3y = 3; \\ & 2x + 3y = 2. \end{aligned}$$

On solving the above equations we get $x = \frac{1}{2}, y = \frac{1}{3}$. Therefore, the critical point is $\left(\frac{1}{2}, \frac{1}{3}\right)$. Now, by differentiating $f_x(x, y)$ and $f_y(x, y)$ again with respect to x and y we get:

$$f_{xx}(x, y) = 6xy^2 - 12x^2y^2 - 6xy^3, \quad f_{xy}(x, y) = 6x^2y - 8x^3y - 9x^2y^2,$$
$$f_{yy}(x, y) = 2x^3 - 2x^4 - 6x^3y.$$

Now at critical point $\left(\frac{1}{2}, \frac{1}{3}\right)$ we have

$$\begin{aligned} D\left(\frac{1}{2}, \frac{1}{3}\right) &= f_{xx}\left(\frac{1}{2}, \frac{1}{3}\right) f_{yy}\left(\frac{1}{2}, \frac{1}{3}\right) - \left[f_{xy}\left(\frac{1}{2}, \frac{1}{3}\right)\right]^2 \\ &= \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right) - \left(-\frac{1}{12}\right)^2 = \frac{1}{72} - \frac{1}{144} \\ &> 0. \end{aligned}$$

Since $D\left(\frac{1}{2}, \frac{1}{3}\right) > 0$, there is a maxima or minima at the critical point $\left(\frac{1}{2}, \frac{1}{3}\right)$. Since $f_{xx}\left(\frac{1}{2}, \frac{1}{3}\right) = -\frac{1}{9} < 0$, there is a maxima of function f and its maximum value is

$$f_{\max} = f\left(\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{72} \cdot \frac{1}{6} = \frac{1}{432}.$$

□

Example 2.25. Discuss the maxima and minima of $f(x, y) = \sin x \sin y \sin(x + y)$.

Solution. Given function is $f(x, y) = \sin x \sin y \sin(x + y)$. Differentiating partially with respect to x and y we get:

$$\begin{aligned} f_x(x, y) &= \cos x \sin y \sin(x + y) + \sin x \sin y \cos(x + y) \\ &= \sin y \sin(2x + y). \end{aligned}$$

By symmetry of f in x and y we have

$$f_y(x, y) = \sin x \sin(x + 2y).$$

First, we find the critical point. Then:

$$f_x(x, y) = 0, \quad f_y(x, y) = 0 \implies \sin y \sin(2x + y) = 0, \quad \sin x \sin(x + 2y) = 0.$$

Since f is symmetric in x and y, a solution of the above system is $x = y$. Putting $x = y$ in the above equation we get:

$$\begin{aligned} \sin x \sin 3x = 0 &\implies \sin x = 0 \text{ or } \sin 3x = 0 \\ &\implies x = 0, \pi, \frac{\pi}{3}, \frac{2\pi}{3}. \end{aligned}$$

Since $x = y$, we get four critical points $(0, 0)$, (π, π), $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$. Now, by differentiating $f_x(x, y)$ and $f_y(x, y)$ again with respect to x and y we get:

$$f_{xx}(x, y) = 2 \sin y \cos(2x+y), \quad f_{xy}(x, y) = \sin(2x+2y), \quad f_{yy}(x, y) = 2 \sin x \cos(x+2y)$$

Now we find D at each critical point. Then:

(i).

$$\begin{aligned} D(0, 0) &= f_{xx}(0, 0) f_{yy}(0, 0) - \left[f_{xy}(0, 0)\right]^2 \\ &= 0 \cdot 0 - \left[0\right]^2 \\ &= 0. \end{aligned}$$

Since $D(0, 0) = 0$, we cannot draw any conclusions and further investigations are required.

(ii).

$$\begin{aligned} D(\pi, \pi) &= f_{xx}(\pi, \pi) f_{yy}(\pi, \pi) - \left[f_{xy}(\pi, \pi)\right]^2 \\ &= 0 \cdot 0 - \left[0\right]^2 \\ &= 0. \end{aligned}$$

Since $D(0,0) = 0$, we cannot draw any conclusions and further investigations are required.
(iii).

$$\begin{aligned} D\left(\frac{\pi}{3},\frac{\pi}{3}\right) &= f_{xx}\left(\frac{\pi}{3},\frac{\pi}{3}\right) f_{yy}\left(\frac{\pi}{3},\frac{\pi}{3}\right) - \left[f_{xy}\left(\frac{\pi}{3},\frac{\pi}{3}\right)\right]^2 \\ &= 2\cdot\frac{\sqrt{3}}{2}(-1)\cdot 2\cdot\frac{\sqrt{3}}{2}(-1) - \left[-\frac{\sqrt{3}}{2}\right]^2 \\ &= \frac{9}{4}. \end{aligned}$$

Since $D\left(\frac{\pi}{3},\frac{\pi}{3}\right) > 0$, there is a maxima or minima at the critical point $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$. Now $f_{xx}\left(\frac{\pi}{3},\frac{\pi}{3}\right) = 2\cdot\frac{\sqrt{3}}{2}(-1) = -\sqrt{3} < 0$, and so, there is a maxima of function f and its maximum value is

$$f_{\max} = f\left(\frac{\pi}{3},\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)\cdot\sin\left(\frac{\pi}{3}\right)\cdot\sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{8}.$$

(iv).

$$\begin{aligned} D\left(\frac{2\pi}{3},\frac{2\pi}{3}\right) &= f_{xx}\left(\frac{2\pi}{3},\frac{2\pi}{3}\right) f_{yy}\left(\frac{2\pi}{3},\frac{2\pi}{3}\right) - \left[f_{xy}\left(\frac{2\pi}{3},\frac{2\pi}{3}\right)\right]^2 \\ &= 2\cdot\frac{\sqrt{3}}{2}\cdot 1\cdot 2\cdot\frac{\sqrt{3}}{2}\cdot 1 - \left[\frac{\sqrt{3}}{2}\right]^2 \\ &= \frac{9}{4}. \end{aligned}$$

Since $D\left(\frac{2\pi}{3},\frac{2\pi}{3}\right) > 0$, there is a maxima or minima at the critical point $\left(\frac{2\pi}{3},\frac{2\pi}{3}\right)$. Now $f_{xx}\left(\frac{2\pi}{3},\frac{2\pi}{3}\right) = 2\cdot\frac{\sqrt{3}}{2}\cdot 1 = \sqrt{3} > 0$, and so, there is a maxima of function f and its minimum value is

$$f_{\min} = f\left(\frac{2\pi}{3},\frac{2\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right)\cdot\sin\left(\frac{2\pi}{3}\right)\cdot\sin\left(\frac{2\pi}{3}+\frac{2\pi}{3}\right) = -\frac{3\sqrt{3}}{8}. \qquad \square$$

Example 2.26. Discuss the maxima and minima of $f(x,y) = \sin x + \sin y + \sin(x+y)$.

Solution. Given function is $f(x,y) = \sin x + \sin y + \sin(x+y)$. Differentiating partially with respect to x and y we get:

$$f_x(x,y) = \cos x + \cos(x+y).$$

By symmetry of f in x and y we have

$$f_y(x,y) = \cos y + \cos(x+y).$$

First, we find the critical point. Then:

$$f_x(x,y)=0, \;\; f_y(x,y)=0 \implies \cos x + \cos(x+y) = 0, \;\; \cos y + \cos(x+y) = 0.$$

Since f is symmetric in x and y, a solution of the above system is $x = y$. Putting $x = y$ in the above equation we get:

$$\begin{aligned} \cos x + \cos(2x) = 0 &\implies \cos x + 2\cos^2 x - 1 = 0 \\ &\implies 2\cos^2 x + \cos x - 1 = 0 \\ &\implies \cos x = \frac{-1 \pm \sqrt{1+8}}{4} = -1, \frac{1}{2} \\ &\implies x = \pi, \frac{\pi}{3}. \end{aligned}$$

Since $x = y$, we two three critical points (π,π), $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right)$. Now, by differentiating $f_x(x,y)$ and $f_y(x,y)$ again with respect to x and y we get:

$$f_{xx}(x,y) = -\sin x - \sin(x+y), \;\; f_{xy}(x,y) = -\sin(x+y), \;\; f_{yy}(x,y) = -\sin y - \sin(x+$$

Now we find D at each critical point. Then:

(i).

$$\begin{aligned} D(\pi,\pi) &= f_{xx}(\pi,\pi) f_{yy}(\pi,\pi) - \left[f_{xy}(\pi,\pi)\right]^2 \\ &= 0 \cdot 0 - [0]^2 \\ &= 0. \end{aligned}$$

Since $D(0,0) = 0$, we cannot draw any conclusions and further investigations are required.

(ii).

$$\begin{aligned} D\left(\frac{\pi}{3}, \frac{\pi}{3}\right) &= f_{xx}\left(\frac{\pi}{3}, \frac{\pi}{3}\right) f_{yy}\left(\frac{\pi}{3}, \frac{\pi}{3}\right) - \left[f_{xy}\left(\frac{\pi}{3}, \frac{\pi}{3}\right)\right]^2 \\ &= (-\sqrt{3}) \cdot (-\sqrt{3}) - \left[-\frac{\sqrt{3}}{2}\right]^2 \\ &= \frac{9}{4} > 0. \end{aligned}$$

Since $D\left(\frac{\pi}{3},\frac{\pi}{3}\right)>0$, there is a maxima or minima at the critical point $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$. Now $f_{xx}\left(\frac{\pi}{3},\frac{\pi}{3}\right)=-\sqrt{3}<0$, and so, there is a maxima of function f and its maximum value is

$$f_{\max}=f\left(\frac{\pi}{3},\frac{\pi}{3}\right)=\sin\left(\frac{\pi}{3}\right)+\sin\left(\frac{\pi}{3}\right)+\sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right)=\frac{3\sqrt{3}}{2}.$$

(iii).

$$\begin{aligned} D\left(\frac{5\pi}{3},\frac{5\pi}{3}\right) &= f_{xx}\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)f_{yy}\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)-\left[f_{xy}\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)\right]^2\\ &= (\sqrt{3})\cdot(\sqrt{3})-\left[\frac{\sqrt{3}}{2}\right]^2\\ &= \frac{9}{4}>0. \end{aligned}$$

Since $D\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)>0$, there is a maxima or minima at the critical point $\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)$. Now $f_{xx}\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)=\sqrt{3}>0$, and so, there is a minima of function f and its minimum value is

$$f_{\min}=f\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)=\sin\left(\frac{5\pi}{3}\right)+\sin\left(\frac{5\pi}{3}\right)+\sin\left(\frac{5\pi}{3}+\frac{5\pi}{3}\right)=-\frac{3\sqrt{3}}{2}.$$

□

Example 2.27. Discuss the maxima or minima of $\sin x\sin y\sin z$, where x,y and z are the angles of a triangle.

Solution. Since x,y and z are the angles of the triangle, we have $x+y+z=\pi$ or $z=\pi-(x+y)$. Now the given function is $f(x,y)=\sin x\sin y\sin z$. On putting the value of z we have

$$f(x,y)=\sin x\sin y\sin\left[\pi-(x+y)\right]=\sin x\sin y\sin(x+y).$$

Now follow the process of Example 2.25. □

Example 2.28. Find the point on the surface $z^2=xy+1$ nearest to the origin.

Solution. Suppose the required point on the surface $z^2=xy+1$ is (x,y,z).

Then we have to find this point such that its distance from the origin, i.e.

$$d = \sqrt{x^2 + y^2 + z^2}$$

is minimum. Since d and d^2 get their minimum values together, for simplicity, we calculate the point of minima of

$$d^2 = x^2 + y^2 + z^2.$$

Since the point (x, y, z) is situated on the surface therefore $z^2 = xy + 1$. On putting this value in the above equation we get: $d^2 = f(x, y) = x^2+y^2+xy+1$. Differentiating partially with respect to x and y we get:

$$f_x(x, y) \;=\; 2x + y.$$

By symmetry of f in x and y we have

$$f_y(x, y) = 2y + x.$$

First, we find the critical point. Then:

$$f_x(x, y) = 0, \;\; f_y(x, y) = 0 \;\implies\; 2x + y = 0, \;\; 2y + x = 0.$$

Since f is symmetric in x and y, a solution of the above system is $x = y$. Putting $x = y$ in the above equation we get:

$$3x = 0 \implies x = 0.$$

Since $x = y$, the critical point is $(0, 0)$. Now, by differentiating $f_x(x, y)$ and $f_y(x, y)$ again with respect to x and y we get:

$$f_{xx}(x, y) = 2, \;\; f_{xy}(x, y) = 1, \;\; f_{yy}(x, y) = 2.$$

Now we find D at critical point $(0, 0)$. Then:

$$\begin{aligned} D(0,0) &= f_{xx}(0,0)f_{yy}(0,0) - \left[f_{xy}(0,0)\right]^2 \\ &= 2 \cdot 2 - \left[1\right]^2 \\ &= 3. \end{aligned}$$

Since $D(0, 0) = 3 > 0$, there are maxima or minima at the critical point $(0, 0)$. Now $f_{xx}(0, 0) = 2 > 0$, and so, there is a minima of function f at point $x = y = 0$, and from the equation of surface $z^2 = xy + 1$, at this point we have $x = y = 0$ and so $z^2 = 0 \cdot 01$, i.e., $z = \pm 1$. Thus, the distance of point $(0, 0, \pm 1)$ of the surface will be minimum from the origin. □

Example 2.29. If the perimeter of a triangle is constant, prove that the area of this triangle is maximum when the triangle is equilateral.

Solution. We know that the area of a triangle is given by

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

where a, b, c are the sides of triangle and $2s = a + b + c$. We have to maximize the area Δ. Since Δ and Δ^2 get their maximum values together, for simplicity, we calculate the point of minima of

$$\Delta^2 = s(s-a)(s-b)(s-c).$$

Since the perimeter is constant we have $c = 2s - a - b$. On putting this value in the above equation we get: $\Delta^2 = f(a,b) = s(s-a)(s-b)(a+b-s)$. Differentiating partially with respect to a and b we get:

$$\begin{aligned} f_a(a,b) &= s(s-b)[-(a+b-s)+(s-a)] \\ &= s(s-b)(2s-2a-b). \end{aligned} \tag{2.30}$$

By symmetry of f in x and y we have

$$f_b(a,b) = s(s-a)(2s-2b-a).$$

First, we find the critical point. Then, $f_a(a,b) = 0, \quad f_b(a,b) = 0$ implies that

$$s(s-b)(2s-2a-b) = 0, \quad s(s-a)(2s-2b-a) = 0.$$

Since f is symmetric in a and b, a solution of the above system is $a = b$. Putting $a = b$ in the above equation we get:

$$s(s-a)(2s-2a-a) = 0 \implies s = 0, s = a, a = \frac{2s}{3}.$$

Since $s = 0$, $s = a$ are not possible, we have $a = b = \frac{2s}{3}$, and so, the critical point is $\left(\frac{2s}{3}, \frac{2s}{3}\right)$. Now, by differentiating $f_a(a,b)$ and $f_b(a,b)$ again with respect to a and b we get:

$$f_{aa}(a,b) = -2s(s-b), \quad f_{ab}(a,b) = s(2a+2b-3s), \quad f_{bb}(a,b) = -2s(s-a).$$

Now we find D at each critical point. Then:

$$\begin{aligned} D\left(\frac{2s}{3},\frac{2s}{3}\right) &= f_{aa}\left(\frac{2s}{3},\frac{2s}{3}\right) f_{bb}\left(\frac{2s}{3},\frac{2s}{3}\right) - \left[f_{ab}\left(\frac{2s}{3},\frac{2s}{3}\right)\right]^2 \\ &= \left(-\frac{2s^2}{3}\right)\left(-\frac{2s^2}{3}\right) - \left[-\frac{s^2}{3}\right]^2 = \frac{s^4}{3}. \end{aligned}$$

Since $D\left(\frac{2s}{3},\frac{2s}{3}\right) = \frac{s^4}{3} > 0$, there is a maxima or minima at the critical point $\left(\frac{2s}{3},\frac{2s}{3}\right)$. Now $f_{aa}\left(\frac{2s}{3},\frac{2s}{3}\right) = -\frac{2s^2}{3} < 0$, and so, there is a maxima of function f at point $x = y = \frac{2s}{3}$, i.e., the area is maximum. Also, since $2s = a+b+c$, at point $\left(\frac{2s}{3},\frac{2s}{3}\right)$ we have $c = 2s - a - b = \frac{2s}{3}$. Therefore, for maximum area we have $a = b = c = \frac{2s}{3}$, i.e., the triangle is equilateral. □

2.8 Exercise

(Q.1) Discuss the maxima or minima of the function $f(x,y) = x^3 - 3xy^2 - 15x^2 - 15y^2 + 72x$.

Ans. Critical point $(6,0)$ (minima), with $f_{\min} = f(6,0) = 108$, $(4,0)$ (maxima), with $f_{\max} = f(4,0) = 112$, $(5,1)$ and $(5,-1)$ are saddle points.

(Q.2) Discuss the maxima or minima of the function $f(x,y) = x^3 - 4xy + 2y^2$.

Ans. Critical point $(0,0)$ (saddle point) and $(4/3,4/3)$ (minima), with $f_{\min} = f(4/3,4/3) = -\frac{32}{27}$.

(Q.3) Discuss the maxima or minima of the function $f(x,y) = \cos x \cos y \cos z$, where x, y and z are the angles of a triangle.

Hint. Since $x + y + z = \pi$ the given function is reduced to $f(x,y) = -\cos x \cos y \cos(x+y)$.

(Q.4) Discuss the maxima or minima of the function $f(x,y) = \cos x + \cos y + \cos z$, where x, y and z are the angles of a triangle.

Hint. Since $x + y + z = \pi$ the given function is reduced to $f(x,y) = \cos x + \cos y - \cos(x+y)$.

(Q.5) Discuss the maxima and minima of $f(x,y) = x^3 + y^3 - 3xy$.

Ans. Critical point $(0,0)$ (saddle point) and $(1,1)$ (minima), $f_{\min} = f(a,a) = -1$.

Unit-III

Matrices, determinants, rank, normal form. Systems of linear equations and their solutions.

3.1 Vectors and their linear combination and generated space.

We consider the three-dimensional Euclidian space and denote it by $\mathbb{R}^3$. Each vector of this space can be represented by its position vector. For example, if O is the origin and $\overrightarrow{OP}$ is the vector with tail O and head P, where coordinates of P are (x_1, x_2, x_3), then $\overrightarrow{OP} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$ is completely described by its coordinates (x_1, x_2, x_3), i.e., all the information about $\overrightarrow{OP}$ is contained in the coordinates (x_1, x_2, x_3). Therefore, all the vectors of $\mathbb{R}^3$ are represented by their coordinates and we write $\mathbb{R}^3 = \{(x_1, x_2, x_3) \colon x_1, x_2, x_3 \in \mathbb{R}\}$. In further discussion, we represent a vector by $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, since it same as (x_1, x_2, x_3) in the sense that both the notations give the same information about the vector. The vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is also called a column vector or column matrix. Similarly, we can define a row vector or a row matrix by $X' = [\ x_1,\ x_2,\ x_3\ ]$.

Suppose, $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ are two three dimensional vectors and $a_1, a_2 \in \mathbb{R}$. Then, the expression

$$X = a_1X_1 + a_2X_2 = a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ a_2 \\ 0 \end{bmatrix}$$

is called a linear combination of the vectors X_1, X_2. Obviously, by changing the values of a_1 and a_2 we can find infinitely many linear combinations of X_1 and X_2. The set of all linear combinations of X_1 and X_2 is called the space generated by the vectors X_1 and X_2. These notions can be generalized for an arbitrary number of vectors of n-dimensional vectors.

3.1.1 Linear independence and dependence of vectors.

Vectors $X_1, X_2, \ldots, X_n$ are called linearly dependent if any one of them is a linear combination of other vectors, otherwise vectors are called linearly independent.

3.1.2 Echelon form of a matrix.

We can always reduce the given matrix into a matrix which is in the following form:

(1) All nonzero rows are above any zero row;

(2) the pivot (first nonzero entry from the left) of any row is always strictly to the right of the pivot of the row above it.

Such a form of matrix is called the echelon form.

3.1.3 Rank of a Matrix.

Suppose that

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [X_1 \; X_2 \; \ldots X_n]$$

where $X_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, X_2 = \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \ldots, X_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$ are the colomn vectors.
Then the number of linearly independent column vectors in $X_1, X_2, \ldots, X_n$ is called the column rank of matrix A. Similarly, the matrix A can be written as $A = \begin{bmatrix} X'_1 \\ X'_2 \\ \vdots \\ X'_m \end{bmatrix}$ where $X'_1 = [a_{11}\; a_{12} \cdots a_{1n}]$, $X'_1 = [a_{21}\; a_{22} \cdots a_{2n}], \ldots, X'_m = [a_{m1}\; a_{m2} \cdots a_{mn}]$. Then, the number of linearly independent row vectors in $X'_1, X'_2, \ldots, X$ is called the row rank of matrix A. An interesting property of matrices says that the row and column ranks of a matrix are always equal and this common value is called the rank of a matrix and is denoted by $\rho(A)$.

How to find the rank of a matrix. We use the following two methods of finding the rank of matrices:

(I). Method of determinants. The rank of a matrix can also be calculated using determinants. The rank of a matrix is the order of the largest square submatrix of the given matrix with a nonzero determinant. To find this, we search for a submatrix with a nonzero determinant, and we start with the largest possible submatrix of the given matrix. If this largest submatrix has a nonzero determinant, then the order of the submatrix is the rank of the given matrix. If the determinant of largest is zero, then we go to the submatrix of order less than 1 from the largest submatrix and repeat this process till we get a submatrix with nonzero determinant.

(II). Method of Echelon form. In this method, we first reduce the given metric into echelon form by applying the elementary transformations, and then, the number of nonzero rows in the reduced echelon form is the rank of the given matrix.

Example 3.1. Find one nonzero minor of the highest order of the matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{bmatrix}$, hence find its rank.

Solution. We start with the highest order minor, i.e., the minor of order 3

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{vmatrix} \\ &= 1(28+2) + 2(-14-1) + 3(-4+4) \\ &= 30 - 30 = 0. \end{aligned}$$

Therefore, the highest minor of order 3 is zero. We next consider the second highest order minor, i.e., the minor of order 2:

$$= \begin{vmatrix} -2 & 3 \\ 4 & -1 \end{vmatrix} = 2 - 12 = -10 \neq 0.$$

Thus, the highest order nonzero minor of A is of order 2, and so, $\rho(A) = 2$. □

Example 3.2. In each case, find the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ bc & ca & ab \end{bmatrix}$

Solution. Again, we start with the highest order minor of A, then:

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ bc & ca & ab \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 & 0 \\ b+c & a-b & a-c \\ bc & c(a-b) & b(a-c) \end{vmatrix} \qquad \text{(applying } C_2 \to C_2 - C_1, C_3 \to C_3 - C_1\text{)} \\ &= \begin{vmatrix} a-b & a-c \\ c(a-b) & b(a-c) \end{vmatrix} \\ &= -(a-b)(b-c)(c-a). \end{aligned}$$

We consider the following cases:
Case I. When $a = b = c$. In this case $|A| = 0$, and so, $\rho(A) < 3$. Also, for $a = b = c$ we have $A = \begin{vmatrix} 1 & 1 & 1 \\ 2a & 2a & 2a \\ a^2 & a^2 & a^2 \end{vmatrix}$. Clearly, all the minors of order 2 of A are zero, and so, $\rho(A) < 2$. Now, clearly the minor of order 1 of A is nonzero , therefore $\rho(A) = 1$.
Case II. When $a = b \neq c$. In this case $|A| = 0$, and so, $\rho(A) < 3$. Also, for $a = b \neq c$ we have $A = \begin{vmatrix} 1 & 1 & 1 \\ a+c & a+c & 2a \\ ac & ac & a^2 \end{vmatrix}$, and a minor of order 2 $= \begin{vmatrix} 1 & 1 \\ a+c & 2a \end{vmatrix} = a - c \neq 0$. Therefore, $\rho(A) = 2$. Since A is symmetric in a, b, c, therefore, if any two of a, b, c are equal, and the remaining is not equal to the first two, the rank of A remains 2.
Case III. When $a \neq b \neq c$. In this case $|A| \neq 0$, and so, $\rho(A) = 3$.

Nullity of a square Matrix. The nullity of a matrix is the excess of the order of the matrix over its rank, and it is denoted by $\nu(A)$. If A is a square matrix of order n and $\rho(A) = r$, the $\nu(A) = n - \rho(A) = n - r$. □

Example 3.3. Determine the rank and nullity of the following matrices:

$$A = \begin{bmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{bmatrix}.$$

Solution. Given matrix is: $A = \begin{bmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{bmatrix}.$

Applying $R_4 \to R_4 - R_3, R_3 \to R_3 - R_2; R_2 \to R_2 - R_1$ we obtain:

$$A \sim \begin{bmatrix} 1 & 4 & 9 & 16 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 11 \\ 7 & 9 & 11 & 13 \end{bmatrix}.$$

Applying $R_4 \to R_4 - R_3, R_3 \to R_3 - R_2; R_2 \to R_2 - R_1$ we obtain:

$$A \sim \begin{bmatrix} 1 & 4 & 9 & 16 \\ 2 & 1 & -2 & -7 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}.$$

Applying $R_3 \to \frac{1}{2}R_3; R_3 \leftrightarrow R_1$ we obtain:

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -2 & -7 \\ 1 & 4 & 9 & 16 \\ 2 & 2 & 2 & 2 \end{bmatrix}.$$

Applying $R_4 \to R_4 - 2R_1, R_3 \to R_3 - R_1; R_2 \to R_2 - 2R_1$ we obtain:

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -4 & -9 \\ 0 & 3 & 8 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Applying $R_3 \to R_3 + 3R_2$ we obtain:

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -4 & -9 \\ 0 & 0 & -4 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The above matrix is in the echelon form, therefore, the rank of matrix A:

$\rho(A) =$ no. of nonzero rows in the echelon form$= 3$ and $\nu(A) = 4 - 3 = 1$. □

Example 3.4. Determine the rank and nullity of the following matrices:

$$(i)\ \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix} \qquad (ii)\ \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Solution. (i). Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$. Applying the transformation $R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1$ we have:

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}.$$

Applying the transformation $R_3 \to R_3 - R_2$ we have:

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The above matrix is in the echelon form, therefore, the rank of matrix A:

$\rho(A) =$ no. of nonzero rows in the echelon form$= 2$ and $\nu(A) = 3 - 2 = 1$.

(II). Let $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$. Applying the transformation $R_1 \leftrightarrow R_2$ we have

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}.$$

Applying $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - R_1$ we have:

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}.$$

Applying $R_3 \to R_3 - R_2$, $R_4 \to R_4 - R_2$ we have:

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The above matrix is in the echelon form, therefore, the rank of matrix A: $\rho(A) =$ no. of nonzero rows in the echelon form$= 2$ and $\nu(A) = 4 - 2 = 2$. □

3.2 Normal form of a Matrix

By elementary row and column transformation, every matrix can be reduced into one of the following forms:

$$\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}; \quad \begin{bmatrix} I_r \\ \mathbf{0} \end{bmatrix}; \quad \begin{bmatrix} I_r & \mathbf{0} \end{bmatrix}; \quad [I_r]$$

where r is the rank of the matrix. The above four forms are called the normal form of the matrix.

Example 3.5. Reduce the following matrix into the normal form and find its rank and nullity:

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}.$$

Solution. Applying $R_1 \leftrightarrow R_2$:

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}.$$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 6R_1$:

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}.$$

Applying $R_2 \to R_2 - R_3$:

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}.$$

Applying $R_3 \to R_3 - 4R_2$, $R_4 \to R_4 - 9R_2$:

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}.$$

Applying $R_4 \to R_4 - 2R_3$:

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Applying the following series of operations in order:

$$C_2 \to C_2 + C_1, C_3 \to C_3 + 2C_1, C_4 \to C_4 + 4C_1;$$

$$C_3 \to C_3 + 6C_2, C_4 \to C_4 + 3C_2; \quad C_4 \to C_4 - \frac{2}{3}C_3; \quad C_3 \to \frac{1}{33}C_3 :$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

It is the required normal form of the matrix and the rank of matrix $\rho(A) = 3$, and the nullity $\nu(A) = 4 - 3 = 1$. □

3.3 Solution of System of Linear Equations.

Consider a system of m linear equations in n variables:

$$\left.\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m. \end{array}\right\} \tag{3.1}$$

In matrix form, this system can be written as $AX = B$, where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is the coefficient matrix, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is the variable vector and $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ is the constant vector.

We denote the augmented matrix by $[\, A \,|\, B \,]$ and

$$[\, A \,|\, B \,] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{12} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right].$$

Then, we follow the following procedure to test the consistency and obtain the solution of the system (3.1):

(A) If $\rho\left([\, A \,|\, B \,]\right) \neq \rho(A)$, then system is inconsistent, and has no solution.

(B) If $\rho\left([\, A \,|\, B \,]\right) = \rho(A) = n$, then the system is consistent and has a unique solution.

(C) If $\rho\left([\, A \,|\, B \,]\right) = \rho(A) < n$, then the system is consistent and has infinitely many solutions with $n - \rho(A)$ independent variables.

Example 3.6. Show that the following system is consistent and solve it:

$$\begin{aligned} x + y + z &= 3 \\ x + 2y + 3z &= 4 \\ x + 4y + 9z &= 6. \end{aligned}$$

Solution. Write the system as: $AX = B$, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$. Now, augmented matrix will be

$$[\, A \,|\, B \,] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{array}\right].$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$[\,A\,|\,B\,] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{array}\right].$$

Applying $R_3 \to R_3 - 3R_2$

$$[\,A\,|\,B\,] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right].$$

It is the echelon form of the augmented matrix and it is clear that $\rho\left([\,A\,|\,B\,]\right) = \rho(A) = 3$, which is equal to the number of unknown variables. Therefore, the given system is consistent and has a unique solution. From the echelon form we have the equations:

$$\begin{aligned} x + y + z &= 3 \\ y + 2z &= 1 \\ 2z &= 0. \end{aligned}$$

Therefore, the solution is: $x = 2,\ y = 1,\ z = 0.$ □

Example 3.7. Test for consistency and solve (if consistent):

$$\begin{aligned} 5x + 3y + 7z &= 4 \\ 3x + 26y + 2z &= 9 \\ 7x + 2y + 10z &= 5. \end{aligned}$$

Solution. Write the system as: $AX = B$, $A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$. Now, augmented matrix will be

$$[\,A\,|\,B\,] = \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right].$$

Applying $R_1 \to R_1 - 2R_2$

$$[\,A\,|\,B\,] \sim \left[\begin{array}{ccc|c} -1 & -49 & 3 & -14 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right].$$

Applying $R_2 \to R_2 + 3R_1, R_3 \to R_3 + 7R_1$

$$[\,A\,|\,B\,] \sim \left[\begin{array}{ccc|c} -1 & -49 & 3 & -14 \\ 0 & -121 & 11 & -33 \\ 0 & -341 & 31 & -93 \end{array}\right].$$

Applying $R_3 \to R_3 - \frac{31}{11}R_2$

$$[\,A\,|\,B\,] \sim \left[\begin{array}{ccc|c} -1 & -49 & 3 & -14 \\ 0 & -121 & 11 & -33 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

It is the echelon form of the augmented matrix and it is clear that $\rho\left([\,A\,|\,B\,]\right) = \rho(A) = 2$, which is less than the number of unknown variables (3). Therefore, the given system is consistent and has infinitely many solutions. Since $n - \rho(A) = 3 - 2 = 1$, so one variable in the given system is independent. From the echelon form we have two equations:

$$\begin{aligned} -x - 49y + 3z &= -14 \\ -121y + 11z &= -33. \end{aligned}$$

Since one variable is independent, let $z = k$, then from the above equations:

$$x = \frac{7}{11} - \frac{16}{11}k, \; y = \frac{3+k}{11}, \; z = k. \qquad \square$$

Example 3.8. Show that the following system is inconsistent:

$$\begin{aligned} x - 2y + z - w &= -1 \\ 3x - 2z + 3w &= -4 \\ 5x - 4y + w &= -3. \end{aligned}$$

Solution. The augmented matrix of the given system is:

$$[\,A\,|\,B\,] = \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & -1 \\ 3 & 0 & -2 & 3 & -4 \\ 5 & -4 & 0 & 1 & -3 \end{array}\right].$$

Applying $R_2 \to R_2 - 3R_1, R_3 \to R_3 - 5R_1$

$$\begin{bmatrix} A \mid B \end{bmatrix} \sim \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & -1 \\ 0 & 6 & -5 & 6 & -1 \\ 0 & 6 & -5 & 6 & 2 \end{array}\right].$$

Applying $R_3 \to R_3 - R_2$

$$\begin{bmatrix} A \mid B \end{bmatrix} \sim \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & -1 \\ 0 & 6 & -5 & 6 & -1 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right].$$

It is the echelon form of the augmented matrix and it is clear that $\rho\left(\begin{bmatrix} A \mid B \end{bmatrix}\right) = 3 \neq \rho(A) = 2$. Therefore, the given system is inconsistent and has no solution. □

Example 3.9. Investigate the values of λ and μ so that the equations:

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + \lambda z &= \mu, \end{aligned}$$

have (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

Solution. Write the system as $AX = B$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$. Now, augmented matrix will be

$$\begin{bmatrix} A \mid B \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right].$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} A \mid B \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{array}\right].$$

Applying $R_3 \to R_3 - R_2$

$$[\,A \,|\, B\,] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right].$$

It is the echelon form of the augmented matrix. Now, we consider the following cases:

(i) **System has no solution:** It is possible only when $\rho\left([\,A \,|\, B\,]\right) \neq \rho(A)$, i.e., when $\lambda = 3$ and $\mu \neq 10$.

(ii) **System has a unique solution:** It is possible only when $\rho\left([\,A \,|\, B\,]\right) = \rho(A) = 3$, i.e., $\lambda \neq 3$ and $\mu \in \mathbb{R}$.

(iii) **System has infinitely many solutions:** It is possible only when $\rho\left([\,A \,|\, B\,]\right) = \rho(A) < 3$, i.e., $\lambda = 3$ and $\mu = 10$.

□

Example 3.10. For what value(s) of k the equations:

$$\begin{aligned} x + y + z &= 1 \\ x + 2y + 4z &= k \\ x + 4y + 10z &= k^2, \end{aligned}$$

have a solution and solve completely in each case.

Solution. The augmented matrix of the given system is:

$$[\,A \,|\, B\,] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & k \\ 1 & 4 & 10 & k^2 \end{array}\right].$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$[\,A \,|\, B\,] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & k-1 \\ 0 & 3 & 9 & k^2 - 1 \end{array}\right].$$

Applying $R_3 \to R_3 - 3R_2$

$$[\,A \,|\, B\,] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & k-1 \\ 0 & 0 & 0 & k^2 - 3k + 2 \end{array}\right]. \tag{3.2}$$

The system of equations will have a solution if it is consistent, i.e., if $\rho\left(\left[\ A\,|\,B\ \right]\right) = \rho(A)$, and it is possible only when $k^2 - 3k + 2 = 0$, i.e., $k = 1$ or $k = 2$. In both cases we have $\rho\left(\left[\ A\,|\,B\ \right]\right) = \rho(A) = 2 < 3$ (no. of variables). Therefore, in both cases, we have $3-2 = 1$ independent variable and infinitely many solutions of the given system. We consider the following cases:

Case I. If $k = 1$, then from (3.2) we have the following equations:

$$\begin{aligned} x + y + z &= 1 \\ y + 3z &= 0. \end{aligned}$$

Since one variable is independent, choose $z = a$ we have the following solution:

$$x = 1 + 2a, \quad y = -3a, \quad z = a.$$

Case II. If $k = 2$, then from (3.2) we have the following equations:

$$\begin{aligned} x + y + z &= 1 \\ y + 3z &= 1. \end{aligned}$$

Since one variable is independent, choose $z = b$ we have the following solution:

$$x = 2b, \quad y = 1 - 3b, \quad z = b.$$

□

Example 3.11. For what value of k the equations:

$$\begin{aligned} 2x - 3y + 6z - 5t &= 3 \\ y - 4z + t &= 1 \\ 4x - 5y + 8z - 9t &= k, \end{aligned}$$

(i) have no solution (ii) have infinitely many solutions.

Solution. The augmented matrix of the given system is:

$$\left[\ A\,|\,B\ \right] = \left[\begin{array}{cccc|c} 2 & -3 & 6 & -5 & 3 \\ 0 & 1 & -4 & 1 & 1 \\ 4 & -5 & 8 & -9 & k \end{array}\right].$$

Applying $R_3 \to R_3 - 2R_1$

$$\left[\ A\,|\,B\ \right] \sim \left[\begin{array}{cccc|c} 2 & -3 & 6 & -5 & 3 \\ 0 & 1 & -4 & 1 & 1 \\ 0 & 1 & -4 & 1 & k-6 \end{array}\right].$$

Applying $R_3 \to R_3 - R_2$

$$\left[\,A\,|\,B\,\right] \sim \left[\begin{array}{cccc|c} 2 & -3 & 6 & -5 & 3 \\ 0 & 1 & -4 & 1 & 1 \\ 0 & 0 & 0 & 0 & k-7 \end{array}\right].$$

The system of equations will have a solution if it is consistent, i.e., if $\rho\left(\left[\,A\,|\,B\,\right]\right) = \rho(A)$, and it is possible only when $k - 7 = 0$, i.e., $k = 7$. In this case, we have $\rho\left(\left[\,A\,|\,B\,\right]\right) = \rho(A) = 2 < 4$ (no. of variables). Therefore, we have $4 - 2 = 1$ independent variables and infinitely many solutions of the given system. Now, for k=7, the above system reduces to:

$$\begin{array}{rcl} 2x - 3y + 6z - 5t & = & 3 \\ y - 4z + t & = & 1. \end{array}$$

Since two variables are independent, choose $z = a$ and $t = b$ we have the following solution:

$$x = 3 + 3a + b, \quad y = 1 + 4a - b, \quad z = a, \quad t = b. \qquad \square$$

3.4 Homogeneous system of equations

We consider the following system of equations:

$$\left.\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & 0 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & 0. \end{array}\right\} \tag{3.3}$$

Note that, in the above system all the constants of R.H.S. are zero and such a system is called the homogeneous system. In matrix form, it can be written as $AX = \mathbf{0}$, where A is the coefficient matrix and X is the variable vector. In such systems the augmented matrix is $\left[\,A\,|\,B\,\right] = \left[\,A\,|\,\mathbf{0}\,\right]$. Note that, in any case, the rank of the coefficient matrix A and the rank of the augmented matrix $\left[\,A\,|\,B\,\right]$ are equal. Therefore, homogeneous systems are always consistent and have a solution, namely, $x_1 = x_2 = \cdots = x_n = 0$ is always a solution of system (3.3) and it is called the zero solution or trivial solution.

Note: (A) If $\rho(A) = n$ =number of variables, then system (3.3) has only the trivial solution.

(B) If $\rho(A) < n$ =number of variables, then system (3.3) has a nontrivial solution.

(C) For a homogeneous system, we find the rank of only A, not of $\left[\,A\,|\,B\,\right]$.

Example 3.12. Solve the equations: $x + 3y - 2z = 0,\ 2x - y + 4z = 0,\ x - 11y + 14z = 0$.

Solution. The coefficient matrix of the given system is:

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}.$$

Applying $R_3 \to R_3 - 2R_2$

$$A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is the echelon form. Clearly, $\rho(A) = 2 < 3 =$number of variables. Therefore, the system has a nontrivial solution and the $n - \rho(A) = 3 - 2 = 1$ variable is independent. By the echelon form, we have the following equations:

$$\begin{aligned} x + 3y - 2z &= 0 \\ -7y + 8z &= 0. \end{aligned}$$

Since one variable is independent, choose $z = k$, we obtain from the above equations:

$$x = -\frac{10k}{7},\ y = \frac{8k}{7},\ z = k.$$ □

Example 3.13. For which value of 'b' the following system:

$$\begin{aligned} 2x + y + 2z &= 0 \\ x + y + 3z &= 0 \\ 4x + 3y + bz &= 0 \end{aligned}$$

has (i) trivial solution (ii) nontrivial solution. Find the nontrivial solution.

Solution. The coefficient matrix of the given system is:

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & b \end{bmatrix}$$

Applying $R_1 \leftrightarrow R_2$; then $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 4R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & -1 & b-12 \end{bmatrix}.$$

Applying $R_3 \to R_3 - R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & b-8 \end{bmatrix}.$$

It is the echelon form. We consider the following cases:
(i) If $b \neq 8$, then obviously $\rho(A) = 3 =$number of variables and so system has a trivial solution.

(ii) If $b = 8$, then obviously $\rho(A) = 2 < 3 =$number of variables and so system has nontrivial solution. In this case, $n - \rho(A) = 3 - 2 = 1$ variable will be independent. By the echelon form, we have the following equations:

$$\begin{aligned} x + y + z &= 0 \\ -y - 4z &= 0. \end{aligned}$$

Since one variable is independent, choose $z = k$, we obtain from the above equations:

$$x = k,\ y = -4k,\ z = k.$$ □

3.5 Exercise

(Q.1) Investigate the values of λ and μ so that the equations:

$$\begin{aligned} 2x + 3y + 5z &= 9 \\ 7x + 3y - 2z &= 8 \\ 2x + 3y + \lambda z &= \mu, \end{aligned}$$

have (i) no solution and (ii) a unique solution (iii) an infinite number of solutions. **Ans.** (i) $\lambda = 5, \mu \neq 9$ (ii) $\lambda \neq 5, \mu \in \mathbb{R}$ (iii) $\lambda = 5,\ \mu = 9$.

(Q.2) Test the consistency of the system:

$$\begin{aligned} x+2y-z &= 3\\ 2x-2y+3z &= 2\\ 3x-y+2z &= 1\\ x-y+z &= -1. \end{aligned}$$

Ans. Consistent and has a unique solution $x=-1,\ y=4,\ z=4.$

(Q.3) Show that the following system is consistent and sole it:

$$\begin{aligned} x+2y-5z &= -9\\ 3x-y+2z &= 5\\ 2x+3y-z &= 3. \end{aligned}$$

Ans. Consistent and has a unique solution $x=1/2,\ y=3/2,\ z=5/2.$

(Q.4) Examine the consistency of the following system:

$$\begin{aligned} 5x+3y+14z &= 4\\ y+2z &= 1\\ x-y+2z &= 0\\ 2x+y+6z &= 2. \end{aligned}$$

Ans. Inconsistent (has no solution).

(Q.5) Show that the system of equations as given below is consistent if and only if $a+c=2b$ and find the solution(s) when exists:

$$\begin{aligned} 3x+4y+5z &= a\\ 4x+5y+6z &= b\\ 5x+6y+7z &= c. \end{aligned}$$

(Q.6) Find the values of λ so that the system of equations has a non-trivial solution and hence find the non-trivial solution:

$$\begin{aligned} (\lambda-1)x+(3\lambda+1)y+2\lambda z &= 0\\ (\lambda-1)x+(4\lambda-2)y+(\lambda+3)z &= 0\\ 2x+(3\lambda+1)y+3(\lambda-1)z &= 0. \end{aligned}$$

Ans. $\lambda=0,1,3.$

(Q.7) Solve the following system of equations completely:

$$\begin{aligned} 2w+3x-y-z &= 0\\ 4w-6x-2y+2z &= 0\\ -6w+12x+3y-4z &= 0. \end{aligned}$$

Ans. $w=\frac{k_1}{2},\ x=\frac{k_2}{2},\ y=k_1,\ z=k_2.$

Unit-IV

Numerical methods for solving nonlinear equations: method of bisection, secant method, false position, Newton-Raphson's method, fixed point method and its convergence.

4.1 Numerical methods for solving nonlinear equations

Algebraic function: A function $f(x)$ is called an algebraic function if it can be expressed using a finite number of terms, involving only the algebraic operations addition, subtraction, multiplication, division, and raising to a fractional power of the variable x.

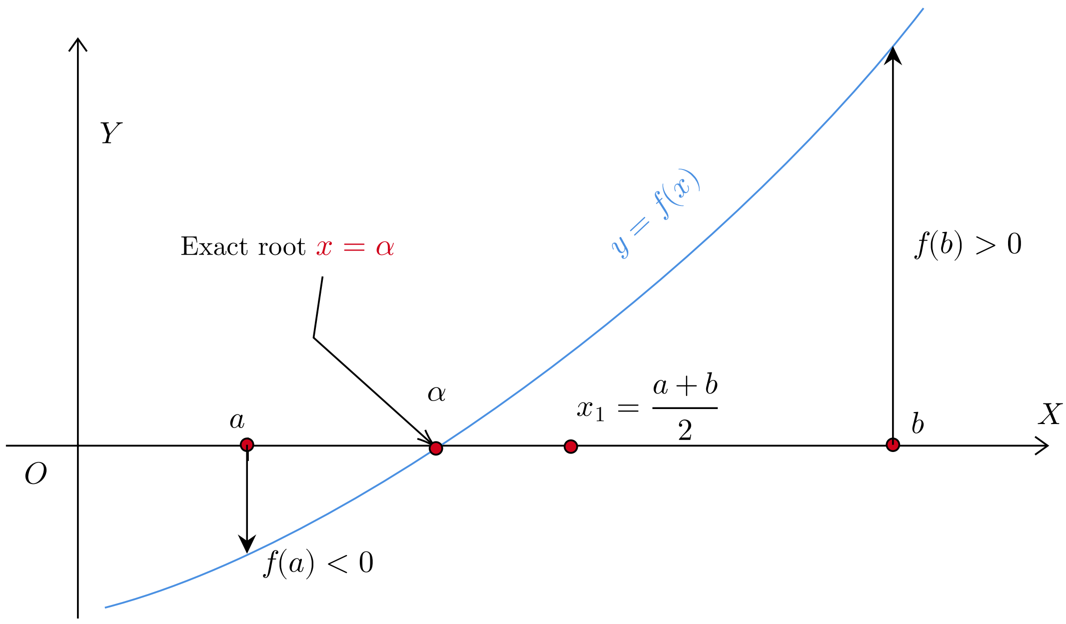
Transcendental function: A function $f(x)$ is called a Transcendental function if it has any root which is not a root of any algebraic function. They are built on functions like logs, exponents, trigonometric functions and inverse trigonometric functions.

Intermediate value theorem: If $f(x)$ is a continuous function in the interval $[a, b]$ and there are two numbers x, y in the interval $[a, b]$ such that $f(x)$ and $f(y)$ have opposite signs (or $f(a) \cdot f(b) < 0$). Then, there is a root of function $f(x)$ between x and y.

Next, we discuss some methods to find the approximate roots of a given function.

4.1.1 Bisection method

This method is based on the intermediate value theorem. In this method, first, we find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs. By intermediate value theorem, there exists a root of the function $f(x)$, between a and b. Now, we bisect this interval by taking mean of a and b and find the point $x_1 = \dfrac{a+b}{2}$. If $f(x_1) = 0$, then we are done. If $f(x_1) \neq 0$, then we find $f(x_1)$. If $f(x_1)$ has an opposite sign to $f(a)$, then the bisected interval is taken $[a, x_1]$, otherwise it is $[x_1, b]$. We repeat the same process with the bisected interval till we get the desired root.



Bisection Method

Example 4.1. Find a root of the equation $x^3 - 4x = 9$ by bisection method correct up to three places of decimals.

Solution. Here $f(x) = x^3 - 4x - 9$. We have to find a root of $f(x)$. Then, since $f(2) = -9 < 0$ and $f(3) = 10 > 0$. Therefore $f(2) \cdot f(3) < 0$, and so, there is a root of $f(x)$ in the interval $[2, 3]$.

Step I. Let

$$x_1 = \frac{2+3}{2} = 2.5$$

and

$$f(x_1) = f(2.5) = -3.375 < 0, \text{ and so, } f(x_1) \cdot f(3) < 0.$$

Therefore, there is a root of $f(x)$ in the interval $[2.5, 3]$.

Step II. Let

$$x_2 = \frac{2.5+3}{2} = 2.75.$$

The

$$f(x_2) = f(2.75) = 0.7969 > 0, \text{ and so, } f(x_1) \cdot f(x_2) < 0.$$

Therefore, there is a root of $f(x)$ in the interval $[2.5, 2.75]$.

Step III. Let

$$x_3 = \frac{2.5+2.75}{2} = 2.625.$$

Then,

$$f(x_3) = f(2.625) = -1.4121 < 0, \text{ and so, } f(x_2) \cdot f(x_3) < 0.$$

Therefore, there is a root of $f(x)$ in the interval $[2.625, 2.75]$.
Step IV. Let

$$x_4 = \frac{2.75 + 2.625}{2} = 2.6875.$$

Then,

$$f(x_4) = f(2.6875) = -0.3391 < 0, \text{ and so, } f(x_2) \cdot f(x_4) < 0.$$

Therefore, there is a root of $f(x)$ in the interval $[2.6875, 2.75]$.
Step V. Let

$$x_5 = \frac{2.75 + 2.6875}{2} = 2.7188.$$

Then

$$f(x_5) = f(2.7188) = 0.2218 > 0, \text{ and so, } f(x_5) \cdot f(x_4) < 0.$$

Therefore, there is a root of $f(x)$ in the interval $[2.6875, 2.7188]$.
Step VI. Let

$$x_6 = \frac{2.6875 + 2.7188}{2} = 2.7031.$$

Then

$$f(x_6) = f(2.7031) = -0.0615 < 0, \text{ and so, } f(x_6) \cdot f(x_5) < 0.$$

Therefore, there is a root of $f(x)$ in the interval $[2.7031, 2.7188]$.
Step VII. Let

$$x_7 = \frac{2.7031 + 2.7188}{2} = 2.711.$$

Then

$$f(x_7) = f(2.711) = 0.0806 > 0, \text{ and so, } f(x_7) \cdot f(x_6) < 0.$$

Therefore, there is a root of $f(x)$ in the interval $[2.7031, 2.711]$.
Step VIII. Let

$$x_8 = \frac{2.7031 + 2.711}{2} = 2.7071.$$

Then

$$f(x_8) = f(2.7071) = 0.01028 > 0, \text{ and so, } f(x_8) \cdot f(x_6) < 0.$$

Therefore, there is a root of $f(x)$ in the interval $[2.7031, 2.7071]$.
Step IX. Let

$$x_9 = \frac{2.7071 + 2.7031}{2} = 2.7051.$$

Then

$$f(x_9) = f(2.7051) = -0.0256 < 0, \text{ and so, } f(x_8) \cdot f(x_9) < 0.$$

Therefore, there is a root of $f(x)$ in the interval $[2.7051, 2.7071]$.
Step X. Let

$$x_{10} = \frac{2.7071 + 2.7051}{2} = 2.7061, \text{ and so, } f(x_{10}) = f(2.7061) = -0.0077 < 0.$$

Therefore, $f(x_{10}) \cdot f(x_8) < 0$, and so, there is a root of $f(x)$ in the interval $[2.7061, 2.7071]$.
Step XI. Let $x_{11} = \frac{2.7061+2.7071}{2} = 2.7066$.
Since $x_{10} = x_{11}$ (up to the three places of the decimal), therefore $x = 2.706$ is the required root of the given equation. □

Example 4.2. By using the bisection method, find an approximate root of the equation $\sin x = 1/x$, that lies between $x = 1$ and $x = 1.5$ (measured in radians). Carry out computation up to 7^{th} stage.

Solution. Let $f(x) = x \sin x - 1$. Then we have to find a root of $f(x)$ lies between 1 and 1.5, i.e., the root lies in the interval $[1, 1.5]$. Then, since $f(1) = -0.1585 < 0$ and $f(1.5) = 0.4962 > 0$. Therefore $f(1) \cdot f(1.5) < 0$, and so, there is a root of $f(x)$ in the interval $[1, 1.5]$.
Step I. Let

$$x_1 = \frac{1 + 1.5}{2} = 1.25$$

and

$$f(x_1) = f(1.25) = 0.1862 > 0, \text{ and so, } f(x_1) \cdot f(1) < 0.$$

Therefore, there is a root of $f(x)$ in the interval $[1, 1.25]$.
Step II. Let

$$x_2 = \frac{1 + 1.25}{2} = 1.125.$$

The

$$f(x_2) = f(1.125) = 0.0150 > 0, \text{ and so, } f(x_2) \cdot f(1) < 0.$$

Therefore, there is a root of $f(x)$ in the interval $[1, 1.125]$.
Step III. Let

$$x_3 = \frac{1 + 1.125}{2} = 1.0625.$$

Then,

$$f(x_3) = f(1.0625) = -0.0718 < 0, \text{ and so, } f(x_2) \cdot f(x_3) < 0.$$

Therefore, there is a root of $f(x)$ in the interval $[1.0625, 1.125]$.
Step IV. Let

$$x_4 = \frac{1.0625 + 1.125}{2} = 1.09375.$$

Then,

$$f(x_4) = f(1.09375) = -0.02836 < 0, \text{ and so, } f(x_2) \cdot f(x_4) < 0.$$

Therefore, there is a root of $f(x)$ in the interval $[1.09375, 1.125]$.
Step V. Let

$$x_5 = \frac{1.09375 + 1.125}{2} = 1.10937.$$

Then

$$f(x_5) = f(1.10937) = -0.00664 < 0, \text{ and so, } f(x_2) \cdot f(x_5) < 0.$$

Therefore, there is a root of $f(x)$ in the interval $[1.10937, 1.125]$.
With a similar process, we obtain:$x_6 = 1.11719$ and $x_7 = 1.11328$. Therefore, $x = 1.11328$ is the required root of the given equation. □

4.1.2 Secant method

Suppose, we have to find a root α of equation $f(x) = 0$ which is between x_0 and x_1. Instead of taking the average (as we do in the Bisection method) we now do a linear approximation to the root α. For this, we join the points $P(x_0, f(x_0))$ and $Q(x_1, f(x_1))$ so that the chord PQ to the curve $y = f(x)$ is constructed. Then, the equation of this chord will be:

$$y - f(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_1).$$

Suppose, this cord PQ intersect the X-axis at point $(x_2, 0)$. Then, from the above equation, we have

$$0 - f(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_1), \quad \text{or:}$$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1).$$

Then, x_2 is the first approximation of α. Now join the points $P'(x_2, f(x_2))$ and $Q(x_1, f(x_2))$ so that the chord $P'Q$ to the curve $y = f(x)$ is constructed. Then, the equation of this chord will be:

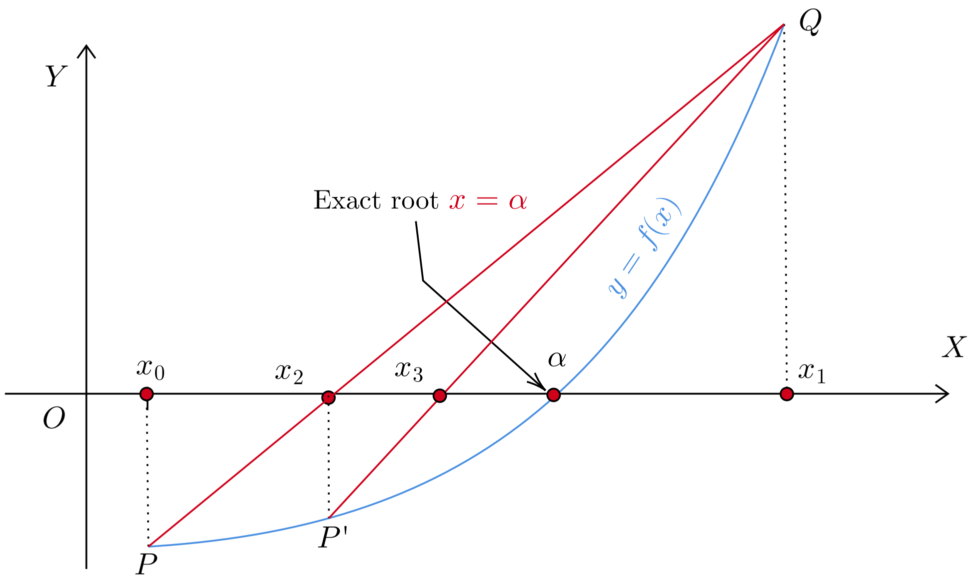
$$y - f(x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_2).$$

Suppose, this cord $P'Q$ intersect the X-axis at point $(x_3, 0)$. Then, from the above equation, we have

$$0 - f(x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x_3 - x_2), \quad \text{or:}$$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2).$$

Then, x_2 is the second approximation of α and obviously, the value x_2 is more close to α than the x_1 (first approximation). We repeat this process till we get the desired accuracy.



Secant Method

4.1.3 General formula for secant method

First, find the values x_0 and x_1 by the Intermediate value theorem (as we have done in the bisection method), then use the following formula for further approximation:

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n), \ n = 1, 2, \dots \ .$$

Example 4.3. Find the root of equation $x \log_{10}(x) = 1.2$ by the Secant method correct up to three places of decimals.

Solution. Let $f(x) = x \log_{10}(x) - 1.2$. Then, we have to find the root of $f(x)$. Note that, $f(1) = -1.2$, $f(2) = -0.598$ and $f(3) = 0.231$. Therefore, there is a root of $f(x)$ between 2 and 3. Let $x_0 = 2$ and $x_1 = 3$. Then the first approximation:

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) = 3 - \frac{3-2}{f(3) - f(2)} f(3) = 3 - \frac{1}{0.231 - (-0.598)} \times 0$$
$$= 2.721.$$

Then, $f(x_2) = f(2.721) = -0.017$. Now, the second approximation:

$$\begin{aligned} x_3 &= x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 2.721 - \frac{2.721 - 3}{f(2.721) - f(3)} f(2.721) \\ &= 2.721 - \frac{-0.279}{-0.017 - 0.231} \times (-0.017) \\ &= 2.740. \end{aligned}$$

Then, $f(x_3) = f(2.740) = -0.0005$. Now, the third approximation:

$$\begin{aligned} x_4 &= x_3 - \frac{x_3 - x_2}{f(x_1) - f(x_2)} f(x_3) = 2.740 - \frac{2.740 - 2.721}{f(2.740) - f(2.721)} f(2.740) \\ &= 2.740 - \frac{0.019}{-0.0005 - (-0.017)} \times (-0.0005) \\ &= 2.740. \end{aligned}$$

Therefore, $\alpha = 2.740$ is the root of the given equation (correct up to the three decimal places). □

Example 4.4. Find the root of equation $x^4 - x - 10 = 0$ by Secant method correct up to four places of decimals.

Solution. Let $f(x) = x^4 - x - 10 = 0$. Then, we have to find the root of $f(x)$. Note that, $f(1.8) = -1.3024$ and $f(2) = 4$. Therefore, there is a root of $f(x)$ between 0 and 2. Let $x_0 = 1.8$ and $x_1 = 2$. Then the first approximation:

$$\begin{aligned} x_2 &= x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) = 2 - \frac{2 - 1.8}{f(2) - f(1.8)} f(2) = 2 - \frac{0.2}{4 - (-1.3024)} \times 4 \\ &= 1.8491. \end{aligned}$$

Then, $f(x_2) = f(1.8491) = -0.1584$. Now, the second approximation:

$$\begin{aligned} x_3 &= x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 1.8491 - \frac{1.8491 - 2}{f(1.8491) - f(2)} f(1.8491) \\ &= 1.8491 - \frac{-0.1509}{-0.1584 - 4} \times (-0.1584) \\ &= 1.8548. \end{aligned}$$

Then, $f(x_3) = f(1.8548) = -0.0192$. Now, the third approximation:

$$\begin{aligned} x_4 &= x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 1.8548 - \frac{1.8548 - 1.8491}{f(1.8548) - f(1.8491)} f(1.8548) \\ &= 1.8548 - \frac{0.0057}{-0.0192 - (-0.1584)} \times (-0.0192) \\ &= 1.8555. \end{aligned}$$

Then, $f(x_4) = f(1.8555) = -0.0021$ Now, the fourth approximation:

$$\begin{aligned} x_5 &= x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4) = 1.8555 - \frac{1.8555 - 1.8548}{f(1.8555) - f(1.8548)} f(1.8555) \\ &= 1.8555 - \frac{0.0007}{-0.0021 - (-0.0192)} \times (-0.0021) \\ &= 1.8555. \end{aligned}$$

Therefore, $\alpha = 1.8555$ is the root of the given equation (correct up to the four decimal places). □

4.1.4 Method of false position (regula-falsi) method

It is the oldest method for finding the real roots of an equation, and this method is a combination of Bisection and Secant methods.

Formula for Regula-falsi method. In this method, for the approximation of the root α of the equation $f(x) = 0$,we use the following formula:

$$c = b - \frac{b - a}{f(b) - f(a)} f(b)$$

where a and b are such that $f(a) < 0$ and $f(b) > 0$. Note that, in this method the values of a and b changes in each step.

Example 4.5. Find the root of the equation $\cos x = xe^x$ using the regula-falsi method correct to four decimal places.

Solution. Let $f(x) = \cos x - xe^x$. Then we have to find the root of $f(x)$. Now since $f(0) = 1 > 0$, $f(1) = -2.1779 < 0$, therefore $a = 1$ and $b = 0$. By the formula, the first approximation for α :

$$c = b - \frac{b - a}{f(b) - f(a)} f(b) = 0 - \frac{0 - 1}{f(0) - f(1)} f(0) = \frac{1}{1 - (-2.1779)} = 0.3146.$$

Then, $f(0.3146) = 0.5198 > 0$. Therefore, the root is between 0.3146 and 1 and, now $a = 1$ and $b = 0.3146$. Then, the second approximation for α :

$$\begin{aligned} c &= b - \frac{b-a}{f(b)-f(a)} f(b) = 0.3146 - \frac{0.3146-1}{f(0.3146)-f(1)} f(0.3146) \\ &= 0.3146 - \frac{-0.6854}{0.5198-(-2.1779)} \times 0.5198 \\ &= 0.4476. \end{aligned}$$

Then, $f(0.4476) = 0.2012 > 0$. Therefore, the root is between 0.4476 and 1 and, now $a = 1$ and $b = 0.4476$. Then, the third approximation for α :

$$\begin{aligned} c &= b - \frac{b-a}{f(b)-f(a)} f(b) = 0.4476 - \frac{0.4476-1}{f(0.4476)-f(1)} f(0.4476) \\ &= 0.4476 - \frac{-0.5524}{0.2012-(-2.1779)} \times 0.2012 \\ &= 0.4943. \end{aligned}$$

Then, $f(0.4943) = 0.0699 > 0$. Therefore, the root is between 0.4943 and 1 and, now $a = 1$ and $b = 0.4943$. Then, the fourth approximation for α :

$$\begin{aligned} c &= b - \frac{b-a}{f(b)-f(a)} f(b) = 0.4943 - \frac{0.4943-1}{f(0.4943)-f(1)} f(0.4943) \\ &= 0.4943 - \frac{-0.5057}{0.0699-(-2.1779)} \times 0.0699 \\ &= 0.5100. \end{aligned}$$

Then, $f(0.5100) = 0.0234 > 0$. Therefore, the root is between 0.5100 and 1 and, now $a = 1$ and $b = 0.5100$. Then, the fifth approximation for α :

$$\begin{aligned} c &= b - \frac{b-a}{f(b)-f(a)} f(b) = 0.5100 - \frac{0.5100-1}{f(0.5100)-f(1)} f(0.5100) \\ &= 0.5100 - \frac{-0.49}{0.0234-(-2.1779)} \times 0.0234 \\ &= 0.5152. \end{aligned}$$

Then, $f(0.5152) = 0.0077 > 0$. Therefore, the root is between 0.5152 and 1 and, now $a = 1$ and $b = 0.5152$. Then, the sixth approximation for α :

$$\begin{aligned} c &= b - \frac{b-a}{f(b)-f(a)} f(b) = 0.5152 - \frac{0.5152-1}{f(0.5152)-f(1)} f(0.5152) \\ &= 0.5152 - \frac{-0.4848}{0.0077-(-2.1779)} \times 0.0077 \\ &= 0.5169. \end{aligned}$$

Then, $f(0.5169) = 0.0026 > 0$. Therefore, the root is between 0.5169 and 1 and, now $a = 1$ and $b = 0.5169$. Then, the seventh approximation for α :

$$\begin{aligned} c &= b - \frac{b-a}{f(b)-f(a)} f(b) = 0.5169 - \frac{0.5169 - 1}{f(0.5169) - f(1)} f(0.5169) \\ &= 0.5169 - \frac{-0.4831}{0.0026 - (-2.1779)} \times 0.0026 \\ &= 0.5174. \end{aligned}$$

Then, $f(0.5174) = 0.001 > 0$. Therefore, the root is between 0.5174 and 1 and, now $a = 1$ and $b = 0.5174$. Then, the eighths approximation for α :

$$\begin{aligned} c &= b - \frac{b-a}{f(b)-f(a)} f(b) = 0.5174 - \frac{0.5174 - 1}{f(0.5174) - f(1)} f(0.5174) \\ &= 0.5174 - \frac{-0.4826}{0.001 - (-2.1779)} \times 0.001 \\ &= 0.5176. \end{aligned}$$

Then, $f(0.5176) = 0.0004 > 0$. Therefore, the root is between 0.5176 and 1 and, now $a = 1$ and $b = 0.5176$. Then, the ninth approximation for α :

$$\begin{aligned} c &= b - \frac{b-a}{f(b)-f(a)} f(b) = 0.5176 - \frac{0.5176 - 1}{f(0.5176) - f(1)} f(0.5176) \\ &= 0.5176 - \frac{-0.4824}{0.0004 - (-2.1779)} \times 0.0004 \\ &= 0.5176. \end{aligned}$$

Therefore, the required root is $\alpha = 0.5176$ (correct to four decimal places). □

Example 4.6. Use the method of false position and find the fourth root of 32 correct to three decimal places.

Solution. Let $x = 32^{1/4}$, i.e., $x^4 = 32$ and $f(x) = x^4 - 32$. Then, the fourth root of 32 is the root of $f(x)$. Now since $f(2) = -16 < 0$ and $f(3) = 49 > 0$, therefore root lies between 2 and 3, and so, $a = 2$ and $b = 3$. By the formula, the first approximation for the root α :

$$c = b - \frac{b-a}{f(b)-f(a)} f(b) = 3 - \frac{3-2}{f(3)-f(2)} f(3) = 3 - \frac{1}{49-(-16)} \times 49 = 2.2461.$$

Then, $f(2.2461) = -6.5483 < 0$. Therefore, the root is between 2.2461 and 3 and, now $a = 2.2461$ and $b = 3$. Then, the second approximation for α :

$$\begin{aligned} c &= b - \frac{b-a}{f(b)-f(a)} f(b) = 3 - \frac{3-2.2461}{f(3)-f(2.2461)} f(3) = 3 - \frac{0.7539}{49-(-6.5483)} \times 49 \\ &= 2.335. \end{aligned}$$

Then, $f(2.335) = -2.2732 < 0$. Therefore, the root is between 2.335 and 3 and, now $a = 2.335$ and $b = 3$. Then, the third approximation for α :

$$\begin{aligned} c &= b - \frac{b-a}{f(b)-f(a)} f(b) = 3 - \frac{3-2.335}{f(3)-f(2.335)} f(3) = 3 - \frac{0.665}{49-(-2.2732)} \times 49 \\ &= 2.3644. \end{aligned}$$

Then, $f(2.3644) = -0.7475 < 0$. Therefore, the root is between 2.3644 and 3 and, now $a = 2.3644$ and $b = 3$. Then, the fourth approximation for α :

$$\begin{aligned} c &= b - \frac{b-a}{f(b)-f(a)} f(b) = 3 - \frac{3-2.3644}{f(3)-f(2.3644)} f(3) = 3 - \frac{0.6356}{49-(-0.7475)} \times 49 \\ &= 2.3739. \end{aligned}$$

Then, $f(2.3739) = -0.2422 < 0$. Therefore, the root is between 2.3739 and 3 and, now $a = 2.3739$ and $b = 3$. Then, the fifth approximation for α :

$$\begin{aligned} c &= b - \frac{b-a}{f(b)-f(a)} f(b) = 3 - \frac{3-2.3739}{f(3)-f(2.3739)} f(3) = 3 - \frac{0.6261}{49-(-0.2422)} \times 49 \\ &= 2.378. \end{aligned}$$

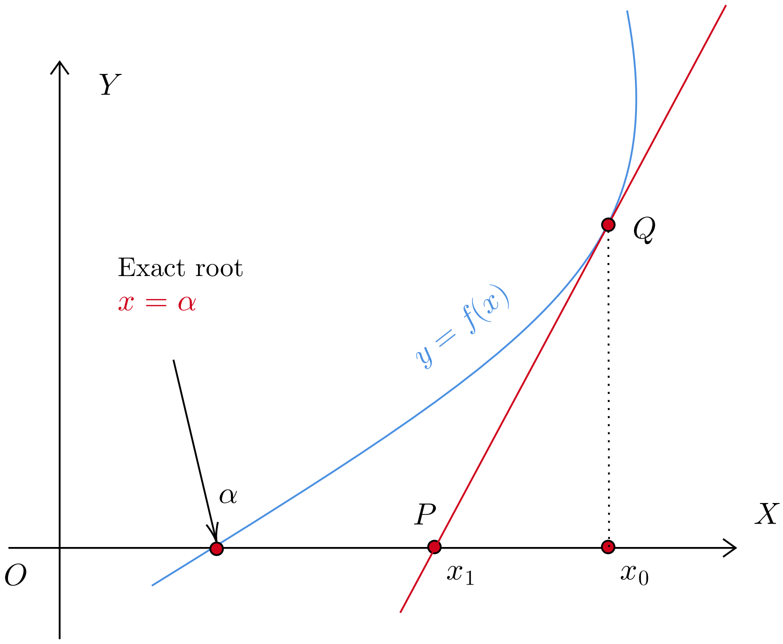
Then, $f(2.378) = -0.0222 < 0$. Therefore, the root is between 2.378 and 3 and, now $a = 2.378$ and $b = 3$. Then, the sixth approximation for α :

$$\begin{aligned} c &= b - \frac{b-a}{f(b)-f(a)} f(b) = 3 - \frac{3-2.378}{f(3)-f(2.378)} f(3) = 3 - \frac{0.622}{49-(-0.0222)} \times 49 \\ &= 2.3783. \end{aligned}$$

Therefore, the required root is $32^{1/4} = \alpha = 0.378$ (correct to three decimal places). □

4.1.5 Newton-Raphson method or Newton's method

The NewtonâĂŞRaphson method was named after English mathematicians Isaac Newton and Joseph Raphson. In this method, we approximate the root of an equation $f(x) = 0$ with a tangential approximation. In the Secant and Regula-Falsi methods, the approximation is done with a chord joining the two initial guesses. Here we start with an initial guess x_0 which is reasonably close to the true root, then the function is approximated by its tangent line drawn at the point $(x_0, f(x_0))$.



Newton-Raphson Method

Suppose, x_0 is the initial guess (initial approximation) of the root α then we draw a tangent at point $(x_0, f(x_0))$ to the curve. Then the slope of this tangent will be $f'(x_0)$ and its equation will be:

$$y - f(x_0) = f'(x_0)(x - x_0).$$

Suppose, this tangent intersect the X-axis at point $P(x_1, 0)$, then we obtain from the above equation: $0 - f(x_0) = f'(x_0)(x_1 - x_0)$, i.e.,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

The quantity x_1 is called the first approximation of the root α and obviously, it is more close to α than the initial approximation x_0. We again draw a tangent at the newly obtained point $(x_1, F(x_1))$, and repeat the same process to get second approximation:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

For further approximations, we repeat this process until we have the root (up to the desired accuracy).

Formula for Newton-Raphson method. First find the two values a and b such that $f(a) < 0$, $f(b) > 0$ by using the Intermediate value theorem and then initial approximation $x_0 = \dfrac{a+b}{2}$ (or, you can choose from a and b which one is closer to the root). Then use the following formula for further approximation:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \; n = 1, 2, \dots \,.$$

Note. Although Newton's method converges faster towards the root of $f(x)$ than the previous method (Secant method), it demands the differentiability of the function $y = f(x)$. If $f'(x)$ is not available, then one can use the secant method.

Example 4.7. Find the positive root of the equation $x^4 - x = 10$ correct to three decimal places, using the Newton-Raphson method.

Solution. Let

$$f(x) = x^4 - x - 10.$$

Then we have to find the positive root of $f(x)$. Since $f(1) = -10 < 0$and $f(2) = 4 > 0$, there is a positive root of f between 1 and 2. Obviously, the root is more closer to 2 than 1. Therefore, let the initial approximation $x_0 = 2$. Now

$$f'(x) = 4x^3 - 1.$$

Then, the first approximation:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{2^4 - 2 - 10}{4 \times 2^3 - 1} = 2 - \frac{4}{31} = 1.871.$$

Now, the second approximation:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.871 - \frac{(1.871)^4 - 1.871 - 10}{4 \times (1.871)^3 - 1} = 1.871 - \frac{0.3835}{25.199} = 1.856.$$

Now, the third approximation:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.856 - \frac{(1.856)^4 - 1.856 - 10}{4 \times (1.856)^3 - 1} = 1.871 - \frac{0.010}{24.574} = 1.856.$$

Therefore, $\alpha = 1.856$ is the positive root of the given equation (correct to three places of decimals). □

Example 4.8. By Newton-Raphson method, find the real root of the equation $3x = \cos x + 1$.

Solution. Let

$$f(x) = 3x - \cos x - 1.$$

Then we have to find the real root of $f(x)$. Since $f(0) = -2 < 0$and $f(1) = 1.4597 > 0$, there is a root of f between 0 and 1. Therefore, let the initial approximation $x_0 = \dfrac{0+1}{2} = 0.5$. Now

$$f'(x) = 3 + \sin x.$$

Then, the first approximation:

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{3 \times 0.5 - \cos(0.5) - 1}{3 + \sin(0.5)} \\ &= 0.6085. \end{aligned}$$

Now, the second approximation:

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 0.6085 - \frac{3 \times 0.6085 - \cos(0.6085) - 1}{3 + \sin(0.6085)} \\ &= 0.6071. \end{aligned}$$

Now, the third approximation:

$$\begin{aligned} x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 0.6071 - \frac{3 \times 0.6071 - \cos(0.6071) - 1}{3 + \sin(0.6071)} \\ &= 0.6071. \end{aligned}$$

Therefore, $\alpha = 0.6071$ is the positive root of the given equation (correct to four places of decimals). □

4.1.6 Fixed point method

Definition 4.1 (Fixed point)**.** A point, say, α is called a fixed point of a function $g(x)$ if it satisfies the equation $g(\alpha) = \alpha$.

Fixed point Method: In this method, the equation $f(x) = 0$ is first converted into the form $x = g(x)$ and then we use the following iterative scheme called the Picard iteration:

$$x_n = g(x_{n-1}), \ n = 1, 2, \ldots$$

with some initial guess x_0. It is also called the fixed point iterative scheme. Again, we first find the values a and b such that $f(a) < 0$ and $f(b) > 0$, and then the initial guess x_0 can be obtained by a similar process as we have used in Newton's method.

Important Note.

- The solution of $f(x) = 0$ is the fixed point of the function $g(x)$, that is why, this method is called the fixed point method.
- In the fixed point method, the convergence of the Picard iterative scheme is a must, and for the convergence of this scheme, we have to ensure the condition of convergence given by
$$|g'(x)| < 1 \quad \text{for all} \quad x \in [a, b].$$
- We observe that the function $g(x)$ is not unique and can be chosen in infinitely many ways. We should choose such a "g" for which the condition $|g'(x)| < 1 \quad \text{for all} \quad x \in [a, b]$ is satisfied.

Example 4.9. Obtain the root of the equation $x^3 - 2x + 5 = 0$ and correct up to four decimal places using the fixed point method.

Solution. The given equation is $f(x) = x^3 - 2x + 5 = 0$. First, we find a and b. Then since $f(-3) = -16 < 0$ and $f(-2) = 1 > 0$, therefore $a = -3$ and $b = -2$. We write this equation in the following form

$$x = (2x - 5)^{1/3} = g(x).$$

Then, $g'(x) = \dfrac{2}{3\,(2x-5)^{2/3}}$. Now, it is easy to see that $g'(x) < 1$ for all x lying in the interval $[-3, -2]$. Therefore, the formula for the Picard iteration will be:

$$x_n = g(x_{n-1}) = (2x_{n-1} - 5)^{1/3}.$$

Let the initial guess be $x_0 = -2$, then by the above formula the first approximation:

$$x_1 = (2x_0 - 5)^{1/3} = (2 \times (-2) - 5)^{1/3} = -2.0800.$$

Then, the second approximation:

$$x_2 = (2x_1 - 5)^{1/3} = (2 \times (-2.0800) - 5)^{1/3} = -2.0923.$$

Then, the third approximation:

$$x_3 = (2x_2 - 5)^{1/3} = (2 \times (-2.0923) - 5)^{1/3} = -2.0942.$$

Then, the fourth approximation:

$$x_4 = (2x_3 - 5)^{1/3} = (2 \times (-2.0942) - 5)^{1/3} = -2.0944.$$

Then, the fifth approximation:

$$x_5 = (2x_4 - 5)^{1/3} = (2 \times (-2.0944) - 5)^{1/3} = -2.0945.$$

Then, the sixth approximation:

$$x_6 = (2x_5 - 5)^{1/3} = (2 \times (-2.0945) - 5)^{1/3} = -2.0945.$$

Therefore, the root of the given equation $\alpha = -2.0945$ (correct up to four decimal places). □

4.2 Exercise

(Q.1) Find the roots of the following equations, using the Bisection method, Secant method and False position method, correct to three decimal places:

(1) $x^3 - 2x - 5$ (2) $x^3 - x^2 - 1 = 0$
(3) $\cos x = xe^x$ (4) $x \log_{10}(x) = 1.2$
(5) $x^3 - x - 11$ which lies between 2 and 3.

Ans. (1) 2.687 (2) 1.46 (3) 0.519 (4) 2.875 (5) 2.375

(Q.2) Using the Newton-Raphson method find a root of the following equations correct to three decimal places:

(1) $x^3 + x - 1 = 0$ (2) $xe^x = 2$
(3) $x^3 - 3x + 1 = 0$ (4) $x \log_{10}(x) = 1.2$

Ans. (1) 0.686 (2) 0.853 (3) 1.532 (4) 2.741

(Q.3) Find the square root of 12 by Newton's method. **Ans. 3.4641**

(Q.4) Obtain the root of the equation $x^3 - 3x - 5 = 0$ correct up to four decimal places using the fixed point method.
Hint: Here $f(2) = -3 < 0$ and $f(3) = 13 > 0$, so, $a = 2, b = 3$. Write the given equation into the following form

$$x = (3x + 5)^{1/3} = g(x).$$

Now you can see that $|g'(x)| = \left|\frac{1}{(3x+5)^{2/3}}\right| < 1$ for all $x \in [2, 3]$. Now apply the Picard iteration scheme and find the solution of the given equation.

Unit-V

Differential equations: formation of differential equations, solution of differential equation of first order and first degree: separation of variable, homogeneous equations, reducible to homogeneous equations, linear equations, reducible to linear equations.

5.1 Differential equations

An equation involving one dependent variable and its derivatives with respect to one or more independent variables is called a differential equation. For example, the following equations are examples of differential equations:

(a) $e^x dx + e^y dy = 0$;

(b) $y = x\,\dfrac{dy}{dx}$;

(c) $\left[\dfrac{d^2y}{dx^2} + 2\right]^{3/2} = 5\,\dfrac{dy}{dx} + y$;

(d) $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

5.1.1 Ordinary differential equations

A differential equation containing the differential coefficients with respect to a single independent variable is called an ordinary differential equation. For example, (i), (ii) and (iii) are ordinary differential equations but not (iv). In this course, we deal with ordinary differential equations only.

5.1.2 Order and degree of differential equations

The order of the highest derivative appearing in a differential equation is called the order of the differential equations. The order of the differential equations (i), (ii) and (iii) are 1,1 and 2 respectively. The degree of a differential equation is the degree of the highest derivative appearing in it after the equation has been expressed in a form free from the radicals and fractions as far as the derivatives are concerned. For example, the degree of differential equation $y = x\,\dfrac{dy}{dx} + \dfrac{x}{dy/dx}$ is 2, while the degree of the differential equation $\left[\dfrac{d^2y}{dx^2} + 2\right]^{3/2} = 5\,\dfrac{dy}{dx} + y$ is 3.

5.2 Formation of differential equations

Consider a particle falling freely under gravity. Suppose it starts falling from rest and moves towards the ground. We want to calculate the distance travelled at any time during its fall. The laws of motion suggest that if the initial velocity of the particle is u and the velocity after time t is v, then the relation between u, v and t is given by:

$$v = u + gt$$

where g is the gravitational acceleration. In the case of free fall from the rest $u = 0$, hence we have $v = gt$, and since $v = \frac{ds}{dt}$, where s represents the distance traveled by particle in time t, hence:

$$\frac{ds}{dt} = gt.$$

The above equation is a differential equation, and its solution gives the distance travelled by the particle at any time t during the motion.

The above discussion illustrates how differential equations are formed in real-life problems. In theoretical mathematics, a differential equation is formed when we eliminate the parameter of a family of curves with the help of derivatives and obtain a relation between the variables and derivatives.

Example 5.1. Form the differential equation of the family of straight lines $y = mx$, where m is the parameter of the family. What are the order and degree of the differential equation?

Solution. The given equation of the family of straight lines is:

$$y = mx. \tag{5.1}$$

Differentiating (5.1) with respect to x we get $\frac{dy}{dx} = m$. On putting this value in (5.1) we get $y = \frac{dy}{dx}x$, i.e.

$$\frac{dy}{dx} = \frac{y}{x}.$$

This is the required differential equation of the family of straight lines. The order and degree of the differential equation both are 1. □

Example 5.2. Form the differential equation of the family of cosine curves $y = A\cos(x + \alpha)$, where A and α denote the parameters of the family. What are the order and degree of the differential equation?

Solution. The given equation of the family of curves is:

$$y = A\cos(x+\alpha). \tag{5.2}$$

Differentiating (5.2) with respect to x we get:

$$\frac{dy}{dx} = -A\sin(x+\alpha).$$

Again differentiating the above we get:

$$\frac{d^2y}{dx^2} = -A\cos(x+\alpha).$$

On putting the value of $A\cos(x+\alpha)$ from the above equation in (5.2) we get:

$$\frac{d^2y}{dx^2} + y = 0.$$

This is the required differential equation of the family of cosine curves. The order of the differential equation is 2 and the degree is 1. □

Example 5.3. Form the differential equation of the family of curves $y = ce^{ax}$, where c and a denote the parameters of the family. What are the order and degree of the differential equation?

Solution. The given equation of the family of curves is:

$$y = ce^{ax}. \tag{5.3}$$

Differentiating (5.3) with respect to x we get:

$$\frac{dy}{dx} = cae^{ax}.$$

On putting the value of ce^{ax} from the above equation in (5.2) we get:

$$\frac{dy}{dx} - ay = 0.$$

This is the required differential equation of the family of cosine curves. The order and degree of the differential equation both are 1. □

5.3 Exercise

(Q.1) Form the differential equation of the simple harmonic motion for the family of cosine waves $x = a\cos(nt+\alpha)$. What is the degree and order of this differential equation?

Ans: $\dfrac{d^2x}{dt^2} + n^2x = 0$. Order is 2 and degree is 1.

(Q.2) Obtain the differential equation of all circles of radius a and the centre (h, k). What is the degree and order of this differential equation?

Ans: $\dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}} = a$. Order and degree both are equal to 2.

(Q.3) Obtain the differential equation of the coaxial circles of the system $x^2+y^2+2ax+c^2=0$, where c is a constant and a is the family parameter. What is the degree and order of this differential equation?

Ans: $2xy\,\dfrac{dy}{dx} = y^2 - x^2 + c^2$. Order and degree both are equal to 1.

(Q.4) Form the differential equation of the family of curves given by: $y = c_1 \cos 2x + c_2 \sin 2x$. What is the degree and order of this differential equation?

Ans: $\dfrac{d^2y}{dx^2} + 4y = 0$. Order is 2 and degree is 1.

(Q.5) Form the differential equation of the family of curves given by the equation: $y = e^x (A\cos x + B \sin x)$. What is the degree and order of this differential equation?

Ans: $\dfrac{d^2y}{dx^2} - 2\,\dfrac{dy}{dx} + 2y = 0$. Order is 2 and degree is 1.

5.4 First order linear differential equations

A differential equation of first order and first degree is called a first-order linear differential equation. A first-order linear differential equation is of the following form:

$$\phi\left(\frac{dy}{dx}, y, x\right) = 0. \tag{5.4}$$

If (5.4) can be written in the following form:

$$\frac{dy}{dx} = \psi(x, y).$$

Then, we use the following techniques to solve such equations:

(i) Variable separable form:

(a) In this form ψ can be factorised in the form $\psi(x,y) = \varphi(x)\upsilon(y)$. Now, the equation can be solved by separating variables.

(b) If ψ cannot be factorised, but is in the form $\psi(x,y) = f(ax+by+c)$, then such equations can be solved by substituting $v = ax+by+c$.

(ii) Homogeneous differential equation:

(a) If ψ cannot be factorised, but $\psi(x,y) = \dfrac{f(x,y)}{g(x,y)}$, where f and g homogeneous functions in x and y of same degree. Such equations can be solved by the substitution $y = vx$.

(b) Differential equation reducible into homogeneous form: It is the following form:

$$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2},\ \frac{a_1}{a_2} \neq \frac{b_1}{b_2}.$$

Such equations can be reduced into variable separable form by substituting $x = X+h, y = Y+k$, where h,k are the constants. While, in case of $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, such equations can be solved by the substitution $ax+by = v$.

(iii) Leibnitz's linear differential equations:

(a) If the differential equation is not in the previous forms, but can be expressed in the form $\dfrac{d\phi(y)}{dx} + P\phi(y) = Q$, where P and Q are the functions of x only. Then, such a form is called the linear differential equation in $\phi(y)$. Its solution is given by

$$\phi(y) \times I.F. = C + \int (Q \times I.F.)dx$$

where the integrating factor $I.F. = e^{\int P dx}$.

(b) Sometimes the equation cannot be reduced in the equation linear in $\phi(y)$, but linear in $\phi(x)$, i.e., the equation is reduced in the form $\dfrac{d\phi(x)}{dy} + P\phi(x) = Q$, where P and Q are the functions of y only. Then, such an equation is solved with the same process, as used, in the previous case, only, the roles of x and y are changed.

(c) *Bernoulli's differential equations*. It is the equation of the following form:

$$\frac{dy}{dx} + Py = Qy^n$$

where P and Q are the functions of only x and $n \neq 1$ (because for $n = 1$ equation reduces into the variable separable form). Such equations can be solved by dividing the equation by y^n, then substituting $y^{1-n} = v$.

5.5 Examples on variable separable form

Example 5.4. Solve: (i) $3e^x \tan y \; dx + (1 - e^x) \sec^2 y \; dy = 0$; (ii) $\dfrac{dy}{dx} = 1 + x + y + xy$.

Solution. (i) The given equation can be written as:

$$\frac{3e^x}{1 - e^x}\, dx = -\frac{\sec^2 y}{\tan y}\, dy.$$

It is a variable separable form, therefore, integrating the above equation we get:

$$\begin{aligned} 3\int \frac{e^x}{1 - e^x}\, dx &= -\int \frac{\sec^2 y}{\tan y}\, dy \\ \Longrightarrow 3\int \frac{e^x}{e^x - 1}\, dx &= -\int \frac{\sec^2 y}{\tan y}\, dy \end{aligned}$$

Process of integration yields:

$$\begin{aligned} 3\ln(e^x - 1) &= \ln(\tan y) + \ln C \\ \Longrightarrow (e^x - 1)^3 &= C \tan y. \end{aligned}$$

(ii) From the given equation we have:

$$\begin{aligned} \frac{dy}{dx} &= 1 + x + y + xy \Longrightarrow \frac{dy}{dx} = (1 + x) + y(1 + x) \\ \Longrightarrow \frac{dy}{dx} &= (1 + x)(1 + y) \Longrightarrow \frac{dy}{1 + x} = \frac{dx}{1 + y}. \end{aligned}$$

□

Example 5.5. Solve: (i) $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$; (ii) $(1 + y^2)dx - xydy = 0$.

Solution. (i) The given differential equation can be written as:

$$\frac{\sec^2 x}{\tan x} dx = -\frac{\sec^2 y}{\tan y} dy.$$

It is a variable separable form, hence integrating we get

$$\begin{aligned} \int \frac{\sec^2 x}{\tan x} dx &= -\int \frac{\sec^2 y}{\tan y} dy \\ \Longrightarrow \ln(\tan x) &= -\ln(\tan y) + C \end{aligned}$$

(ii) The given differential equation can be written as:

$$\frac{1}{x}dx = \frac{y}{1+y^2}dy.$$

It is a variable separable form, hence integrating we get

$$\begin{aligned} \int \frac{1}{x}dx &= \int \frac{y}{1+y^2}dy \\ \implies \ln(x) &= \frac{1}{2}\ln\left(1+y^2\right) + \ln(C) \\ \implies x &= C\sqrt{1+y^2}. \end{aligned}$$

□

Example 5.6. Solve: (i) $\frac{dy}{dx} = (4x+y+1)^2$; (ii) $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$.

Solution. (i) The given differential equation is: $\frac{dy}{dx} = (4x+y+1)^2$, which is of the form $\frac{dy}{dx} = f(ax+by+c)$. Therefore, putting $4x+y+1 = v$ we have $4 + \frac{dy}{dx} = \frac{dv}{dx}$, i.e., $\frac{dy}{dx} = \frac{dv}{dx} - 4$. Putting these values in the given equation, we get

$$\frac{dv}{dx} = 4 + v^2 \implies \frac{dv}{v^2+4} = dx.$$

Integrating, we get

$$\frac{1}{2}\tan^{-1}\left(\frac{v}{2}\right) = x + C \implies \tan^{-1}\left(\frac{4x+y+1}{2}\right) = 2(x+C).$$

(ii) given differential equation is: $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$, which is of the form $\frac{dy}{dx} = f(ax+by+c)$. Therefore, putting $x+y = v$ we have $\frac{dy}{dx} = \frac{dv}{dx} - 1$. Putting these values in the given equation, we get

$$\frac{dy}{dx} = 1 + \cos v + \sin v \implies \frac{dv}{1+\cos v + \sin v} = dx.$$

Integrating, we get $\int \frac{dv}{1+\cos v+\sin v} = \int\ dx + C$. Since

$$1 + \cos v + \sin v = 1 + \frac{1-\tan^2\left(\frac{v}{2}\right)}{1+\tan^2\left(\frac{v}{2}\right)} + \frac{2\tan\left(\frac{v}{2}\right)}{1+\tan^2\left(\frac{v}{2}\right)} = 2\frac{1+\tan\left(\frac{v}{2}\right)}{\sec^2\left(\frac{v}{2}\right)}$$

we obtain

$$\begin{aligned} \frac{1}{2}\int \frac{\sec^2\left(\frac{v}{2}\right)dv}{1+\tan\left(\frac{v}{2}\right)} &= \int dx + C \\ \implies \ln\left[1+\tan\left(\frac{v}{2}\right)\right] &= x + C \\ \implies \ln\left[1+\tan\left(\frac{x+y}{2}\right)\right] &= x + C \end{aligned}$$

□

5.6 Examples on Homogeneous differential equation

Example 5.7. Solve: (i) $xdy - ydx = \sqrt{x^2+y^2}\,dx$; (ii) $x(x-y)dy + y^2dx = 0$.

Solution. (i) The given equation can be written as

$$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}.$$

This is homogeneous differential equation, therefore putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we obtain:

$$\begin{aligned} v + x\frac{dv}{dx} &= \frac{vx+\sqrt{x^2+v^2x^2}}{x} \\ \implies \frac{dx}{x} &= \frac{dv}{\sqrt{1+v^2}}. \end{aligned}$$

This is a variable separable form, therefore integrating we get

$$\begin{aligned} \int \frac{dx}{x} &= \int \frac{dv}{\sqrt{1+v^2}} + C \\ \implies \qquad \ln(x) &= \ln\left(v+\sqrt{1+v^2}\right) + \ln(c) \\ \implies \qquad x &= c\left(v+\sqrt{1+v^2}\right) \\ \implies \qquad x^2 &= c\left(y+\sqrt{x^2+y^2}\right). \end{aligned}$$

(ii) The given equation can be written as

$$\frac{dy}{dx} = -\frac{y^2}{x(x-y)}.$$

It is homogeneous differential equation, therefore putting $y = vx$ and $\dfrac{dy}{dx} = v + x\,\dfrac{dv}{dx}$, we obtain:

$$\begin{aligned} v + x\,\frac{dv}{dx} &= -\frac{v^2x^2}{x(x-vx)} \\ \implies \qquad x\,\frac{dv}{dx} &= -\frac{v^2}{(1-v)} - v \implies \qquad \frac{dx}{x} = \frac{v-1}{v}dv. \end{aligned}$$

It is a variable separable form, therefore integrating we get

$$\begin{aligned} \ln(x) &= v - \ln(v) + \ln(c) \\ \implies \quad vx &= ce^v \\ \implies \quad y &= ce^{y/x}. \end{aligned}$$

□

Example 5.8. Solve: (i) $y^2dx + (xy + x^2)dy = 0$; (ii) $(x^2 - y^2)dx + 2xydy = 0$.

Solution. (i) The given equation can be written as:

$$\frac{dy}{dx} = -\frac{y^2}{xy + x^2}.$$

This is homogeneous differential equation, therefore putting $y = vx$ and $\dfrac{dy}{dx} = v + x\,\dfrac{dv}{dx}$, we obtain:

$$\begin{aligned} v + x\,\frac{dv}{dx} &= -\frac{v^2x^2}{vx^2 + x^2} = -\frac{v^2}{v+1} \\ \implies \quad x\,\frac{dv}{dx} &= -\frac{v + 2v^2}{1+v} \implies \frac{1+v}{v(1+2v)}dv = -\frac{dx}{x}. \end{aligned}$$

This is a variable separable form, therefore integrating we get

$$\begin{aligned} \int \frac{1+v}{v(1+2v)}dv &= -\ln(x) + \ln(c) \\ \implies \quad \int \left[\frac{1}{v} - \frac{1}{2v+1}\right] dv &= -\ln(cx) \\ \implies \quad \ln(v) - \frac{1}{2}\ln(1+2v) &= -\ln(cx) \\ \implies \quad \frac{c^2x^2v^2}{2v+1} &= 1. \end{aligned}$$

Putting $v = y/x$ we get: $\dfrac{c^2y^2x}{2y+x} = 1.$
(ii) The given equation can be written as:

$$\frac{dy}{dx} = -\frac{y^2 - x^2}{2xy}.$$

It is homogeneous differential equation, therefore putting $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$, we obtain:

$$\begin{aligned} v + x\frac{dv}{dx} &= -\frac{v^2-1}{2v} \\ \implies \quad x\frac{dv}{dx} &= -\frac{1+v^2}{2v} \\ \implies \quad \frac{2v}{1+v^2}dv &= \frac{dx}{x}. \end{aligned}$$

It is a variable separable form, therefore integrating we get

$$\begin{aligned} \ln(1+v^2) &= \ln(x) + \ln(c) \\ \implies \quad 1+v^2 &= cx \\ \implies \quad x^2+y^2 &= cx^3. \\ \ln(1+v^2) &= \ln(x) + \ln(c) \\ \implies \quad 1+v^2 &= cx \\ \implies \quad x^2+y^2 &= cx^3. \end{aligned}$$

□

Example 5.9. Solve: $(2x + y + 3)dx = (2y + x + 1)dy$.

Solution. (i) The given equation can be written as

$$\frac{dy}{dx} = \frac{2x+y+3}{x+2y+1}. \tag{5.5}$$

It is the equation of the form $\dfrac{dy}{dx} = \dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$. Here $\dfrac{a_1}{a_2} = \dfrac{2}{1} \neq \dfrac{b_1}{b_2} = \dfrac{1}{2}$, therefore, putting $x = X + h, y = Y + k$ we have $dx = dX, dy = dY$. The equation (5.5) reduces into the following form:

$$\frac{dY}{dX} = \frac{2(X+h)+(Y+k)+3}{(X+h)+2(Y+k)+1}.$$

This shows that

$$\frac{dY}{dX} = \frac{2X + Y + (2h + k + 3)}{X + 2Y + (h + 2k + 1)}. \tag{5.6}$$

Choosing h and k such that

$$\begin{aligned} 2h + k + 3 &= 0 \\ h + 2k + 1 &= 0. \end{aligned}$$

On solving we get $h = -5/3$ and $k = 1/3$. Now (5.6) becomes

$$\frac{dY}{dX} = \frac{2X + Y}{X + 2Y}.$$

This equation is homogeneous in X and Y, therefore, putting $Y = VX$, we get $\frac{dY}{dX} = V + \frac{dV}{dX}$, and so, we have

$$\begin{aligned} V + \frac{dV}{dX} &= \frac{2X + VX}{X + 2VX} = \frac{2 + V}{1 + 2V} \\ \implies 2\frac{dV}{dX} &= \left[\frac{1}{1 - V^2} + \frac{2V}{1 - V^2}\right]. \end{aligned}$$

On integrating we obtain:

$$\begin{aligned} \frac{1}{2}\ln\left(\frac{1 + V}{1 - V}\right) - \ln(1 - V^2) &= 2\ln(X) + \ln(C) \\ \implies \frac{1 + V}{1 - V} &= C^2(1 - V^2)^2 X^4 \\ \implies 1 &= C^2 X^4 (1 - V)^3 (1 + V). \end{aligned}$$

Putting $V = Y/X$, $X = x - h = x + 5/3$ and $Y = y - 1/3$ the solution will be:

$$\left(x + y + \frac{4}{3}\right)(x - y + 2)^3 C^2 = 1.$$

□

Example 5.10. Solve: $(2x + y + 1)dx + (4x + 2y - 1)dy = 0$.

Solution. (i) The given equation can be written as

$$\frac{dy}{dx} = -\frac{2x + y + 1}{4x + 2y - 1}.$$

It is the equation of the form $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$. Here $\frac{a_1}{a_2} = \frac{2}{4} = \frac{b_1}{b_2} = \frac{1}{2}$, therefore, the above equation is written as:

$$\frac{dy}{dx} = -\frac{2x+y+1}{2(2x+y)-1}. \tag{5.7}$$

Putting $2x+y=v$ we have $2+\frac{dy}{dx}=\frac{dv}{dx}$. The equation (5.7) reduces into the following form:

$$\frac{dv}{dx} - 2 = -\frac{v+1}{2v-1} \implies \frac{dv}{dx} = \frac{3(v-1)}{2v-1}.$$

This gives:

$$\frac{2v-1}{v-1}dv = 3dx.$$

On integrating we obtain:

$$\begin{aligned} 3x+c &= \int \frac{2v-1}{v-1}dv = \int \frac{2(v-1)+1}{v-1}dv = \int \left[2+\frac{1}{v-1}\right] dv \\ &= 2v + \ln(v-1). \end{aligned}$$

Putting $v = 2x+y$ the solution will be:

$$x + 2y + \ln(2x+y-1) = c.$$

□

Example 5.11. Solve: $\left(1+e^{x/y}\right) dx + e^{x/y}\left(1-\frac{x}{y}\right) dy = 0.$

Solution. The given equation can be written as:

$$\left(1+e^{x/y}\right)\frac{dx}{dy} = -e^{x/y}\left(1-\frac{x}{y}\right).$$

Putting $x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$ in the given equation we have

$$\begin{aligned} (1+e^v)\,(v+y\frac{dv}{dy}) = -e^v\,(1-v) &\implies v+e^v+(1+e^v)y\frac{dv}{dy} = 0 \\ &\implies (1+e^v)y\frac{dv}{dy} = -(v+e^v) \\ &\implies \frac{1+e^v}{v+e^v}\,dv = -\frac{dy}{y}. \end{aligned}$$

It is the variable separable form, therefore, on integrating we obtain:

$$\begin{aligned} \ln(v+e^v)+\ln(y) &= \ln(C) \\ \implies \ln\left[(v+e^v)y\right] &= \ln(C) \\ \implies (v+e^v)y &= C. \end{aligned}$$

Putting $v = x/y$ the solution will be:

$$x + ye^{x/y} = C. \qquad \square$$

5.7 Exercise

(Q.1) Solve: $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$.

Ans. Variable separable, $y = C(1-ay)(x+a)$.

(Q.2) Solve: $\dfrac{dy}{dx} = e^{x-y} + x^2e^{-y}$.

Ans. Variable separable, $e^y = e^x + \dfrac{x^3}{3} + C$.

(Q.3) Solve: $(e^y+1)\cos x\ dx + e^y \sin x\ dy = 0$.

Ans. Variable separable, $\sin x(e^y+1) = C$.

(Q.4) Solve: $\dfrac{dy}{dx} = \cos(x+y+1)$.

Hint. Put $x+y+1 = v$. **Ans.** $\tan\left(\frac{x+y+1}{2}\right) = x + c$.

(Q.5) Solve: $(x+2y)(dx-dy) = dx+dy$.

Hint. Write it $\dfrac{dy}{dx} = \dfrac{x+2y-1}{x+2y+1}$ and put $x+2y = v$

Ans. $\frac{1}{3}\left[x+2y+\frac{4}{3}\ln(3x+6y-1)\right] = x + C$.

(Q.6) Solve: $\dfrac{dy}{dx} = \dfrac{y}{x} + \sin\left(\dfrac{y}{x}\right)$.

Hint. Put $\frac{y}{x} = v$. **Ans.** $\operatorname{cosec}\left(\frac{y}{x}\right) + \cot\left(\frac{y}{x}\right) = cx$.

(Q.7) Solve: $\dfrac{dy}{dx} = \dfrac{y-x+1}{y+x-5}$.

Hint. Solve as Example 5.10.

Ans. $\tan^{-1}\left(\frac{y-2}{x-3}\right) + \frac{1}{2}\ln\left[1+\left(\frac{y-2}{x-3}\right)^2\right] = -\ln(x-3) + \ln(C)$.

(Q.8) Solve: $\dfrac{y}{x}\dfrac{dy}{dx} + \dfrac{x^2+y^2-1}{2(x^2+y^2)+1} = 0$.

Hint. Put $x^2 + y^2 = v$, and $\frac{y}{x}\frac{dy}{dx} = \frac{1}{2x}\frac{dv}{dx} - 1$.

5.8 Examples on linear differential equations

Theorem 8. If P and Q are two integrable functions of x, then prove that the solution of differential equation $\frac{dy}{dx} + Py = Q$ is given by

$$y \cdot e^{\int P dx} = C + \int Q e^{\int P dx} dx.$$

where C is an arbitrary constant.

Proof. The given differential equation is:

$$\frac{dy}{dx} + Py = Q. \tag{5.8}$$

On multiplying (5.8) by $e^{\int P dx}$ we get:

$$e^{\int P dx}\frac{dy}{dx} + P e^{\int P dx} y = e^{\int P dx} Q.$$

The above equation can be written as

$$\frac{d}{dx}\left(y \cdot e^{\int P dx}\right) = e^{\int P dx} Q.$$

Integration of the above equation gives:

$$y \cdot e^{\int P dx} = C + \int Q e^{\int P dx} dx.$$

□

Example 5.12. Solve: (i) $\sec x \frac{dy}{dx} = y + \sin x$ (ii) $\frac{dy}{dx} = -\frac{x + y\cos x}{1 + \sin x}$.

Solution. (i) The given equation can be written as

$$\frac{dy}{dx} - (\cos x)y = \sin x \cos x.$$

It is a linear differential equation in y. Here $P = -\cos x$, $Q = \sin x \cos x$. Therefore,

$$\text{I.F.} = e^{\int P dx} = e^{\int -\cos x \, dx} = e^{-\sin x}.$$

Therefore, the solution of the given equation will be:

$$\begin{aligned} y \times \text{I.F.} &= C + \int Q \times \text{I.F.}\ dx \\ \implies ye^{-\sin x} &= C + \int \sin x \cos x \times e^{-\sin x}\ dx \\ \implies ye^{-\sin x} &= C + \int t \times e^{-t}\ dx \qquad (\text{putting } \sin x = t) \\ \implies ye^{-\sin x} &= C - te^{-t} - e^{-t} \\ \implies ye^{-\sin x} &= C - \sin x e^{-\sin x} - e^{-\sin x}. \end{aligned}$$

(ii) The given equation can be written as:

$$\frac{dy}{dx} + \frac{\cos x}{1 + \sin x} y = -\frac{x}{1 + \sin x}.$$

It is a linear differential equation in y. Here $P = \dfrac{\cos x}{1 + \sin x}$, $Q = -\dfrac{x}{1 + \sin x}$. Therefore,

$$\text{I.F.} = e^{\int P dx} = e^{\int \frac{\cos x}{1+\sin x}\ dx} = e^{\ln(1+\sin x)} = 1 + \sin x.$$

Therefore, the solution of the given equation will be:

$$\begin{aligned} y \times \text{I.F.} &= C + \int Q \times \text{I.F.}\ dx \\ \implies y(1 + \sin x) &= C - \int \frac{x}{1 + \sin x} \times (1 + \sin x)\ dx \\ \implies y(1 + \sin x) &= C - \frac{x^2}{2}. \end{aligned}$$

□

Example 5.13. Solve: $\dfrac{dy}{dx} + 2y \tan x = \sin x$, given that $y\left(\dfrac{\pi}{3}\right) = 0$.

Solution. The given equation is a linear differential equation in y. Here $P = 2\tan x$, $Q = \sin x$. Therefore,

$$\text{I.F.} = e^{\int P dx} = e^{\int 2\tan x\ dx} = e^{2\ln(\sec x)} = \sec^2 x.$$

Therefore, the solution will be $y \times \text{I.F.} = C + \int Q \times \text{I.F.}\ dx$, i.e.

$$\begin{aligned} \implies y \sec^2 x &= C + \int \sin x \times \sec^2 x\ dx \\ \implies y \sec^2 x &= C + \sec x. \end{aligned}$$

Applying the condition $y\left(\frac{\pi}{3}\right) = 0$, i.e., putting $x = \frac{\pi}{3}$ and $y = 0$ in the above equation we get:

$$0 = 2 + C \implies C = -2.$$

Therefore, the solution will be: $y \sec^2 x = -2 + \sec x$. □

Example 5.14. Solve: (i) $\sqrt{1-y^2}dx = (\sin^{-1} y - x)dy$. (ii) $(y - x)\frac{dy}{dx} = a^2$.

Solution. (i) One can see that the given equation cannot be written in a form so that it is linear in y. But the equation can be written as

$$\frac{dx}{dy} + \frac{1}{\sqrt{1-y^2}}\,x = \frac{\sin^{-1} y}{\sqrt{1-y^2}}.$$

It is linear in x. Here $P = \frac{1}{\sqrt{1-y^2}}$ and $Q = \frac{\sin^{-1} y}{\sqrt{1-y^2}}$. Therefore

$$\text{I.F.} = e^{\int P dy} = e^{\int \frac{1}{\sqrt{1-y^2}} dy} = e^{\sin^{-1} y}.$$

Hence, the solution of the given equation will be:

$$\begin{aligned}
x \times \text{I.F.} &= C + \int Q \times \text{I.F.}\ dy \\
\implies x e^{\sin^{-1} y} &= C + \int \frac{\sin^{-1} y}{\sqrt{1-y^2}} \sin^{-1} y\ dy \\
\implies x e^{\sin^{-1} y} &= C + \int t e^t\ dt \qquad (t = \sin^{-1} y) \\
\implies x e^{\sin^{-1} y} &= C + t e^t - e^t \\
\implies x &= C e^{-\sin^{-1}} + \sin^{-1} y - 1.
\end{aligned}$$

(ii) One can see that the given equation cannot be written in a form so that it is linear in y. But the equation can be written as

$$\frac{dx}{dy} + \frac{1}{a^2}\,x = \frac{y}{a^2}.$$

It is linear in x. Here $P = \frac{1}{a^2}$ and $Q = \frac{y}{a^2}$. Therefore

$$\text{I.F.} = e^{\int P dy} = e^{\int \frac{1}{a^2} dy} = e^{y/a^2}.$$

Therefore, the solution will be: $x \times \text{I.F.} = C + \int Q \times \text{I.F.}\ dy$, i.e.

$$\begin{aligned}
\implies xe^{y/a^2} &= C + \int \frac{y}{a^2}\, e^{y/a^2}\, dy \\
\implies xe^{y/a^2} &= C + \frac{1}{a^2}\left[y\,\frac{e^{y/a^2}}{1/a^2} - \int 1 \cdot \frac{e^{y/a^2}}{1/a^2}\right] \\
\implies xe^{y/a^2} &= C + ye^{y/a^2} - \frac{e^{y/a^2}}{1/a^2} \\
\implies x &= Ce^{-y/a^2} + y - a^2.
\end{aligned}$$

□

Example 5.15. Solve: (i) $x\,\frac{dy}{dx} + y = x^3y^6$. (ii) $\frac{dy}{dx} + x\sin(2y) = x^3\cos^2 y$.

Solution. (i) Dividing by xy^6 the given equation we get

$$y^{-6}\,\frac{dy}{dx} + \frac{1}{x}y^{-5} = x^2.$$

It is the Bernoulli's differential equations, therefore, putting $y^{-5} = v$, i.e., $y^{-6}\,\frac{dy}{dx} = -\frac{1}{5}\,\frac{dv}{dx}$ in the above equation, we get:

$$\frac{dv}{dx} - \frac{5}{x}v = -5x^2.$$

It is linear in v. Here, $P = -\frac{5}{x}$ and $Q = -5x^2$ therefore,

$$\text{I.F.} = e^{\int P dx} = e^{-\int \frac{5}{x} dx} = \frac{1}{x^5}.$$

Therefore, the solution will be:

$$\begin{aligned}
v \times \text{I.F.} &= C + \int Q \times \text{I.F.}\ dx \\
\implies v \times \frac{1}{x^5} &= C + \int -5x^2 \times \frac{1}{x^5}\ dx \\
\implies \frac{v}{x^5} &= C + \frac{5}{2x^2}.
\end{aligned}$$

Putting $v = y^{-5}$, the solution of given equation will be:

$$\frac{1}{x^5y^5} = C + \frac{5}{2x^2}.$$

(ii) Dividing by $\cos^2 y$ the given equation we get

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3.$$

Putting $\tan y = v$, i.e., $\sec^2 y \frac{dy}{dx} = \frac{dv}{dx}$, the above equation is reduced into the following form:

$$\frac{dv}{dx} + 2xv = x^3.$$

The above equation is linear in v. Here $P = 2x$, $Q = x^3$. Therefore, I.F. $= e^{\int P dx} = e^{\int 2x dx} = e^{x^2}$.

The solution will be: $v \times \text{I.F.} = C + \displaystyle\int Q \times \text{I.F.}\ dx$, i.e.:

$$\begin{aligned} ve^{x^2} &= C + \int x^3 e^{x^2}\ dx \\ \implies ve^{x^2} &= C + \int te^t \frac{dt}{2} \qquad (t = x^2) \\ \implies ve^{x^2} &= C + \frac{1}{2}(e^t - e^t) \\ \implies (\tan y)e^{x^2} &= C + \frac{1}{2}(x^2 e^{x^2} - e^{x^2}). \end{aligned}$$

□

Example 5.16. Solve: (i) $\frac{dy}{dx}(x^2y^3 + xy) = 1$. (ii) $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \cdot \sec y$.

Solution. (i) The given differential equation cannot be arranged in the form of linear or Bernoulli's differential equations in which y is the dependent variable. But, it can be written as:

$$\frac{dx}{dy} - xy = x^2y^3.$$

The above equation is of Bernoulli type with x as the dependent variable. Therefore, by dividing by x^2 we get

$$x^{-2}\frac{dx}{dy} - x^{-1}y = y^3.$$

Substituting $x^{-1} = v$, i.e., $x^{-2}\frac{dx}{dy} = -\frac{dv}{dy}$ the above equation reduced into the following form:

$$\frac{dv}{dy} + yv = -y^3.$$

This equation is linear in v. Here $P = y$, $Q = -y^3$, therefore I.F. $= e^{\int P dy} = e^{\int y dy} = e^{y^2/2}$, and the solution will be: $v \times$ I.F. $= C + \int Q \times$ I.F. dy

$$\begin{aligned}
\implies ve^{y^2/2} &= C + \int -y^3 e^{y^2/2}\, dy \\
\implies ve^{y^2/2} &= C + \int -2te^t\, dt \qquad (t = y^2/2) \\
\implies ve^{y^2/2} &= C + -2[te^t - e^t] \\
\implies ve^{y^2/2} &= C - 2\left[\frac{y^2}{2}e^{y^2/2} - e^{y^2/2}\right]
\end{aligned}$$

Hence, the solution is: $\frac{1}{x} = Ce^{-y^2/2} + (2 - y^2)$.
(ii) Dividing by $\sec y$ the given equation:

$$\cos x \frac{dy}{dx} - \frac{1}{1+x}\sin y = e^x(1+x).$$

Substitute $\sin y = v$, i.e., $\cos y \dfrac{dy}{dx} = \dfrac{dv}{dx}$, the above equation becomes

$$\frac{dv}{dx} - \frac{1}{1+x}v = e^x(1+x).$$

This equation is linear in v. Here $P = -\dfrac{1}{1+x}$, $Q = e^x(1+x)$ and

$$\text{I.F.} = e^{\int P dx} = e^{\int -\frac{1}{1+x}dx} = \frac{1}{1+x}$$

and the solution will be:

$$\begin{aligned}
v \times \text{I.F.} &= C + \int Q \times \text{I.F.}\, dx \\
\implies v\frac{1}{1+x} &= C + \int e^x(1+x)\frac{1}{1+x}\, dx \\
\implies v\frac{1}{1+x} &= C + e^x.
\end{aligned}$$

Hence, the solution is: $\sin y = C(1+x)(1+x)e^x$. □

5.9 Exercise

(Q.1) Solve: $\cos x\, dy = (\sin x - y)dx$.

Hint: Linear in y. Ans. $y(\sec x + \tan x) = \sec x + \tan x - x + C$.

(Q.2) Solve: $x\ln(x)\,\dfrac{dy}{dx} + y = \dfrac{2}{x}\ln(x)$.

Hint: Linear in y. Ans. $y\ln(x) = C - \frac{2}{x}(1+\ln(x))$.

(Q.3) Solve: $\cosh x\,\dfrac{dy}{dx} + y\sinh x = 2\cosh^2 x \sinh x$.

Hint: Linear in y. Ans. $y\cosh x = C + \frac{2}{3}\cosh^3 x$.

(Q.4) Solve: $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2$, given $y = 1$, when $x = 1$.

Hint: Linear in y. Ans. $xy = \frac{3}{4} + \frac{1}{4}x^4$.

(Q.5) Solve: $(1+x^2)\,\dfrac{dy}{dx} + 2xy - 4x^2 = 0$, subject to $y(0) = 0$.

Hint: Linear in y. Ans. $y(1+x^2) = \frac{4x^2}{3}$.

(Q.6) Solve: $x\,\dfrac{dy}{dx} + y = y^2\ln(x)$.

Hint: Bernoulli's differential equations, arrange it in the standard form, then put $y^{-1} = v$. Ans. $y\big(1+\ln(x)+Cx\big) = 1$.

(Q.7) Solve: $e^y\left(\dfrac{dy}{dx} + 1\right) = e^x$.

Hint: Put $e^y = v$. Ans. $e^{x+y} = C + \frac{1}{2}e^{2x}$.

(Q.8) Solve: $\dfrac{dy}{dx} = \dfrac{x^2+y^2+1}{2xy}$.

Hint: Rewrite the equation: $2y\dfrac{dy}{dx} - \dfrac{y^2}{x} = x + \dfrac{1}{x}$, then put $y^2 = v$.
Ans. $y^2 = Cx + x^2 - 1$.

(Q.9) Solve: $y(2xy + e^x)dx = e^x dy$.

Hint: Rewrite the equation: $y^{-2}\dfrac{dy}{dx} - y^{-1} = \frac{2x}{e^x}$, then put $y^{-1} = v$.
Ans. $e^x = Cy + x^2$.

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