CHENNAI Mathematical Institute

Graduate Programme in Mathematics

Entrance Examination, 2010

Part A

State whether True or False and give brief reasons in the sheets provided (e.g., if you feel that a statement is "False" then give a counter-example). Marks will be given only when reasons are provided.

1. Suppose A is an $m \times n$ matrix, V an $m \times 1$ matrix, with both A and V having rational entries. If the equation $AX = V$ has a solution in $\mathbb{R}^n$, then the equation has a solution with rational entries. (Here and in Question 5 below of Part A, $\mathbb{R}^n$ is identified with the space of $n \times 1$ real matrices.)

2. A closed and bounded subset of a complete metric space is compact.

3. Let p be a prime number. If P is a p-Sylow subgroup of some finite group G, then for every subgroup H of G, $H \cap P$ is a p-Sylow subgroup of H.

4. There exists a real 3×3 orthogonal matrix with only non-zero entries.

5. A 5×5 real matrix has an eigenvector in $\mathbb{R}^5$.

6. A continuous function on $\mathbb{Q} \cap [0,1]$ can be extended to a continuous function on $[0,1]$.

7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then $f'(x)$ is continuous.

8. There is a continuous onto function from the unit sphere in $\mathbb{R}^3$ to the complex plane $\mathbb{C}$.

9. $f\colon \mathbb{C} \to \mathbb{C}$ is an entire function such that the function $g(z)$ given by $g(z) = f(\frac{1}{z})$ has a pole at 0. Then f is a surjective map.

10. Every finite group of order 17 is abelian.

11. Let $n \geq 2$ be an integer. Given an integer k there exists an $n \times n$ matrix A with integer entries such that $\det A = k$ and the first row of A is $(1, 2, \ldots, n)$.

12. There is a finite Galois extension of $\mathbb{R}$ whose Galois group is nonabelian.

13. There is a non-constant continuous function from the open unit disc

$$D = \{z \in \mathbb{C} \mid |z| < 1\}$$

to $\mathbb{R}$ which takes only irrational values.

14. There is a field of order 121.

Part B

Answer all questions.

1. Let α, β be two complex numbers with $\beta \neq 0$, and $f(z)$ a polynomial function on $\mathbb{C}$ such that $f(z) = \alpha$ whenever $z^5 = \beta$. What can you say about the degree of the polynomial $f(z)$?

2. Let $f, g \colon \mathbb{Z}/5\mathbb{Z} \to S_5$ be two non-trivial group homomorphisms. Show that there is a $\sigma \in S_5$ such that $f(x) = \sigma g(x) \sigma^{-1}$, for every $x \in \mathbb{Z}/5\mathbb{Z}$.

3. Suppose f is continuous on $[0, \infty)$, differentiable on $(0, \infty)$ and $f(0) \geq 0$. Suppose $f'(x) \geq f(x)$ for all $x \in (0, \infty)$. Show that $f(x) \geq 0$ for all $x \in (0, \infty)$.

4. A linear transformation $T \colon \mathbb{R}^8 \to \mathbb{R}^8$ is defined on the standard basis $e_1, \ldots, e_8$ by

$$\begin{aligned} Te_j &= e_{j+1} \qquad j = 1, \ldots, 5 \\ Te_6 &= e_7 \\ Te_7 &= e_6 \\ Te_8 &= e_2 + e_4 + e_6 + e_8. \end{aligned}$$

   What is the nullity of T?

5. If f and g are continuous functions on $[0, 1]$ satisfying $f(x) \geq g(x)$ for every $0 \leq x \leq 1$, and if $\int_0^1 f(x)dx = \int_0^1 g(x)dx$, then show that $f = g$.

6. Let $\{a_n\}$ and $\{b_n\}$ be sequences of complex numbers such that each a_n is non-zero, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$, and such that for every natural number k,

$$\lim_{n\to\infty} \frac{b_n}{a_n^k} = 0.$$

   Suppose f is an analytic function on a connected open subset U of $\mathbb{C}$ which contains 0 and all the a_n. Show that if $f(a_n) = b_n$ for every natural number n, then $b_n = 0$ for every natural number n.

7. Let $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal transformation such that $\det T = 1$ and T is not the identity linear transformation. Let $S \subset \mathbb{R}^3$ be the unit sphere, i.e.,

$$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}.$$

   Show that T fixes exactly two points on S.

8. Compute

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\,dx.$$

9. Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with integer coefficients and whose degree is at least 2. Suppose each a_i $(0 \le i \le n-1)$ is of the form

$$a_i = \pm\frac{17!}{r!(17-r)!}$$

with $1 \le r \le 16$. Show that $f(m)$ is not equal to zero for any integer m.

10. Suppose $\varphi = (\varphi_2, \ldots, \varphi_n)\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ is a C^2 function, i.e. all second order partial derivatives of the φ_i exist and are continuous. Show that the symbolic determinant

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_1} & \cdots & \frac{\partial \varphi_n}{\partial x_1} \\ \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial x_n} & \frac{\partial \varphi_2}{\partial x_n} & \cdots & \frac{\partial \varphi_n}{\partial x_n} \end{vmatrix}$$

vanishes indentically.

CHENNAI MATHEMATICAL INSTITUTE

Graduate Programme in Mathematics

Entrance Examination, 2011

Part A

State whether True or False and give brief reasons. Marks will be given only when reasons are provided. Answer any 10 questions in this part. All questions carry 5 marks.

1. There is a sequence of open intervals $I_n \subset \mathbb{R}$ such that $\bigcap_{n=1}^{\infty} I_n = [0, 1]$.
2. The set S of real numbers of the form $\dfrac{m}{10^n}$ with $m, n \in \mathbb{Z}$ and $n \geq 0$ is a dense subset of $\mathbb{R}$.
3. There is a continuous bijection from $\mathbb{R}^2 \to \mathbb{R}$.
4. There is a bijection between $\mathbb{Q}$ and $\mathbb{Q} \times \mathbb{Q}$.
5. If $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive real numbers with the first converging to zero, and the second diverging to ∞, then the sequence of complex numbers $c_n = a_n e^{ib_n}$ also converges to zero.
6. For any polynomial $f(x)$ with real coefficients and of degree 2011, there is a real number b such that $f(b) = f'(b)$.
7. If $f\colon [0, 1] \to [-\pi, \pi]$ is a continuous bijection then it is a homeomorphism.
8. For any $n \geq 2$ there is an $n \times n$ matrix A with real entries such that $A^2 = A$ and $\operatorname{trace}(A) = n + 1$.
9. There is 2×2 real matrix with characteristic polynomial $x^2 + 1$.
10. There is a field with 10 elements.
11. There are at least three non-isomorphic rings with 4 elements.
12. The group $(\mathbb{Q}, +)$ is a finitely generated abelian group.
13. $\mathbb{Q}(\sqrt{7})$ and $\mathbb{Q}(\sqrt{17})$ are isomorphic as fields.
14. A vector space of dimension ≥ 2 can be expressed as a union of two proper subspaces.
15. There is a bijective analytic function from the complex plane to the upper half-plane.
16. There is a non-constant bounded analytic function on $\mathbb{C} \setminus \{0\}$.

Part B

Answer any five questions. All questions carry 10 marks

1. (a) Consider the ring R of polynomials in n variables with integer coefficients. Prove that the polynomial $f(x_1, x_2, \ldots, x_n) = x_1 x_2 \cdots x_n$ has $2^{n+1} - 2$ non-constant polynomials in R dividing it.

   (b) Let $p_1, p_2, \ldots, p_n$ be distinct prime numbers. Then show that the number $N = p_1 p_2^2 p_3^3 \cdots p_n^n$ has $(n+1)!$ positive divisors.

2. Let $f(x) = (x^2 - 2)(x^2 - 3)(x^2 - 6)$. For every prime number p, show that $f(x) \equiv 0 \pmod{p}$ has a solution in $\mathbb{Z}$.

3. Let **S** denote the group of all those permutations of the English alphabet that fix the letters T,E,N,D,U,L,K,A and R. Other letters may or may not be fixed. Show that **S** has elements σ, τ of order 36 and 39 respectively, but does not have any element of order 37 or 38.

4. Show that there are at least two non-isomorphic groups of order 198. Show that in all those groups the number of elements of order 11 is the same.

5. Suppose f, g, h are functions from the set of positive real numbers into itself satisfying $f(x)g(y) = h(\sqrt{x^2 + y^2}\,)$ for all $x, y \in (0, \infty)$. Show that the three functions $f(x)/g(x), g(x)/h(x)$, and $h(x)/f(x)$ are all constant.

6. Let $a, b > 0$.

   (a) Prove that $\lim_{n\to\infty}(a^n + b^n)^{1/n} = \max\{a, b\}$.

   (b) Define a sequence by $x_1 = a, x_2 = b$ and $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for $n > 2$. Show that $\{x_n\}$ is a convergent sequence.

7. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function with the following property: *In the power series expansion around any $a \in \mathbb{C}$, given as $f(z) = \sum_{n=o}^{\infty} c_n(a)(z-a)^n$, the coefficient $c_n(a)$ is zero for some n (with n depending on a).* Show that $f(z)$ is in fact a polynomial.

8. (a) Show that in a Hausdorff topological space any compact set is closed.

   (b) If (X, d_1) and (Y, d_2) are two metric spaces that are homeomorphic then does completeness of (X, d_1) imply the completeness of (Y, d_2)? Give reasons for your answer.

9. Fix an integer $n > 1$. Show that there is a real $n \times n$ diagonal matrix D such that the condition $AD = DA$ is valid only for a diagonal matrix A.

CHENNAI MATHEMATICAL INSTITUTE

Graduate Programme in Mathematics - M.Sc./Ph.D.

Entrance Examination, 2012

100

Part A

State whether True or False and give brief reasons in the sheets provided. Marks will be given only when reasons are provided. Try to answer 10 questions. Each question carries 5 marks.

1. The function $f : \mathbb{R}^n \to \mathbb{R}$, defined as $f(x_1, \cdots, x_n) = Max\{|x_i|\}, i = 1, \cdots, n$, is uniformly continuous.
2. Let x_n be a sequence with the following property: Every subsequence of x_n has a further subsequence which converges to x. Then the sequence x_n converges to x.
3. Let $f : (0, \infty) \longrightarrow \mathbb{R}$ be a continuous function. Then f maps any Cauchy sequence to a Cauchy sequence.
4. Let $\{f_n : \mathbb{R} \longrightarrow \mathbb{R}\}$ be a sequence of continuous functions. Let $x_n \longrightarrow x$ be a convergent sequence of reals. If $f_n \longrightarrow f$ uniformly then $f_n(x_n) \longrightarrow f(x)$.
5. Let $K \subset \mathbb{R}^n$ such that every real valued continuous function on K is bounded. Then K is compact (i.e closed and bounded).
6. If $A \subset \mathbb{R}^2$ is a countable set, then $\mathbb{R}^2 \setminus A$ is connected.
7. The set $A = \{(z, w) \in \mathbb{C}^2 \mid z^2 + w^2 = 1\}$ is bounded in $\mathbb{C}^2$.
8. Let $f, g : \mathbb{C} \longrightarrow \mathbb{C}$ be complex analytic, and let $h : [0, 1] \longrightarrow \mathbb{C}$ be a non-constant continuous map. Suppose $f(z) = g(z)$ for every $z \in \text{Im } h$, then $f = g$. (Here Im h denotes the image of the function h.)
9. There is a field with 121 elements.
10. The matrix $\begin{pmatrix} \pi & \pi \\ 0 & \frac{22}{7} \end{pmatrix}$ is diagonalizable over $\mathbb{C}$.
11. There are no infinite group with subgroups of index 5.
12. Every finite group of odd order is isomorphic to a subgroup of A_n, the group of all even permutations.
13. Every group of order 6 abelian.

14. Two abelian groups of the same order are isomorphic.

15. There is a non-constant continuous function $f : \mathbb{R} \to \mathbb{R}$ whose image is contained in $\mathbb{Q}$.

Part B

Each question carries 10 marks. Try to answer 5 questions.

1. Suppose $f : \mathbb{R} \mapsto \mathbb{R}^n$ be a differentiable mapping satisfying $\|f(t)\| = 1$ for all $t \in \mathbb{R}$. Show that $\langle f'(t), f(t)\rangle = 0$ for all $t \in \mathbb{R}$. (Here $\|.\|$ denotes standard norm or length of a vector in $\mathbb{R}^n$, and $\langle ., .\rangle$ denotes the standard inner product (or scalar product) in $\mathbb{R}^n$.)

2. Let $A, B \subset \mathbb{R}^n$ and define $A + B = \{a + b;\ a \in A,\ b \in B\}$. If A and B are open, is $A + B$ open? If A and B are closed, is $A + B$ closed? Justify your answers.

3. Let $f : X \mapsto Y$ be continuous map onto Y, and let X be compact. Also $g : Y \mapsto Z$ is such that $g \circ f$ is continuous. Show g is continuous.

4. Let A be a $n \times m$ matrix with real entries, and let $B = AA^t$ and let α be the supremum of $x^t Bx$ where supremum is taken over all vectors $x \in \mathbb{R}^n$ with norm less than or equal to 1. Consider
$$C_k = I + \sum_{j=1}^{k} B^j.$$
Show that the sequence of matrices C_k converges if and only if $\alpha < 1$.

5. Show that a power series $\sum_{n\geq 0} a_n z^n$ where $a_n \to 0$ as $n \to \infty$ cannot have a pole on the unit circle. Is the statement true with the hypothesis that (a_n) is a bounded sequence?

6. Show that a biholomorphic map of the unit ball onto itself which fixes the origin is necessarily a rotation.

7. (i) Let $G = GL(2, \mathbb{F}_p)$. Prove that there is a Sylow $p-$subgroup H of G whose normalizer $N_G(H)$ is the group of all upper triangular matrices in G.

   (ii) Hence prove that the number of Sylow subgroups of G is $1 + p$.

8. Calculate the minimal polynomial of $\sqrt{2}e^{\frac{2\pi i}{3}}$ over $\mathbb{Q}$.

9. Let G be a group $\mathbb{F}$ a field and n a positive integer. A *linear action* of G on $\mathbb{F}^n$ is a map $\alpha : G \times \mathbb{F}^n \to \mathbb{F}^n$ such that $\alpha(g, v) = \rho(g)v$ for some group homomorphism $\rho : G \to \mathrm{GL}_n(\mathbb{F})$. Show that for every finite group G, there is an n such that there is a linear action α of G on $\mathbb{F}^n$ and such that there is a nonzero vector $v \in \mathbb{F}^n$ such that $\alpha(g, v) = v$ for all $g \in G$.

10. Let R be an integral domain containing a field F as a subring. Show that if R is a finite-dimensional vector space over F, then R is a field.

Chennai Mathematical Institute

MSc/PhD Entrance Examination, 2013

15th May 2013

Problems in Part A will be used for screening purposes. Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. However, the scores in both the sections will be taken into account to decide whether you qualify for the interview.

Notation: $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ stand, respectively, for the sets of integers, of the real numbers, and of the complex numbers.

Part A

This section consists of <u>fifteen</u> (15) multiple-choice questions, each with one or more correct answers. Record your answers on the attached bubble-sheet by filling in the appropriate circles. Every question is worth <u>four</u> (4) marks. A solution receives credit if and only if <u>all the correct answers</u> are chosen, and <u>no incorrect answer</u> is chosen.

1. Pick the correct statement(s) below.
   (a) There exists a group of order 44 with a subgroup isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.
   (b) There exists a group of order 44 with a subgroup isomorphic to $\mathbb{Z}/4$.
   (c) There exists a group of order 44 with a subgroup isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ and a subgroup isomorphic to $\mathbb{Z}/4$.
   (d) There exists a group of order 44 without any subgroup isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ or to $\mathbb{Z}/4$.

2. Let G be group. The following statements hold.
   (a) If G has nontrivial centre C, then G/C has trivial centre.
   (b) If $G \neq 1$, there exists a nontrivial homomorphism $h : \mathbb{Z} \to G$.
   (c) If $|G| = p^3$, for p a prime, then G is abelian.
   (d) If G is nonabelian, then it has a nontrivial automorphism.

3. Let $C[0, 1]$ be the space of continuous real-valued functions on the interval $[0, 1]$. This is a ring under point-wise addition and multiplication. The following are true.
   (a) For any $x \in [0, 1]$, the ideal $M(x) = \{f \in C[0, 1] \mid f(x) = 0\}$ is maximal.
   (b) $C[0, 1]$ is an integral domain.
   (c) The group of units of $C[0, 1]$ is cyclic.
   (d) The linear functions form a vector-space basis of $C[0, 1]$ over $\mathbb{R}$.

4. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with eigenvalues $\frac{2}{3}$ and $\frac{9}{5}$. Then, there exists a non-zero vector $v \in \mathbb{R}^2$ such that
   (a) $\|Av\| > 2\|v\|$;
   (b) $\|Av\| < \frac{1}{2}\|v\|$;
   (c) $\|Av\| = \|v\|$;
   (d) $Av = 0$;

5. Let F be a field with 256 elements, and $f \in F[x]$ a polynomial with all its roots in F. Then,
   (a) $f \neq x^{15} - 1$;
   (b) $f \neq x^{63} - 1$;
   (c) $f \neq x^2 + x + 1$;
   (d) if f has no multiple roots, then f is a factor of $x^{256} - x$.

6. Let $h : \mathbb{C} \to \mathbb{C}$ be an analytic function such that $h(0) = 0$; $h(\frac{1}{2}) = 5$, and $|h(z)| < 10$ for $|z| < 1$. Then,
   (a) the set $\{z : |h(z)| = 5\}$ is unbounded by the Maximum Principle;
   (b) the set $\{z : |h'(z)| = 5\}$ is a circle of strictly positive radius;
   (c) $h(1) = 10$;
   (d) regardless of what h' is, $h'' \equiv 0$.

7. Suppose that $f(z)$ is analytic, and satisfies the condition $|f(z)^2 - 1| = |f(z) - 1| \cdot |f(z) + 1| < 1$ on a non-empty connected open set U. Then,
   (a) f is constant.
   (b) The imaginary part of f, $Im(f)$, is positive on U.
   (c) The real part of f, $Re(f)$, is non-zero on U.
   (d) $Re(f)$ is of fixed sign on U.

8. Consider the following subsets of $\mathbb{R}^2$: $X_1 = \{(x, \sin\frac{1}{x}) | 0 < x < 1\}$, $X_2 = [0, 1] \times \{0\}$, and $X_3 = \{(0, 1)\}$. Then,
   (a) $X_1 \cup X_2 \cup X_3$ is a connected set;
   (b) $X_1 \cup X_2 \cup X_3$ is a path-connected set;
   (c) $X_1 \cup X_2 \cup X_3$ is not path-connected, but $X_1 \cup X_2$ is path-connected;
   (d) $X_1 \cup X_2$ is not path-connected, but every open neighbourhood of a point in this set contains a smaller open neighbourhood which is path-connected.

9. For a set $A \subset \mathbb{R}$, denote by $Cl(A)$ the *closure* of A, and by $Int(A)$ the *interior* of A. There is a set $A \subset \mathbb{R}$ such that
   (a) A, $Cl(A)$, and $Int(A)$ are pairwise distinct;
   (b) A, $Cl(A)$, $Int(A)$, and $Cl(Int(A))$ are pairwise distinct;
   (c) A, $Cl(A)$, $Int(A)$, and $Int(Cl(A))$ are pairwise distinct;
   (d) A, $Cl(A)$, $Int(A)$, $Int(Cl(A))$, and $Cl(Int(A))$ are pairwise distinct.

10. Let $f, g : [0,1] \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational;} \end{cases}$$

$$g(x) := \begin{cases} 1/q & \text{if } x = \frac{p}{q} \text{ is rational, with } gcd(p,q) = 1, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Then,

(a) g is Riemann integrable, but not f;

(b) both f and g are Riemann integrable;

(c) the Riemann integral $\int_0^1 f(x)dx = 0$;

(d) the Riemann integral $\int_0^1 g(x)dx = 0$.

11. Let C be the ellipse $24x^2 + xy + 5y^2 + 3x + 2y + 1 = 0$. Then, the line integral $\oint(x^2ydy + xy^2dx)$

(a) lies in $(0,1)$;

(b) is 1;

(c) is either 1 or -1 depending on whether C is traversed clockwise or counterclockwise;

(d) is 0.

12. The series $\sum_{n=1}^{\infty} a_n$ where $a_n = (-1)^{n+1}n^4e^{-n^2}$

(a) has unbounded partial sums;

(b) is absolutely convergent;

(c) is convergent but not absolutely convergent;

(d) is not convergent, but partial sums oscillate between -1 and $+1$.

13. Let f be continuously differentiable on $\mathbb{R}$. Let $f_n(x) = n\left(f(x+\frac{1}{n}) - f(x)\right)$. Then,

(a) f_n converges uniformly on $\mathbb{R}$;

(b) f_n converges on $\mathbb{R}$, but not necessarily uniformly;

(c) f_n converges to the derivative of f uniformly on $[0,1]$;

(d) there is no guarantee that f_n converges on any open interval.

14. Let $f : X \to Y$ be a nonconstant continuous map of topological spaces. Which of the following statements are true?

(a) If $Y = \mathbb{R}$ and X is connected then X is uncountable.

(b) If X is Hausdorff then $f(X)$ is Hausdorff.

(c) If X is compact then $f(X)$ is compact.

(d) If X is connected then $f(X)$ is connected.

15. Let X be a set with the property that for any two metrics d_1, and d_2 on X, the identity map

$$id : (X, d_1) \to (X, d_2)$$

is continuous. Which of the following are true?

(a) X must be a singleton.

(b) X can be any finite set.

(c) X cannot be infinite.

(d) X may be infinite but not uncountable.

Part B

Solve six (6) problems from below, **clearly indicating** which problems you would like us to mark. Every problem is worth ten (10) marks. Justify all your arguments to receive credit.

1. Let G be a finite group, p the smallest prime divisor of $|G|$, and $x \in G$ an element of order p. Suppose $h \in G$ is such that $hxh^{-1} = x^{10}$. Show that $p = 3$.

2. (a) Show that there exists a 3×3 invertible matrix $M \neq I_3$ with entries in the field $\mathbb{F}_2$ such that $M^7 = I_3$.

   (b) Let A be an $m \times n$ matrix, and $\mathbf{b}$ an $m \times 1$ vector, both with integer entries.

      1. Suppose that there exists a prime number p such that the equation $A\mathbf{x} = \mathbf{b}$ seen as an equation over the finite field $\mathbb{F}_p$ has a solution. Then does there exist a solution to $A\mathbf{x} = \mathbf{b}$ over the real numbers?
      2. If $A\mathbf{x} = \mathbf{b}$ has a solution over $\mathbb{F}_p$ for every prime p, is a real solution guaranteed?

3. Let $M_n(\mathbb{C})$ denote the set of $n \times n$ matrices over $\mathbb{C}$. Think of $M_n(\mathbb{C})$ as the $2n^2$-dimensional Euclidean space $\mathbb{R}^{2n^2}$. Show that the set of all diagonalizable $n \times n$ matrices is dense in $M_n(\mathbb{C})$.

4. Compute the integral
$$\int_{-\infty}^{\infty} \frac{x}{(x^2 + 2x + 2)(x^2 + 4)} dx.$$

5. Show that there does not exists an analytic function f defined in open unit disk for which $f(\frac{1}{n})$ is 2^{-n}.

6. Let f be a real valued continuous function on $[0, 2]$ which is differentiable at every point except possibly at $x = 1$. Suppose that $\lim_{x \to 1} f'(x) = 2013$. Show that f is differentiable at x.

7. (a) Show that there exists no bijective map $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that f and f^{-1} are differentiable.

   (b) Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a differentiable map such that the derivative $Df(x)$ is surjective for all x. Is f surjective?

8. (a) Let $f \in \mathbb{Z}[x]$ be a non-constant polynomial with integer coefficients. Show that as a varies over the integers, the set of divisors of $f(a)$ includes infinitely many different primes.

   (b) Assume known the following result: If G is a finite group of order n such that for integer $d > 0$, $d|n$, there is no more than one subgroup of G of order d, then G is cyclic. Using this (or otherwise) prove that the multiplicative group of units in any finite field is cyclic.

9. Let $K_1 \supset K_2 \supset \ldots$ be a sequence of connected compact subsets of $\mathbb{R}^2$. Is it true that their intersection $K = \cap_{i=1}^{\infty} K_i$ is connected also? Provide either a proof or a counterexample.

10. Let A be a subset of $\mathbb{R}^2$ with the property that every continuous function $f : A \to \mathbb{R}$ has a maximum in A. Prove that A is compact.

CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
15 May 2014

Instructions:

- Enter your *Registration Number* here **CMIPG ID–** ☐☐☐☐☐☐
  OR here **M–** ☐☐☐☐
- Enter the name of the city where you write this test: ☐☐☐☐☐☐☐☐☐☐☐☐

- The allowed time is 3 hours.
- This examination has two parts. You may use the blank pages at the end for your rough-work.
- Part A is worth 40 marks and will be used for screening. Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. However, the scores in both the sections will be taken into account while making the final decision. **Record your answers to Part A in the attached bubble-sheet.**
- Part B is worth 60 marks. You should answer six (6) questions in Part B. If you are applying for the PhD Mathematics programme (possibly in addition to other programmes), at least two (2) should be from among the starred questions (17*)–(20*). **Indicate the six questions to be marked in the boxes in the bubble-sheet. Write your solutions to Part B in the page assigned to each question.**
- **Please read the further instructions given before Part A and inside each part carefully.**

For office use only

Part B

No.	Marks	Remarks
11		
12		
13		
14		
15		
16		

Part B (ctd.)

No.	Marks	Remarks
17*		
18*		
19*		
20*		

Further remarks:

Part A	
Part B	
Total	

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CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
15 May 2014

- *Registration Number*: **CMIPG ID–** ☐☐☐☐☐☐ OR **M–** ☐☐☐☐
- City where you write this test: ☐☐☐☐☐☐☐☐☐☐☐☐

Bubble Sheet

Part A

1. Ⓐ Ⓑ Ⓒ Ⓓ
2. Ⓐ Ⓑ Ⓒ Ⓓ
3. Ⓐ Ⓑ Ⓒ Ⓓ
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Part B

In the boxes below, clearly indicate the SIX (6) solutions that should be marked. If you are applying for the PhD Mathematics programme (possibly in addition to other programmes) then you should answer at least two (2) from among the starred questions (17*)–(20*).

For office use only:

Number of correct answers in Part A:	
Marks in Part A:	

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CHENNAI MATHEMATICAL INSTITUTE

Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
15 May 2014

Important: Questions in Part A will be used for **screening**. Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. However, the scores in both the sections will be taken into account while making the final decision.

For qualifying for the PhD interview, you should answer at least two (2) from among the starred questions (17*)–(20*) in Part B.

Notation: $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ stand, respectively, for the sets of the natural numbers, of the rational numbers, of the real numbers, and of the complex numbers. For a field F, $M_{m\times n}(F)$ stands for the set of $m \times n$ matrices over F. We treat $M_{m\times n}(\mathbb{R})$ and $M_{m\times n}(\mathbb{C})$ as metric spaces with the metric $d(A, B) = \sqrt{\sum_{i,j} |a_{ij} - b_{ij}|^2}$ where $A = (a_{ij})$ and $B = (b_{ij})$.

Part A

Instructions: Each of the questions 1–8 has one or more correct answers. Record your answers on the attached bubble-sheet by filling in the appropriate circles. Every question is worth four (4) marks. A solution receives credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x + 1) = f(x)$ for all $x \in \mathbb{R}$. Which of the following statement(s) is/are true?
(A) f is bounded.
(B) f is bounded if it is continuous.
(C) f is differentiable if it is continuous.
(D) f is uniformly continuous if it is continuous.

(2) Let $W \subset \mathbb{R}^n$ be a linear subspace of dimension at most $n - 1$. Which of the following statement(s) is/are true?
(A) W is nowhere dense.
(B) W is closed.
(C) $\mathbb{R}^n \setminus W$ is connected.
(D) $\mathbb{R}^n \setminus W$ is not connected.

(3) Let G be a finite group. An element $a \in G$ is called a *square* if there exists $x \in G$ such that $x^2 = a$. Which of the following statement(s) is/are true?
(A) If $a, b \in G$ are not squares, ab is a square.
(B) Suppose that G is cyclic. Then if $a, b \in G$ are not squares, ab is a square.
(C) G has a normal subgroup.
(D) If every proper subgroup of G is cyclic then G is cyclic.

(4) Let $A \in M_{m\times n}(\mathbb{R})$ and let $b_0 \in \mathbb{R}^m$. Suppose the system of equations $Ax = b_0$ has a unique solution. Which of the following statement(s) is/are true?
(A) $Ax = b$ has a solution for every $b \in \mathbb{R}^m$.
(B) If $Ax = b$ has a solution then it is unique.
(C) $Ax = 0$ has a unique solution.
(D) A has rank m.

(5) Let $A \in M_{n\times n}(\mathbb{C})$. Which of the following statement(s) is/are true?
(A) There exists $B \in M_{n\times n}(\mathbb{C})$ such that $B^2 = A$.
(B) A is diagonalizable.
(C) There exists an invertible matrix P such that PAP^{-1} is upper-triangular.
(D) A has an eigenvalue.

(6) Let $f : \mathbb{C} \to \mathbb{C}$ be a function. Which of the following statement(s) is/are true?
(A) Consider f as a function $(f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$. Suppose that for $i = 1, 2$, both $\frac{\partial f_i}{\partial X}$ and $\frac{\partial f_i}{\partial Y}$ exist and are continuous. Then f is entire.
(B) Assume that f is entire and $|f(z)| < 1$ for all $z \in \mathbb{C}$. Then f is constant.
(C) Assume that f is entire and $\mathrm{Im}(f(z)) > 0$ for all $z \in \mathbb{C}$. Then f is constant.

(7) Let $\mathcal{C}(\mathbb{R})$ be the $\mathbb{R}$-vector space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Let a_1, a_2, a_3 be distinct real numbers. For $i = 1, 2, 3$, let $f_i \in \mathcal{C}(\mathbb{R})$ be the function $f_i(t) = e^{a_i t}$. Which of the following statement(s) is/are true?
(A) f_1, f_2 and f_3 are linearly independent.
(B) f_1, f_2 and f_3 are linearly dependent.
(C) f_1, f_2 and f_3 form a basis of $\mathcal{C}(\mathbb{R})$.

(8) Which of the following statement(s) is/are true?
(A) The series $\sum\limits_{n=1}^{\infty} \mathrm{e}^{-n^2}$ converges.
(B) The series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.
(C) The series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges absolutely.
(D) The series $\sum\limits_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ converges uniformly on $\mathbb{R}$.

Instructions: The answers to questions 9 and 10 are integers. You are required to write the answers in decimal form in the attached bubble-sheet. Every question is worth four (4) marks.

(9) What is the dimension of the ring $\mathbb{Q}[x]/((x + 1)^2)$ as a $\mathbb{Q}$-vector space?

(10) Evaluate $\lim\limits_{n\to\infty} \left[\frac{\pi \sum_{i=1}^{n} \sin\left(\frac{i\pi}{n}\right)}{n}\right]$.

Part B

Instructions: Answer six (6) questions from below. Provide sufficient justification. Write your solutions on the page assigned to each question. Each of the questions is worth ten (10) marks. **For qualifying for the PhD interview, you should answer at least two (2) from among starred questions (17*)–(20*).** Clearly indicate which six questions you would like us to mark in the six boxes in the bubble sheet. If the boxes are unfilled, we will mark the first six solutions that appear in your answer-sheet. If you do not want a solution to be considered, clearly strike it out.

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(11) Show that the set of rank two matrices in $M_{2\times 3}(\mathbb{R})$ is open.

(12) (A) Let F be a finite field extension of $\mathbb{Q}$. Show that any field homomorphism $\phi : F \to F$ is an isomorphism. (Note that $\phi(1) = 1$ by definition.)

(B) Let F be a finite field whose characteristic is not 2. Let $F^\times$ denote the multiplicative group of nonzero elements of F. An element $a \in F^\times$ is called a *square* if there exists $x \in F^\times$ such that $x^2 = a$. Show that exactly half the elements $F^\times$ are squares.

(13) Let $n \in \mathbb{N}$. Show that the determinant map $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ is infinitely differentiable and compute the total derivative $d(\det)$ at every point $A \in M_{n\times n}(\mathbb{R})$. Find a necessary and sufficient condition on the rank of A for $d(\det) = 0$ at A.

(14) Let $a_i, i \in \mathbb{R}$ be non-negative real numbers such that $\sup\left\{ \sum_{i\in F} a_i \mid F \subseteq \mathbb{R} \text{ a finite subset}\right\}$ is finite. Show that $a_i = 0$ except for countably many $i \in \mathbb{R}$. Give an example to show that 'countably' cannot be replaced by 'finite'. (Hint: consider $F_n := \{i \mid a_i \geq \frac{1}{n}\}$.)

(15) Let G be a finite group of order $2n$ for some integer n. Consider the map $\phi : G \to G$ given by $\phi(a) = a^2$. Show that ϕ is not surjective.

(16) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.
(A) Construct a sequence $\{z_n\}$ in $\mathbb{C}$ such that $|z_n| \to \infty$ and $e^{z_n} \to 1$.
(B) Show that the function $g(z) = f(e^z)$ is not a polynomial.

(17*) For $F = \mathbb{R}$ and $F = \mathbb{C}$, let $O_n(F) = \{A \in M_{n\times n}(F) \mid AA^t = I_n\}$.
(A) Show that $O_n(\mathbb{R})$ is compact.
(B) Is $O_n(\mathbb{R})$ connected? Justify.
(C) Is $O_n(\mathbb{C})$ compact? Justify.

(18*) Let Ω be a region in $\mathbb{C}$. Let $\{a_n\}$ be a sequence of nonzero elements in Ω such that $a_n \to 0$ as $n \to \infty$. Let $\{b_n\}$ be a sequence of complex numbers such that $\lim_{n\to\infty} \frac{b_n}{a_n^k} = 0$ for every nonnegative integer k. Suppose that $f : \Omega \to \mathbb{C}$ is an entire function such that $f(a_n) = b_n$ for all n. Show that $b_n = 0$ for every n.

(19*) Let G be a finite group of order n and let H be a subgroup of G of order m. Assume that $(\frac{n}{m})! < 2n$. Show that G is not simple, that is: G has a nontrivial proper normal subgroup. (Hint: Think along the lines of Cayley's theorem.)

(20*) Let

$$\mathcal{C}_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}, \lim_{x\to\infty} |f(x)| = 0 \text{ and } \lim_{x\to -\infty} |f(x)| = 0\}, \text{ and}$$

$$\mathcal{C}_0^\infty(\mathbb{R}) = \{f \in \mathcal{C}_0(\mathbb{R}) \mid f \text{ is infinitely differentiable}\}.$$

Let $\phi \in \mathcal{C}_0(\mathbb{R})$. For $f \in \mathcal{C}_0(\mathbb{R})$, define $\phi^*(f) = f \circ \phi$.
(A) Show that $\phi^*(f) \in \mathcal{C}_0(\mathbb{R})$ if $f \in \mathcal{C}_0(\mathbb{R})$.
(B) If $\phi^*(\mathcal{C}_0^\infty(\mathbb{R})) \subseteq \mathcal{C}_0^\infty(\mathbb{R})$, then show that ϕ is infinitely differentiable.

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (11)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (12)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (13)

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Solution to Question (14)

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Solution to Question (15)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (16)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (17*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (18*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (19*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (20*)

CHENNAI MATHEMATICAL INSTITUTE

Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
18 May 2015

Instructions:

- Enter your *Registration Number* here: **CMI–PG–** | | | | | | | | |
- Enter the name of the city where you write this test: | | | | | | | | | | | | |

- The allowed time is 3 hours.
- This examination has two parts. You may use the blank pages at the end for your rough-work.
- Part A is worth 40 marks and will be used for screening. There will be a cut-off for Part A, which will not be more than twenty (20) marks (out of 40). Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.) However, note that the scores in both the sections will be taken into account while making the final decision. **Record your answers to Part A in the attached bubble-sheet.**
- Part B is worth 60 marks. You should answer six (6) questions in Part B. In order to qualify the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions (17*)–(20*). **Indicate the six questions to be marked in the boxes in the bubble-sheet. Write your solutions to Part B in the page assigned to each question.**
- **Please read the further instructions given before Part A and inside each part carefully.**

For office use only

Part B

No.	Marks	Remarks
11		
12		
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Part B (ctd.)

No.	Marks	Remarks
17*		
18*		
19*		
20*		

Part A	
Part B	
Total	

Further remarks:

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CHENNAI MATHEMATICAL INSTITUTE
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18 May 2015

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Bubble Sheet

Part A

1. Ⓐ Ⓑ Ⓒ Ⓓ
2. Ⓐ Ⓑ Ⓒ
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8. ________
9. ________
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Part B

In the boxes below, clearly indicate the SIX (6) solutions that should be marked. In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions (17*)–(20*).

For office use only:

Number of correct answers in Part A:	
Marks in Part A:	

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Important: Questions in Part A will be used for **screening**. There will be a cut-off for Part A, which will not be more than twenty (20) marks (out of 40). Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.) However, note that the scores in both the sections will be taken into account while making the final decision. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions** (17*)–(20*)**.**

Notation: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ stand, respectively, for the sets of the natural numbers, of the integers, of the rational numbers, of the real numbers, and of the complex numbers. For a prime number p, $\mathbb{F}_p$ is the field with p elements. For a field F, $M_{m\times n}(F)$ stands for the set of $m \times n$ matrices over F and $\mathrm{GL}_n(F)$ is the set of invertible $n \times n$ matrices over F. If $F = \mathbb{R}$ or $F = \mathbb{C}$, we treat these sets as metric spaces with the metric $d(A, B) = \sqrt{\sum_{i,j} |a_{ij} - b_{ij}|^2}$ where $A = (a_{ij})$ and $B = (b_{ij})$.

Part A

Instructions: Each of the questions 1–7 has one or more correct answers. Record your answers on the attached bubble-sheet by filling in the appropriate circles. Every question is worth four (4) marks. A solution receives credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

(1) Which of the following topological spaces is/are connected?
(A) $\mathrm{GL}_1(\mathbb{R})$
(B) $\mathrm{GL}_1(\mathbb{C})$
(C) $\mathrm{GL}_2(\mathbb{R})$
(D) $\left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} : x, y \in \mathbb{R}, x^2 + y^2 = 1 \right\}$

(2) Consider $f : \{z \in \mathbb{C} : |z| > 1\} \longrightarrow \mathbb{C}, f(z) = \frac{1}{z}$. Choose the correct statement(s):
(A) There are infinitely many entire functions g such that $g(z) = f(z)$ for every $z \in \mathbb{C}$ with $|z| > 1$.
(B) There does not exist an entire function g such that $g(z) = f(z)$ for every $z \in \mathbb{C}$ with $|z| > 1$.
(C) $g : \mathbb{C} \longrightarrow \mathbb{C}$ with
$$g(z) = \begin{cases} 1 - \frac{1}{2}z^2, & |z| \le 1 \\ \frac{1}{z}, & |z| > 1 \end{cases}$$
is an entire function such that $g(z) = f(z)$ for every $z \in \mathbb{C}$ with $|z| > 1$.

(3) Let
$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a > 0 \right\}, \quad N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}.$$
Which of the following are true?
(A) G/N is isomorphic to $\mathbb{R}$ under addition.
(B) G/N is isomorphic to $\{a \in \mathbb{R} : a > 0\}$ under multiplication.
(C) There is a proper normal subgroup N' of G which properly contains N.
(D) N is isomorphic to $\mathbb{R}$ under addition.

(4) Choose the correct statement(s):
(A) There is a continuous surjective function from $[0, 1)$ to $\mathbb{R}$;
(B) $\mathbb{R}$ and $[0, 1)$ are homeomorphic to each other;
(C) There is a bijective function from $[0, 1)$ to $\mathbb{R}$;
(D) Bounded subspaces of $\mathbb{R}$ cannot be homeomorphic to $\mathbb{R}$.

(5) Which of the following complex numbers has/have a prime number as the degree of its minimal polynomial over $\mathbb{Q}$?
(A) ζ_7, a primitive 7th root of unity;
(B) $\sqrt{2} + \sqrt{3}$;
(C) $\sqrt{-1}$;
(D) $\sqrt[3]{2}$.

(6) Let R be an integral domain such that every non-zero prime ideal of $R[X]$ (where X is an indeterminate) is maximal. Choose the correct statement(s):
(A) R is a field;
(B) R contains $\mathbb{Z}$ as a subring;
(C) Every ideal in $R[X]$ is principal;
(D) R contains $\mathbb{F}_p$ as a subring for some prime number p.

(7) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be such that $\int_{-\infty}^{\infty} |f(x)|\mathrm{d}x < \infty$. Define $F : \mathbb{R} \longrightarrow \mathbb{R}$ by $F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t$. Choose the correct statement(s):
(A) f is continuous;
(B) F is continuous;
(C) F is uniformly continuous;
(D) There exists a positive real number M such that $|f(x)| < M$ for all $x \in \mathbb{R}$.

Instructions: The answers to questions 8–10 are integers. You are required to write the answers in decimal form in the attached bubble-sheet. Every question is worth <u>four</u> (4) marks.

(8) Let $\omega \in \mathbb{C}$ be a primitive third root of unity. How many distinct possible images of ω are there under all the field homomorphisms $\mathbb{Q}(\omega) \longrightarrow \mathbb{C}$.

(9) Let $C := \{z \in \mathbb{C} : |z| = 5\}$. What is value of M such that

$$2\pi\imath M = \int_C \frac{1}{z^2 - 5z + 6}\mathrm{d}z?$$

(10) Consider the set $\mathbb{R}[X]$ of polynomials in X with real coefficients as a real vector space. Let T be the $\mathbb{R}$-linear operator on $\mathbb{R}[X]$ given by

$$T(f) = \frac{\mathrm{d}^2 f}{\mathrm{d}X^2} - \frac{\mathrm{d}f}{\mathrm{d}X} + f.$$

What is the nullity of f?

Part B

Instructions: Answer six (6) questions from below. Provide sufficient justification. Write your solutions on the page assigned to each question. Each of the questions is worth ten (10) marks. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions** (17*)–(20*)**.** Clearly indicate which six questions you would like us to mark in the six boxes in the bubble sheet. If the boxes are unfilled, we will mark the first six solutions that appear in your answer-sheet. If you do not want a solution to be considered, clearly strike it out.

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(11) Let $f \in \mathbb{R}[x, y]$ be such that there exists a non-empty open set $U \subseteq \mathbb{R}^2$ such that $f(x, y) = 0$ for every $(x, y) \in U$. Show that $f = 0$.

(12) Let $A \in M_{n\times n}(\mathbb{C})$.
(a) Suppose that $A^2 = 0$. Show that λ is an eigenvalue of $(I_n + A)$ if and only if $\lambda = 1$. (I_n is the $n \times n$ identity matrix.)
(b) Suppose that $A^2 = -1$. Determine (with proof) whether A is diagonalizable.

(13) Let f be a non-constant entire function satisfying the following conditions:
(a) $f(0) = 0$;
(b) For every positive real number M, the set $\{z : |f(z) < M\}$ is connected.
Prove that $f(z) = cz^n$ for some constant c and positive integer n.

(14) Let $(a_{mn})_{m\geq 1, n\geq 1}$ be a double sequence of real numbers such that
(a) For every n, $b_n := \lim_{m\to\infty} a_{mn}$ exists;
(b) For all strictly increasing sequences $(m_k)_{k\geq 1}$ and $(n_k)_{k\geq 1}$ of positive integers, $\lim_{k\to\infty} a_{m_k n_k} = 1$.
Show that the sequence $(b_n)_{n\geq 1}$ converges to 1.

(15) Let $f \in \mathbb{C}[x, y]$ be such that $f(x, y) = f(y, x)$. Show that there is a $g \in \mathbb{C}[x, y]$ such that $f(x, y) = g(x + y, xy)$.

(16) Let X be a topological space and $f : X \longrightarrow [0, 1]$ be a closed continuous surjective map such that $f^{-1}(a)$ is compact for every $0 \leq a \leq 1$. Prove or disprove: X is compact. (A map is said to be *closed* if it takes closed sets to closed sets.)

(17*) Determine the cardinality of set of subrings of $\mathbb{Q}$. (Hint: For a set P of positive prime numbers, consider the smallest subring of $\mathbb{Q}$ that contains $\{\frac{1}{p} \mid p \in P\}$.)

(18*) Let

$$f(x) = \sum_{n\geq 1} \frac{\sin(\frac{x}{n})}{n}.$$

Show that f is continuous. Determine (with justification) whether f differentiable.

(19*) Let m and n be positive integers and p a prime number. Let $G \subseteq \mathrm{GL}_m(\mathbb{F}_p)$ be a subgroup of order p^n. Let $U \subseteq \mathrm{GL}_m(\mathbb{F}_p)$ be the subgroup that consists of all the matrices with 1's on the diagonal and 0's below the diagonal. Show that there exists $A \in \mathrm{GL}_m(\mathbb{F}_p)$ such that $AGA^{-1} \subseteq U$.

(20*) Let m and n be positive integers and $0 \leq k \leq \min\{m, n\}$ an integer. Prove or disprove: The subspace of $M_{m\times n}(\mathbb{C})$ consisting of all matrices of rank equal to k is connected. (You may use the following fact: For $t \geq 2$, $\mathrm{GL}_t(\mathbb{C})$ is connected.)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (11)

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Solution to Question (12)

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Solution to Question (20*)

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Part B

No.	Marks	Remarks
11		
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Part B (ctd.)

No.	Marks	Remarks
17*		
18*		
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Further remarks:

Part A	
Part B	
Total	

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Bubble Sheet

Part A

1. Ⓐ Ⓑ Ⓒ Ⓓ
2. Ⓐ Ⓑ Ⓒ Ⓓ
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9. ____________
10. ____________

Part B

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MSc/PhD Entrance Examination
18 May 2016

Important: Questions in Part A will be used for **screening**. There will be a cut-off for Part A, which will not be more than twenty (20) marks (out of 40). Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.) However, note that the scores in both the sections will be taken into account while making the final decision. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions** (17*)–(20*).

Notation: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ stand, respectively, for the sets of natural numbers, of integers, of rational numbers, of real numbers, and of complex numbers. For a prime number p, $\mathbb{F}_p$ is the field with p elements. For a field F, $M_{m\times n}(F)$ stands for the set of $m \times n$ matrices over F and $\mathrm{GL}_n(F)$ is the set of invertible $n \times n$ matrices over F. The symbol $\imath$ denotes a square-root of -1.

Part A

Instructions: Each of the questions 1–8 has one or more correct answers. Record your answers on the attached bubble-sheet by filling in the appropriate circles. Every question is worth four (4) marks. A solution receives credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

(1) We say that two subsets X and Y of $\mathbb{R}$ are *order-isomorphic* if there is a bijective map $\phi : X \longrightarrow Y$ such that for every $x_1 \leq x_2 \in X$, $\phi(x_1) \leq \phi(x_2)$, where '$\leq$' denotes the usual order on $\mathbb{R}$. Choose the correct statement(s) from below:
(A) $\mathbb{N}$ and $\mathbb{Z}$ are not order-isomorphic;
(B) $\mathbb{N}$ and $\mathbb{Q}$ are order-isomorphic;
(C) $\mathbb{Z}$ and $\mathbb{Q}$ are order-isomorphic;
(D) The sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are pairwise non-order-isomorphic.

(2) Let $x_n = \left(1 - \frac{1}{n}\right) \sin \frac{n\pi}{3}$, $n \geq 1$. Write $l = \liminf x_n$ and $s = \limsup x_n$. Choose the correct statement(s) from below:
(A) $-\frac{\sqrt{3}}{2} \leq l < s \leq \frac{\sqrt{3}}{2}$;
(B) $-\frac{1}{2} \leq l < s \leq \frac{1}{2}$;
(C) $l = -1$ and $s = 1$;
(D) $l = s = 0$.

(3) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as
$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}), & \text{if } x \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$
Choose the correct statement(s) from below:
(A) f is continuous;
(B) f is discontinuous at 0;
(C) f is differentiable;
(D) f is continuously differentiable.

(4) Let $A \in M_{m\times n}(\mathbb{R})$ be of rank m. Choose the correct statement(s) from below:
 (A) The map $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ given by $v \mapsto Av$ is injective;
 (B) There exist matrices $B \in M_m(\mathbb{R})$ and $C \in M_n(\mathbb{R})$ such that $BAC = \left[I_m \mid \mathbf{0}_{n-m}\right]$;
 (C) There exist matrices $B \in \mathrm{GL}_m(\mathbb{R})$ and $C \in \mathrm{GL}_n(\mathbb{R})$ such that $BAC = \left[I_m \mid \mathbf{0}_{n-m}\right]$;
 (D) For every $(B, C) \in M_m(\mathbb{R}) \times M_n(\mathbb{R})$ such that $BAC = \left[I_m \mid \mathbf{0}_{n-m}\right]$, C is uniquely determined by B.

(5) Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function such that $f(z+1) = f(z+\imath) = f(z)$ for every $z \in \mathbb{C}$. Choose the correct statement(s) from below:
 (A) f is constant;
 (B) $f(z) = 0$ for every $z \in \mathbb{C}$;
 (C) There exist complex numbers a, b such that for every $x, y \in \mathbb{R}$, $f(x+\imath y) = a\sin(x) + \imath b\cos(y)$;
 (D) f is not necessarily constant but $|f(z)|$ is constant.

(6) What is the cardinality of the centre of $O_2(\mathbb{R})$? (Definition: The *centre* of a group G is $\{g \in G \mid gh = hg \text{ for every } h \in G\}$. Hint: Reflection matrices and permutation matrices are orthogonal.)
 (A) 1;
 (B) 2;
 (C) The cardinality of $\mathbb{N}$;
 (D) The cardinality of $\mathbb{R}$.

(7) Let $U \subseteq \mathbb{R}$ be a non-empty open subset. Choose the correct statement(s) from below:
 (A) U is uncountable;
 (B) U contains a closed interval as a proper subset;
 (C) U is a countable union of disjoint open intervals;
 (D) U contains a convergent sequence of real numbers.

(8) Let R be a commutative ring. The *characteristic* of R is the smallest positive integer n such that $a + a + \cdots + a$ (n times) is zero for every $a \in R$, if such an integer exists, and zero, otherwise. Choose the correct statement(s) from below:
 (A) For every $n \in \mathbb{N}$, there exists a commutative ring whose characteristic is n;
 (B) There exists a integral domain with characteristic 57;
 (C) The characteristic of a field is either 0 or a prime number;
 (D) For every prime number p, every commutative ring of characteristic p contains $\mathbb{F}_p$ as a subring.

Instructions: The answers to questions 9–10 are integers. You are required to write the answers in decimal form in the attached bubble-sheet. Every question is worth <u>four</u> (4) marks.

(9) Consider the $\mathbb{Q}$-vector-space

$$\{f : \mathbb{R} \longrightarrow \mathbb{R} \mid f \text{ is continuous and } \mathrm{Image}(f) \subseteq \mathbb{Q}\}.$$

What is its dimension?

(10) Let p be a prime number and F a field of p^{23} elements. Let $\phi : F \longrightarrow F$ be the field automorphism of F sending a to a^p. Let $K := \{a \in F \mid \phi(a) = a\}$. What is the value of $[K : \mathbb{F}_p]$?

Part B

Instructions: Answer six (6) questions from below. Provide sufficient justification. Write your solutions on the page assigned to each question. Each of the questions is worth ten (10) marks. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions** (17*)–(20*)**.** Clearly indicate which six questions you would like us to mark in the six boxes in the bubble sheet. If the boxes are unfilled, we will mark the first six solutions that appear in your answer-sheet. If you do not want a solution to be considered, clearly strike it out.

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(11) Let $U = \{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 4\}$. Let $p, q \in U$. Show that there is a continuous map $\gamma : [0, 1] \longrightarrow U$ such that $\gamma(0) = p$ and $\gamma(1) = q$ and such that γ is differentiable on $(0, 1)$.

(12) If I, J are two maximal ideals in a PID that is not a field, then show that IJ is never a prime ideal.

(13) Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function. Suppose that $f(z) \in \mathbb{R}$ if z is on the real axis or on the imaginary axis. Show that $f'(z) = 0$ at $z = 0$.

(14) Let $A \subseteq \mathbb{R}^n$ be a closed proper subset. For $x, y \in \mathbb{R}^n$, denote the usual (Euclidean) distance between them by $d(x, y)$. Let $x \in \mathbb{R}^n \smallsetminus A$; define $\delta := \inf\{d(x, y) \mid y \in A\}$. Show that there exists $y \in A$ such that $\delta = d(x, y)$.

(15) Let F be a field and V a finite-dimensional vector-space over F. Let $T : V \longrightarrow V$ be a linear transformation, such that for every $v \in V$, there exists $n \in \mathbb{N}$ such that $T^n(v) = v$.
(A) Show that if $F = \mathbb{C}$, then T is diagonalizable.
(B) Show that if $\mathrm{char}(F) > 0$, then there exists a non-diagonalizable T satisfying the above hypothesis.

(16) Let $F = \mathbb{Q}(\omega, \sqrt[3]{2})$, where $\omega \in \mathbb{C}$ is a primitive cube-root of unity. Find a $\mathbb{Q}$-basis for F (with proof). Let $\mu : F \longrightarrow F$ be the $\mathbb{Q}$-linear map given by $\mu(a) = \omega^2 a$. Find the matrix of μ with respect to the basis obtained above.

(17*) Let G be a non-trivial subgroup of the group $(\mathbb{R}, +)$. Show that either G is dense in $\mathbb{R}$ or that $G = \mathbb{Z} \cdot l$ where $l = \inf\{x \in G \mid x > 0\}$.

(18*) Let G be a subgroup of the group of permutations on a finite set X. Let F be the $\mathbb{C}$-vector-space of all the functions from X to $\mathbb{C}$. G acts on F by $(g \cdot f) : x \mapsto f(g^{-1}(x))$. Show that there is an $\phi \in F$ such that $g \cdot \phi = \phi$ for every $g \in G$. Show that there is a subspace F' of F such that $F = F' \oplus \mathbb{C}\langle\phi\rangle$ and such that $g \cdot f \in F'$ for every $g \in G$ and $f \in F'$.

(19*) (A) Let A and B be $n \times n$ matrices with entries in $\mathbb{N}$. Show that if $B = A^{-1}$ then A and B are permutation matrices. (A *permutation matrix* is a matrix obtained by permuting the rows of the identity matrix.)
(B) Let A be an $n \times n$ complex matrix that is not a scalar multiple of I_n. Show that A is similar to a matrix B such that $B_{1,1}$ (i.e. the top left entry of B) is 0.

(20*) Let $S^1 = \{z \in \mathbf{C} : |z| = 1\}$. Consider the map $\mathrm{Sq} : S^1 \longrightarrow S^1$,

$$\mathrm{Sq}(z) = z^2.$$

Show that there does not exist a continuous map $\mathrm{Sqrt} : S^1 \longrightarrow S^1$ such that $\mathrm{Sq} \circ \mathrm{Sqrt} = Id_{S^1}$? (That is, $(\mathrm{Sqrt}(w))^2 = w$.) (Hint: If such a map existed, show that there would be a bijective continuous map $S^1 \times \{1, -1\} \longrightarrow S^1$.)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (11)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (12)

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Solution to Question (13)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (14)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (15)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (16)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (17*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (18*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (19*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (20*)

CHENNAI MATHEMATICAL INSTITUTE

Postgraduate Programme in Mathematics

MSc/PhD Entrance Examination

18 May 2017

Instructions:

- Enter your *Admit Card Number* here: **M** – ☐☐☐ – ☐☐☐☐
- The allowed time is 3 hours.
- This examination has two parts. You may use the blank pages at the end for your rough-work.
- Part A is worth 40 marks and will be used for screening. There will be a cut-off for Part A, which will not be more than twenty (20) marks (out of 40). Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.) However, note that the scores in both the sections will be taken into account while making the final decision. **Record your answers to Part A in the attached bubble-sheet.**
- Part B is worth 60 marks. You should answer six (6) questions in Part B. In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions (17*)–(20*). **Indicate the six questions to be marked in the boxes in the bubble-sheet. Write your solutions to Part B in the page assigned to each question.**
- **Please read the further instructions given before Part A and inside each part carefully.**

For office use only

Part B

No.	Marks	Remarks
11		
12		
13		
14		
15		
16		

Part B (ctd.)

No.	Marks	Remarks
17*		
18*		
19*		
20*		

Further remarks:

Part A	
Part B	
Total	

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CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
18 May 2017

Admit Card Number: M – ☐☐☐ – ☐☐☐☐

Bubble Sheet

Part A

1. Ⓐ Ⓑ Ⓒ Ⓓ
2. Ⓐ Ⓑ Ⓒ Ⓓ
3. Ⓐ Ⓑ Ⓒ Ⓓ
4. Ⓐ Ⓑ Ⓒ Ⓓ
5. Ⓐ Ⓑ Ⓒ Ⓓ
6. Ⓐ Ⓑ Ⓒ Ⓓ
7. Ⓐ Ⓑ Ⓒ Ⓓ
8. Ⓐ Ⓑ Ⓒ Ⓓ
9. Ⓐ Ⓑ Ⓒ Ⓓ
10. ____________

Part B

In the boxes below, clearly indicate the SIX (6) solutions that should be marked. In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions (17^*)–(20^*).

For office use only:

Number of correct answers in Part A:	
Marks in Part A:	

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CHENNAI MATHEMATICAL INSTITUTE

Postgraduate Programme in Mathematics

MSc/PhD Entrance Examination

18 May 2017

Important: Questions in Part A will be used for **screening**. There will be a cut-off for Part A, which will not be more than twenty (20) marks (out of 40). Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.) However, note that the scores in both the sections will be taken into account while making the final decision. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions** (17*)–(20*)**.**

Notation: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ stand, respectively, for the sets of non-negative integers, of integers, of rational numbers, of real numbers, and of complex numbers. For a prime number p, $\mathbb{F}_p$ is the field with p elements. For a field F, $\mathrm{GL}_n(F)$ is the set of invertible $n \times n$ matrices over F. The symbol $\imath$ denotes a square-root of -1. When considered as a topological space, $\mathbb{R}^n$ is taken with the euclidean topology.

Part A

Instructions: Each of the questions 1–9 has one or more correct answers. Record your answers on the attached bubble-sheet by filling in the appropriate circles. Every question is worth four (4) marks. A solution receives credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

(1) Let G be a finite subgroup of $\mathrm{GL}_n(\Bbbk)$ where $\Bbbk$ is an algebraically closed field. Choose the correct statement(s) from below:
- (A) Every element of G is diagonalizable;
- (B) Every element of G is diagonalizable if $\Bbbk$ is an algebraic closure of $\mathbb{Q}$;
- (C) Every element of G is diagonalizable if $\Bbbk$ is an algebraic closure of $\mathbb{F}_p$;
- (D) There exists a basis of $\Bbbk^n$ with respect to which every element of G is a diagonal matrix.

(2) Consider the ideal $I := (ux, uy, vx, uv)$ in the polynomial ring $\mathbb{Q}[u, v, x, y]$, where u, v, x, y are indeterminates. Choose the correct statement(s) from below:
- (A) Every prime ideal containing I contains the ideal (x, y);
- (B) Every prime ideal containing I contains the ideal (x, y) or the ideal (u, v);
- (C) Every maximal ideal containing I contains the ideal (u, v);
- (D) Every maximal ideal containing I contains the ideal (u, v, x, y).

(3) Let f be an irreducible cubic polynomial over $\mathbb{Q}$ with at most one real root and $\Bbbk$ the smallest subfield of $\mathbb{C}$ containing the roots of f. Choose the correct statement(s) from below:
- (A) $\sigma(K) \subseteq K$ where σ denotes complex conjugation;
- (B) $[K : \mathbb{Q}]$ is an even number;
- (C) $[(K \cap \mathbb{R}) : \mathbb{Q}]$ is an even number;
- (D) K is uncountable.

(4) For a positive integer n, let S_n denote the permutation group on n symbols. Choose the correct statement(s) from below:
   (A) For every positive integer n and for every m with $1 \leq m \leq n$, S_n has a cyclic subgroup of order m;
   (B) For every positive integer n and for every m with $n < m < n!$, S_n has a cyclic subgroup of order m;
   (C) There exist positive integers n and m with $n < m < n!$ such that S_n has a cyclic subgroup of order m;
   (D) For every positive integer n and for every group G of order n, G is isomorphic to a subgroup of S_n.

(5) Let $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\}$, both taken with the subspace topology of $\mathbb{R}^2$. Choose the correct statement(s) from below:
   (A) Every continuous function from A to $\mathbb{R}$ has bounded image;
   (B) There exists a non-constant continuous function from B to $\mathbb{N}$ (in the subspace topology of $\mathbb{R}$);
   (C) For every surjective continuous function from $A \cup B$ to a topological space X, X has at most two connected components;
   (D) B is homeomorphic to the unit circle.

(6) Let (X, d) be a metric space. Choose the correct statement(s) from below:
   (A) There exists a metric $\tilde{d}$ on X such that d and $\tilde{d}$ define the same topology and such that $\tilde{d}$ is bounded (i.e., there exists a real number M such that $\tilde{d}(x, y) < M$ for all $x, y \in X$.);
   (B) Every closed subset of X that is bounded with respect to d is compact;
   (C) X is connected;
   (D) For every $x \in X$, there exists $y \in X$ such that $d(x, y)$ is a non-zero rational number.

(7) Which of the following are equivalence relations on $\mathbb{R}$?
   (A) $a \sim b$ if and only if $|a - b| \leq 25$;
   (B) $a \sim b$ if and only if $a - b$ is rational;
   (C) $a \sim b$ if and only if $a - b$ is irrational;
   (D) $a \sim b$ if and only if $f(a) = f(b)$ for every continuous $f : \mathbb{R} \longrightarrow \mathbb{R}$.

(8) Let $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be two differentiable functions such that $f(x+1, y) = f(x, y+1) = f(x, y)$ and $g(x+1, y) = g(x, y+1) = g(x, y)$ for all $(x, y) \in \mathbb{R}^2$. Choose the correct statement(s) from below:
(A) f is uniformly continuous;
(B) f is bounded;
(C) The function $(f, g) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is differentiable;
(D) If $\partial f/\partial x = \partial g/\partial y$ and $\partial f/\partial y = -\partial g/\partial x$, then the function $\mathbb{C} \longrightarrow \mathbb{C}$ sending $(x + \imath y) \longrightarrow f(x, y) + \imath g(x, y)$ (with $x, y \in \mathbb{R}$) is constant.

(9) Consider the equation
$$\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}.$$
Choose the correct statement(s) from below:
(A) There exists $(a, b) \in \mathbb{R}^2$ satisfying the above equation;
(B) There exists $(a, b) \in \mathbb{C}^2$ satisfying the above equation;
(C) There exists $(a, b) \in \mathbb{C}^2$ with $a = b$ satisfying the above equation;
(D) There exists $(a, b) \in (\mathbb{F}_3)^2$ with $a = b$ satisfying the above equation.

Instructions: The answer to Question 10 is an integer. You are required to write the answer in decimal form in the attached bubble-sheet. The question is worth four (4) marks.

(10) Let $p = (0, 0), q = (0, 1), r = (\imath, 0)$ be points of $\mathbb{C}^2$. What is the dimension of the $\mathbb{C}$-vector space
$$\{f(X, Y) \in \mathbb{C}[X, Y] \mid \deg f \leq 2 \text{ and } f(p) = f(q) = f(r) = 0\},$$
where by $\deg f$, we mean the total degree of the polynomial f?

Part B

Instructions: Answer six (6) questions from below. Provide sufficient justification. Write your solutions on the page assigned to each question. Each of the questions is worth ten (10) marks. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions** (17*)–(20*)**.** Clearly indicate which six questions you would like us to mark in the six boxes in the bubble sheet. If the boxes are unfilled, we will mark the first six solutions that appear in your answer-sheet. If you do not want a solution to be considered, clearly strike it out.

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(11) Let (X, τ) be a topological space and $d : X \times X \longrightarrow \mathbb{R}_{\geq 0}$ a continuous function where $X \times X$ has the product topology and $\mathbb{R}_{\geq 0}$ is the set of non-negative real numbers, with the subspace topology of the usual topology of $\mathbb{R}$. Assume that $d^{-1}(0) = \{(x, x) \mid x \in X\}$, and that $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$. Show the following:
(A) (X, τ) is Hausdorff.
(B) The sets $B_{x,\epsilon} := \{y \in X \mid d(x, y) < \epsilon\}$, $0 < \epsilon \in \mathbb{R}$ is the basis for a topology τ' on X.
(C) τ' is coarser than τ (i.e., every set open in τ' is open in τ).

(12) (A) Let f be an entire function such that $|f(z)| \leq |z|$. Show that f is a polynomial of degree ≤ 1.
(B) Let Γ be a closed differentiable contour oriented counterclockwise and let

$$\int_\Gamma \overline{z}\, \mathrm{d}z = A.$$

What is the integral

$$\int_\Gamma (x + y)\, \mathrm{d}z$$

(where x and y, respectively, are the real and imaginary parts of z) in terms of A?

(13) Let f_n, f be real-valued functions on $[0, 1]$ with f continuous. Suppose that for all convergent sequences $\{x_n : n \geq 1\} \subseteq [0, 1]$ with $x = \lim_{n\to\infty} x_n$ one has

$$\lim_{n\to\infty} f_n(x_n) = f(x).$$

Show that f_n converges to f uniformly.

(14) (A) Show that for any positive rational number r, the sequence $\{\frac{\log n}{n^r} : n \geq 1\}$ is bounded.
(B) Show that the series

$$\sum_{n\geq 10} \frac{(\log n)^2 (\log \log n)}{n^2}$$

is convergent.

(15) For a group G, let $\mathrm{Aut}(G)$ denote the group of group automorphisms of G. (The group operation of $\mathrm{Aut}(G)$ is composition.) Let p be prime number. Show that the multiplicative group $\mathbb{F}_p \smallsetminus \{0\}$ is isomorphic to $\mathrm{Aut}((\mathbb{F}_p, +))$ under the map $a \mapsto [b \mapsto ab]$ ($a \in \mathbb{F}_p \smallsetminus \{0\}$, $b \in \mathbb{F}_p$).

(16) Let $\Bbbk$ be a field, X an indeterminate and $R = \Bbbk[X]/(X^7 - 1)$. Determine the set

$$\{\dim_{\Bbbk} R/\mathfrak{m} \mid \mathfrak{m} \text{ is a maximal ideal in } R\}$$

in the following three cases: $\Bbbk = \mathbb{Q}$; $\Bbbk = \mathbb{C}$; $\Bbbk$ is a field of characteristic 7.

(17*) For a 3×3 matrix A, say that a point p on the unit sphere centred at the origin in $\mathbb{R}^3$ is a *pole* of A if $Ap = p$. Denote by SO_3 the subgroup of $\mathrm{GL}_3(\mathbb{R})$ consisting of all the orthogonal matrices with determinant 1.
(A) Show that if $A \in \mathrm{SO}_3$, then A has a pole.
(B) Let G be a subgroup of SO_3. Show that G acts on the set

$$\{p \in \mathbb{S}^2 \mid p \text{ is a pole for some matrix } A \in G\}.$$

(18*) Let $f : X \longrightarrow Y$ be a continuous surjective map such that for every closed $A \subseteq X$, $f(A)$ is closed in Y. Show that if Y and all the fibres $f^{-1}(y)$, $y \in Y$ are compact, then X is compact. Show that if Y is Hausdorff and X is compact, then Y and the all fibres $f^{-1}(y)$, $y \in Y$ are compact.

(19*) Let $\Bbbk$ be an algebraically closed uncountable field and $\mathfrak{m}$ a maximal ideal in the polynomial ring $R := \Bbbk[x_1, \ldots, x_n]$ in the indeterminates $x_1, \ldots, x_n$. Show that the composite map $\Bbbk \longrightarrow R \longrightarrow R/\mathfrak{m}$ is a field isomorphism. You may use without proof the following fact from linear algebra: If a vector space has a countable spanning set, it cannot have a linearly independent uncountable set in it. (Hint: If t is transcendental over $\Bbbk$, then consider the set $\{\frac{1}{t-\alpha} \mid \alpha \in \Bbbk\}$.)

(20*) Prove that for every $z \in \mathbb{C}$, the series $\sum_{n=1}^{\infty} \frac{\sin(z/n)}{n}$ converges. For $z \in \mathbb{C}$, let $f(z) = \sum_{n=1}^{\infty} \frac{\sin(z/n)}{n}$. Prove that f is entire.

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (11)

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- Part A is worth 40 marks and will be used for screening. There will be a cut-off for Part A, which will not be more than twenty (20) marks (out of 40). Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.) However, note that the scores in both the sections will be taken into account while making the final decision. **Record your answers to Part A in the attached bubble-sheet.**
- Part B is worth 60 marks. You should answer six (6) questions in Part B. In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions (17*)–(20*). **Indicate the six questions to be marked in the boxes in the bubble-sheet. Write your solutions to Part B in the page assigned to each question.**
- **Please read the further instructions given before Part A and inside each part carefully.**

For office use only

Part B

No.	Marks	Remarks
11		
12		
13		
14		
15		
16		

Part B (ctd.)

No.	Marks	Remarks
17*		
18*		
19*		
20*		

Further remarks:

Part A	
Part B	
Total	

CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
15 May 2018

Admit Card Number: | M | – | | | | – | | | | |

Bubble Sheet

Part A

1. Ⓐ Ⓑ Ⓒ Ⓓ
2. Ⓐ Ⓑ Ⓒ Ⓓ
3. Ⓐ Ⓑ Ⓒ Ⓓ
4. Ⓐ Ⓑ Ⓒ Ⓓ
5. Ⓐ Ⓑ Ⓒ Ⓓ
6. Ⓐ Ⓑ Ⓒ Ⓓ
7. Ⓐ Ⓑ Ⓒ Ⓓ
8. Ⓐ Ⓑ Ⓒ Ⓓ
9. Ⓐ Ⓑ Ⓒ Ⓓ
10. ________

Part B

In the boxes below, clearly indicate the SIX (6) solutions that should be marked. In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions (17^*)–(20^*).

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Number of correct answers in Part A:	
Marks in Part A:	

CHENNAI MATHEMATICAL INSTITUTE

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15 May 2018

Important: Questions in Part A will be used for **screening**. There will be a cut-off for Part A, which will not be more than twenty (20) marks (out of 40). Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.) However, note that the scores in both the sections will be taken into account while making the final decision. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions** (17*)–(20*).

Notation: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ stand, respectively, for the sets of non-negative integers, of integers, of rational numbers, of real numbers, of positive real numbers, and of complex numbers. For a prime number p, $\mathbb{F}_p$ is the field with p elements. For a field F, $M_n(F)$ stands for the set of $n \times n$ matrices over F and $\mathrm{GL}_n(F)$ is the set of invertible $n \times n$ matrices over F. The symbol $\imath$ denotes a square-root of -1. When considered as topological spaces, $\mathbb{R}^n$ or $\mathbb{C}$ are taken with the euclidean topology. When $M_n(\mathbb{R})$ is considered as a topological space, it is identified with $\mathbb{R}^{n^2}$.

Part A

Instructions: Each of the questions 1–9 has one or more correct answers. Record your answers on the attached bubble-sheet by filling in the appropriate circles. Every question is worth four (4) marks. A solution receives credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

(1) Let G be a group of order 6. Let $C_1, C_2, \ldots, C_k$ be the distinct conjugacy classes of G. Which of the following sequences of integers are possible values of $(|C_1|, |C_2|, \ldots, |C_k|)$?
(A) $(1, 1, 1, 1, 1, 1)$;
(B) $(1, 5)$;
(C) $(3, 3)$;
(D) $(1, 2, 3)$.

(2) Let $R = \mathbb{F}_2[X]$. Choose the correct statement(s) from below:
(A) R has uncountably many maximal ideals;
(B) Every maximal ideal of R has infinitely many elements;
(C) For all maximal ideals $\mathfrak{m}$ of R, $R/\mathfrak{m}$ is a finite field;
(D) For every integer n, every ideal of R has only finitely many elements of degree $\leq n$.

(3) Which of the following spaces are connected?
(A) $\{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$ as a subspace of $\mathbb{R}^2$;
(B) The set of upper triangular matrices as a subspace of $M_n(\mathbb{R})$;
(C) The set of invertible diagonal matrices as a subspace of $M_n(\mathbb{R})$;
(D) $\{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0, z^2 \geq x^2 + y^2\}$ as a subspace of $\mathbb{R}^3$.

(4) Let A be an $n \times n$ nilpotent real matrix A. Define

$$e^A = I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots .$$

Choose the correct statement(s) from below:
(A) For every real number t, e^{tA} is invertible;
(B) There exists a basis of $\mathbb{R}^n$ such that e^A is upper-triangular;
(C) There exist $B, P \in \mathrm{GL}_n(\mathbb{R})$ such that $B = Pe^A P^{-1}$ and $\mathrm{trace}(B) = 0$;
(D) There exists a basis of $\mathbb{R}^n$ such that A is lower-triangular.

(5) Let $f(w, x, y, z) = wz - xy$. Choose the correct statement(s) from below:
(A) The directional derivative at $(1, 0, 0, 1)$ in the direction (a, b, c, d) is 0 if $a + d = 0$;
(B) The directional derivative at $(1, 0, 0, 1)$ in the direction (a, b, c, d) is 0 only if $a+d = 0$;
(C) The vector $(0, -1, -1, 0)$ is normal to $f^{-1}(1)$ at the point $(1, 0, 0, 1)$;
(D) The set of points (a, b, c, d) where the total derivative of f is zero is finite.

(6) Choose the correct statement(s) from below:
(A) There exists a subfield F of $\mathbb{C}$ such that $F \not\subseteq \mathbb{R}$ and $F \simeq \mathbb{Q}[X]/(2X^3 - 3X^2 + 6)$;
(B) For every irreducible cubic polynomial $f(X) \in \mathbb{Q}[X]$, there exists a subfield F of $\mathbb{C}$ such that $F \not\subseteq \mathbb{R}$ and $F \simeq \mathbb{Q}[X]/f(X)$;
(C) There exists a subfield F of $\mathbb{R}$ such that $F \simeq \mathbb{Q}[X]/(2X^3 - 3X^2 + 6)$;
(D) For every irreducible cubic polynomial $f(X) \in \mathbb{Q}[X]$, there exists a subfield F of $\mathbb{R}$ such that $F \simeq \mathbb{Q}[X]/f(X)$.

(7) For a continuous function $f : [0, 1] \longrightarrow \mathbb{R}$, define $a_n(f) = \int_0^1 x^n f(x)\mathrm{d}x$. Choose the correct statement(s) from below:
(A) The sequence $\{a_n(f)\}$ is bounded for every continuous function $f : [0, 1] \longrightarrow \mathbb{R}$;
(B) The sequence $\{a_n(f)\}$ is Cauchy for every continuous function $f : [0, 1] \longrightarrow \mathbb{R}$;
(C) The sequence $\{a_n(f)\}$ converges to 0 for every continuous function $f : [0, 1] \longrightarrow \mathbb{R}$;
(D) There exists a continuous function $f : [0, 1] \longrightarrow \mathbb{R}$ such that the sequence $\{a_n(f)\}$ is divergent.

(8) Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a holomorphic function. Choose the correct statement(s) from below:
(A) $f(\overline{z})$ is holomorphic;
(B) Suppose that $f(\mathbb{R}) \subseteq \mathbb{R}$. Then $f(\mathbb{R})$ is open in $\mathbb{R}$;
(C) the map $z \mapsto e^{f(z)}$ is holomorphic;
(D) Suppose that $f(\mathbb{C}) \subset \mathbb{R}$. Then $f(A)$ is closed in $\mathbb{C}$ for every closed subset A of $\mathbb{C}$.

(9) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a twice-differentiable function such that $f(\frac{1}{n}) = 0$ for every positive integer n. Choose the correct statement(s) from below:
(A) $f(0) = 0$;
(B) $f'(0) = 0$;
(C) $f''(0) = 0$;
(D) f is a nonzero polynomial.

Instructions: The answer to Question 10 is an integer. You are required to write the answer in decimal form in the attached bubble-sheet. The question is worth four (4) marks.

(10) Let A be a non-zero 4×4 complex matrix such that $A^2 = 0$. What is the largest possible rank of A?

Part B

Instructions: Answer six (6) questions from below. Provide sufficient justification. Write your solutions on the page assigned to each question. Each of the questions is worth ten (10) marks. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions (17*)–(20*).** Clearly indicate which six questions you would like us to mark in the six boxes in the bubble sheet. If the boxes are unfilled, we will mark the first six solutions that appear in your answer-sheet. If you do not want a solution to be considered, clearly strike it out.

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(11) A subspace Y of $\mathbb{R}$ is said to be a *retract* of $\mathbb{R}$ if there exists a continuous map $r : \mathbb{R} \longrightarrow Y$ such that $r(y) = y$ for every $y \in Y$.
- (A) Show that $[0, 1]$ is a retract of $\mathbb{R}$.
- (B) Determine (with appropriate justification) whether every closed subset of $\mathbb{R}$ is a retract of $\mathbb{R}$.
- (C) Show that $(0, 1)$ is not a retract of $\mathbb{R}$.

(12) Let N be a positive integer and a_n be a complex number for every $-N \leq n \leq N$. Consider the holomorphic function on $\{z \in \mathbb{C} | z \neq 0\}$ given by

$$F(z) = \sum_{n=-N}^{n=N} a_n z^n.$$

Consider the function f defined on the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ by

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{F(\xi)}{\xi - z} d\xi,$$

where Γ is the boundary of the disc, oriented counterclockwise. Write down an expression for f in terms of the coefficients a_n of F.

(13) Let $\phi : [0, 1] \longrightarrow \mathbb{R}$ be a continuous function such that

$$\int_0^1 \phi(t) e^{-at} \mathrm{d}t = 0$$

for every $a \in \mathbb{R}_+$. Show that for every non-negative integer n,

$$\int_0^1 \phi(t) t^n \mathrm{d}t = 0.$$

(14) Let U be a non-empty open subset of $\mathbb{R}$. Suppose that there exists a uniformly continuous homeomorphism $h : U \longrightarrow \mathbb{R}$. Show that $U = \mathbb{R}$.

(15) Let A be 2×2 orthogonal matrix such that $\det(A) = -1$. Show that A represents reflection about a line in $\mathbb{R}^2$.

(16) A subgroup H of a group G is said to be a *characteristic subgroup* if $\sigma(H) = H$ for every group isomorphism $\sigma : G \longrightarrow G$ of G.
  (A) Determine all the characteristic subgroups of $(\mathbb{Q}, +)$ (the additive group).
  (B) Show that every characteristic subgroup of G is normal in G. Determine whether the converse is true.

(17*) Write V for the space of 3×3 skew-symmetric real matrices.
  (A) Show that for $A \in SO_3(\mathbb{R})$ and $M \in V$, $AMA^t \in V$. Write $A \cdot M$ for this action.
  (B) Let $\Phi : \mathbb{R}^3 \longrightarrow V$ be the map
$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} \mapsto \begin{bmatrix} 0 & w & -v \\ -w & 0 & u \\ v & -u & 0 \end{bmatrix}.$$
  With the usual action of $SO_3(\mathbb{R})$ on $\mathbb{R}^3$ and the above action on V, show that $\Phi(Av) = A \cdot \Phi(v)$ for every $A \in SO_3(\mathbb{R})$ and $v \in \mathbb{R}^3$.
  (C) Show that there does not exist $M \in V$, $M \neq 0$ such that for every $A \in SO_3(\mathbb{R})$, $A \cdot M$ belongs to the span of M.

(18*) Let $m > 1$ be an integer and consider the following equivalence relation on $\mathbb{C} \smallsetminus \{0\}$: $z_1 \sim z_2$ if $z_1 = z_2 e^{\frac{2\pi \imath a}{m}}$ for some $a \in \mathbb{Z}$. Write X for the set of equivalence classes and $\pi : \mathbb{C} \smallsetminus \{0\} \longrightarrow X$ for the map that takes z to its equivalence class. Define a topology on X by setting $U \subseteq X$ to be open if and only if $\pi^{-1}(U)$ is open in the euclidean topology of $\mathbb{C} \smallsetminus \{0\}$. Determine (with appropriate justification) whether X is compact.

(19*) Let $\Bbbk$ be a field, n a positive integer and G a *finite* subgroup of $\mathrm{GL}_n(\Bbbk)$ such that $|G| > 1$. Further assume that every $g \in G$ is upper-triangular and all the diagonal entries of g are 1.
  (A) Show that $\operatorname{char} \Bbbk > 0$. (Hint: consider the minimal polynomials of elements of G.)
  (B) Show that the order of g is a power of $\operatorname{char} \Bbbk$, for every $g \in G$.
  (C) Show that the centre of G has at least two elements.

(20*) Let $f : [0, 1] \longrightarrow \mathbb{R}$ be a continuous function. Determine (with appropriate justification) the following limit:
$$\lim_{n \longrightarrow \infty} \int_0^1 n x^n f(x) \mathrm{d}x.$$

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (11)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (12)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (13)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (14)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (15)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (16)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (17*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (18*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (19*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (20*)