# Solving the Green's function Schrodinger equation

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## 1 Abstract

This paper will solve the one-dimensional free particle case of the Schrödinger equation, with an added Dirac delta. The purpose of the Dirac delta, which in this case is in terms of position and time, is to represent an impulse at a certain position and time; the solution represents the wave function response to this impulse. While the homogeneous case of the Schrodinger equation represents a particle peacefully existing, this particle responds to an instantaneous disturbance to a system at some point.

The solution will use a Green's function, and this is different from other solutions because firstly, it pieces together two approaches that both have a step in which an educated guess of the form of a function or component of the solution is made. This guess step is different in both approaches, and by including two, we create a more comprehensive solution. Secondly, most other papers and lecture notes only verify a solution — not solve it through deduction like we do here.

## 2 Keywords

Physics; Quantum Mechanics; Green's Functions; Partial Differential Equations; Linear Algebra.

## 3 Introduction

The Schrödinger equation is a fundamental equation in quantum mechanics that governs the time evolution of quantum states (Shankar, 1994). We solve the one-dimensional time-dependent Schrödinger equation for a free particle subject to a Dirac delta impulse at a specific position and time, using a Green's function. The primary motivation for writing this paper is to offer a self-contained, first-principles derivation that starts from classical Lagrangian mechanics, derives the free-particle Hamiltonian, and proceeds to the quantum mechanical solution. This approach addresses gaps in existing literature by providing a deductive solution rather than mere verification, and by integrating two distinct methods for constructing the Green's function—one based on Fourier transforms and another on the time evolution operator—to yield a more robust and comprehensive understanding. Furthermore, we delve into the non-normalizability issue in the homogeneous free particle case and create a fix by adding the infinite square well potential. Though this is a defined result, it allows for an enhanced understanding of the free particle we are considering.

There is existing literature on Green's functions for the Schrödinger equation. However, classic treatments verifying the form of the Green's function for the free particle often lack detailed deductions from basic principles (Messiah, 1961; Economou, 2006). Additionally, most quantum mechanics literature discusses the Green's function in this context but typically assume prior knowledge of the form or derive it succinctly without combining multiple approaches (Shankar, 1994; Sakurai, 1967). Research on time-dependent Green's functions for quantum systems often focuses on many-body physics or condensed matter, rather than foundational derivations for the simple free particle (Economou, 2006).

Other methods for solving the inhomogeneous Schrödinger equation include the path integral formalism, which provides an alternative framework for quantum propagation (Feynman, 1948), perturbation theory for weak impulses (Sakurai, 1967) and direct numerical integration or Laplace transform techniques (Taylor, 1972). However, the Green's function method is particularly advantageous for point-source impulses, as it directly yields the response function and is integral to understanding quantum propagation and diffusion (Messiah, 1961).

Regarding the infinite square well, this model is a staple in quantum mechanics for illustrating quantization in confined systems (Griffiths, 2018; Liboff, 2003; Cohen-Tannoudji et al., 1977; Merzbacher, 1998). It remedies the non-normalizability of free particle states by imposing boundary conditions that lead to discrete energy eigenvalues, as detailed in introductory texts (Griffiths, 2018). Our treatment extends this to connect with the free particle Green's function analysis, providing a unified view.

This paper thus contributes a thorough, accessible derivation suitable for advanced students and researchers, filling a niche in the pedagogical literature.

# 4 Methods

The derivation begins from classical mechanics by constructing the Lagrangian for a free particle in one dimension, leveraging principles of homogeneity and isotropy of space and time to obtain $L = \frac{1}{2}mv^2$. The Hamiltonian is then derived as $H = \frac{p^2}{2m}$, which is quantized to form the free-particle Schrödinger equation $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$.

For the homogeneous case, separation of variables yields plane-wave solutions $\psi(x,t) = A\exp(ikx - i\omega t) + B\exp(-ikx - i\omega t)$, with $\omega = \frac{\hbar k^2}{2m}$.

The inhomogeneous case, incorporating a Dirac delta source $\delta(x - x')\delta(t - t')$, is solved using Green's functions via two complementary approaches. The first employs a Fourier transform in space, assuming a form $G(x,t;x',t') = \Theta(t - t')\int \frac{dk}{2\pi}\exp(ik(x - x'))g_k(t - t')$, leading to $g_k(\tau) = \exp\left(-i\frac{\hbar k^2}{2m}\tau\right)$ and evaluation via Gaussian integration. The second approach deduces the inclusion of the time evolution operator $U = \exp\left(-iH(t - t')/\hbar\right)$, expressing $G$ in position representation by inserting momentum eigenstates and similarly evaluating the integral.

To address non-normalizability of the free-particle wave function, an infinite square well potential $V(x) = 0$ for $0 < x < a$ and infinite elsewhere is introduced. The time-independent Schrödinger equation is solved with boundary conditions $\Psi(0) = \Psi(a) = 0$, yielding $\Psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$ and discrete energies $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$.

# 5 First principles

Suppose we have an isolated system of one free particle in free space, not subject to any external force. The particle's position can be uniquely defined by three Cartesian coordinates: x, y, z, which describes the position on each axis relative to a fixed reference point. The particle's position will be taken as a vector $r$ with components $x$, $y$ and $z$

$$r(t) = \langle x, y, z \rangle$$

The velocity of this particle $v$ is $v(t) = \dfrac{dr}{dt}$. It is the change in position with respect to time.

# 6 The Lagrangian

We will define an infinitely differentiable function $L(r, v, t)$, which depends on the free particle's position and velocity, and the time. Suppose our free particle moves such that at the instance of time $t_1$, the particle is at $r_1$, and at $t_2$, the particle is at $r_2$. We are interested to see, between this time interval $[t_1, t_2]$, how the particle moves to get from $r_1$ to $r_2$. With this, we can impose another constraint on our function $L$. The particle's movement between the time interval $[t_1, t_2]$, will be such that the integral

$$S = \int_{t_1}^{t_2} L(r, v, t)\, dt$$

is minimized. We don't know yet how the particle will move from $r_1$ to $r_2$, but we are constraining that the integral of this function $L$ from $t_1$ to $t_2$ is minimized for the position and velocity function, $r(t)$ and $v(t)$, taken. $L$ can be seen as the function that maps the physically correct function $r(t)$ to the minimum of $S$.

## 6.1 Euler-Lagrange equation

Since $v(t)$ is simply the time derivative of $r(t)$, picking the function $r$ will constrain $v$. Suppose the function $r = r(t)$ is such that $S$ is minimal. This means that $S$ is increased when $r(t)$ is replaced by any function of the form

$$r(t) \to r(t) + \delta r(t)$$

where $\delta r(t)$ is a small-valued function between $t_1$ and $t_2$. Of course, $\dfrac{d(\delta r)}{dt} = \delta v(t)$. Recalling that we have fixed our initial and final position at $t_1$ and $t_2$ to be respectively $r_1$ and $r_2$, this means that the new replaced function must also satisfy these initial and final positions. Therefore,

$$\delta r(t_1) = 0$$

and

$$\delta r\left(t_2\right)=0$$

We haven't invoked anything else thus far — all of these deductions are with the limited information we know. When $r$ is replaced by $r+\delta r$, the change in $S$ is

$$\delta S=\int_{t_1}^{t_2} L\left(r+\delta r, v+\delta v, t\right) dt\ -\int_{t_1}^{t_2} L\left(r, v, t\right) dt$$

Remember that $L$ is an infinitely differentiable function, so we can expand it in a multivariable Taylor series. If $L$ isn't $C^{\infty}$, then we would get a contradiction somewhere, but later we will see that everything does work out. We have no reason to believe this function is not infinitely differentiable.

The Taylor series expansion of $L$, about $(r,v)$, in powers of $\delta r$ and $\delta v$, is the following:

$$L\left(r+\delta r, v+\delta v, t\right)=L\left(r, v, t\right)+\frac{\partial L}{\partial r}\delta r+\frac{\partial L}{\partial v}\delta v$$

$$+\frac{1}{2}\left(\frac{\partial^2 L}{\partial r^2}\left(\delta r\right)^2+2\frac{\partial^2 L}{\partial r\partial v}\delta r\delta v+\frac{\partial^2 L}{\partial v^2}\left(\delta v\right)^2\right)+O\left(\left\|\delta r,\delta v\right\|^3\right)$$

So our difference is

$$L\left(r+\delta r, v+\delta v, t\right)-L\left(r, v, t\right)=\frac{\partial L}{\partial r}\delta r+\frac{\partial L}{\partial v}\delta v$$

$$+\frac{1}{2}\left(\frac{\partial^2 L}{\partial r^2}\left(\delta r\right)^2+2\frac{\partial^2 L}{\partial r\partial v}\delta r\delta v+\frac{\partial^2 L}{\partial v^2}\left(\delta v\right)^2\right)+O\left(\left\|\delta r,\delta v\right\|^3\right)$$

We can treat the second-order and above $\delta r$ and $\delta v$ terms as zero, since we are physicists. The two first-order terms, however, are not trivial, so the difference resorts to just those two terms.

$$\delta S=\int_{t_1}^{t_2} L\left(r+\delta r, v+\delta v, t\right) dt\ -L\left(r, v, t\right) dt$$

$$\delta S=\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial r}\delta r+\frac{\partial L}{\partial v}\delta v\right) dt$$

We focus on the second term, and apply integration by parts. Integrate $\delta v(t)$ and differentiate $\dfrac{\partial L}{\partial v}$.

$$\int_{t_1}^{t_2}\frac{\partial L}{\partial v}\delta v\ dt=\frac{\partial L}{\partial v}\delta r\ \bigg|_{t_1}^{t_2}-\int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right)\delta r\ dt$$

Remembering that $\delta r\left(t_1\right)=\delta r\left(t_2\right)=0$, the left term evaluates to 0. Therefore, we have

$$\int_{t_1}^{t_2}\frac{\partial L}{\partial v}\delta v\ dt=-\int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right)\delta r\ dt$$

and substituting into the full integral, we get

$$\delta S=\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial r}\delta r-\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right)\delta r\right) dt$$

$$= \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial r} - \frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) \right) \delta r(t) \ dt = 0$$

Now, this integral must be identically zero for all small functions $\delta r(t)$. This can only happen if

$$\frac{\partial L}{\partial r} - \frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) = \frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) - \frac{\partial L}{\partial r} = 0$$

This is known as the Euler-Lagrange equation (Shankar, 1994).

## 6.2 Space is homogeneous

We now introduce some new information: absolutely fundamental properties facts about physical space. This means that the origin of space is arbitrary, and that the laws of physics don't change depending on where in space you are. Absolute position does not matter; it only matters where a particle is relative to other particles. If two particles three metres apart in free space move towards each other, it doesn't matter where in space they are, just where they are *compared* to each other. The homogeneity of space implies the Lagrangian $L$, which is defined based on the way a particle moves between two positions in an interval, cannot depend on position. No matter where the particle is located in free space, its movement will be the same; therefore, the Lagrangian will be the same irrespective of position. It follows that

$$\frac{\partial L}{\partial r} = 0$$

and the Euler-Lagrange equation for a free particle is

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) = 0$$

Since $\frac{\partial L}{\partial v}$ does not vary with time, and $\frac{\partial L}{\partial v}$ is purely a function of velocity, this implies that $v$ is constant over time. Therefore

$$\frac{dr}{dt} = v = \mathbf{constant}$$

and

$$r = vt + r_0$$

which is an equation of motion for a free particle.

## 6.3 The Lagrangian is not unique

This will be useful in determining the Hamiltonian of a free particle as we will need to consider the Lagrangians given different velocity. Suppose we have two Lagrangians, $L(r, v, t)$ and $L'(r, v, t)$. These two differ by a time derivative of a function $f$ of position and time:

$$L'(r, v, t) = L(r, v, t) + \frac{d}{dt} f(q, t)$$

Integrating both sides with respect to $t$, which gives us the action of $L'$:

$$S' = \int_{t_1}^{t_2} L'(r, v, t)\, dt = \int_{t_1}^{t_2} L(r, v, t)\, dt + \int_{t_1}^{t_2} \frac{d}{dt} f(q, t)\, dt$$

$$= S + f\left(r(t), t\right)\big|_{t_1}^{t_2} = S + f\left(r_2, t_2\right) - f\left(r_1, t_1\right)$$
$$= S + f\left(r_2, t_2\right) - f\left(r_1, t_1\right)$$

As we did before, if we vary the function $r$ to $r + \delta r$, then the initial and final positions at $t_1$ and $t_2$ still stay the same and $\delta r\left(t_2\right) = \delta r\left(t_1\right) = 0$. So if we do the same operation, varying $S'$ using $r \to r + \delta r$:

$$\delta S' = \delta S + \left(f\left(r\left(t_2\right) + \delta r\left(t_2\right), t_2\right) - f\left(r\left(t_1\right) + \delta r\left(t_1\right), t_1\right)\right)$$
$$- \left(f\left(r\left(t_2\right), t_2\right) - f\left(r\left(t_1\right), t_1\right)\right)$$
$$\delta S' = \delta S + \left(f\left(r\left(t_2\right), t_2\right) - f\left(r\left(t_1\right), t_1\right)\right) - \left(f\left(r\left(t_2\right), t_2\right) - f\left(r\left(t_1\right), t_1\right)\right)$$
$$\delta S' = \delta S$$

and we knew that $\delta S = 0$. This means the condition for the actions' minimality still holds, as the variation does not change the Lagrangian. This means that you can add a total time derivative of a function of position and time to the Lagrangian, and it will still satisfy all the obtained properties and equations of motion. This will be useful when we vary the Lagrangian to consider a different value of velocity below.

## 6.4 Space is isotropic

The isotropy of space means that absolute angles in space are arbitrary. Only the relative angles between vectors (like velocity) matter. If the aforementioned two particles were rotated in free space, they would still satisfy the same laws of physics. Once again, since the Lagrangian is defined on the action of the free particle, this means that the Lagrangian cannot depend on the direction of velocity — only the magnitude of velocity. Then, we will take the Lagrangian as depending on $v^2$; let us rewrite it as

$$L\left(v^2\right)$$

Now suppose we have two scenarios of a single free particle. In the first, our particle is moving at velocity $v$. In the second, our particle is moving at the velocity $v + \epsilon$ where $\epsilon$ is infinitesimal. We can define a varied Lagrangian $L'$ such that

$$L'\left(v^2\right) = L\left(\left(v + \epsilon\right)^2\right) = L\left(v^2 + 2v \cdot \epsilon + \epsilon^2\right) = L\left(v^2 + 2v \cdot \epsilon\right)$$

As a physicist would do, we once again took the second-order infinitesimal as 0. Expanding the Lagrangian $L$ as a Taylor series centered at $v^2$ in powers of $\epsilon$:

$$L\left(\left(v + \epsilon\right)^2\right) = L\left(v^2\right) + \frac{\partial L}{\partial v^2} 2v \cdot \epsilon$$

Remember that two different Lagrangians can only differ by a total time derivative, in the form of

$$\frac{d}{dt} f\left(q, t\right)$$

$v$ is a total time derivative already, and $\partial L / \partial v^2$ is only in terms of $v^2$. If the $\partial L / \partial v^2 \ 2v \cdot \epsilon$ term was not linear in $v$, it would not be a total time derivative. To illustrate, the integral of a nonlinear function of $v$ with respect to $v$ typically results in a function that depends on $v$, which cannot be expressed as $f(q, t)$ as required for a total time derivative in Lagrangian mechanics.

Therefore, $\partial L/\partial v^2$ is a constant in terms of velocity, which means that the Lagrangian is linear in the velocity squared. We will suppose our constant is $\frac{m}{2}$, so we have

$$L = \frac{1}{2}mv^2$$

and this is the Lagrangian of a free particle (Shankar, 1994).

### 6.5 Time is homogeneous

Similar to how space is homogeneous, time is also homogeneous in that the origin of time is arbitrary. Returning to our two particles in free space moving toward each other, it does not matter at which point in time the situation occurs; the laws of physics will remain the same and they will collide in the same way. This means that the Lagrangian, which is defined by and intertwined with the action — the way a particle moves over time — cannot depend on time because that would mean the way a free particle moves changes depending on what time it started. Therefore, the partial derivative of the Lagrangian with respect to time is 0. Then, the Lagrangian's total differential is

$$dL = \frac{\partial L}{\partial r}dr + \frac{\partial L}{\partial v}dv + (0)dt = \frac{\partial L}{\partial r}dr + \frac{\partial L}{\partial v}dv$$

and

$$\frac{dL}{dt} = \frac{\partial L}{\partial r}\frac{dr}{dt} + \frac{\partial L}{\partial v}\frac{dv}{dt} = \frac{\partial L}{\partial r}v + \frac{\partial L}{\partial v}\frac{dv}{dt}$$

According to the Euler-Lagrange equation, we can replace $\frac{\partial L}{\partial r}$ with $\frac{d}{dt}\frac{\partial L}{\partial v}$:

$$\frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial v}\right)v + \frac{\partial L}{\partial v}\frac{d}{dt}(v)$$

and the right-hand side is clearly the product rule derivative of

$$\frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial v}v\right)$$

Subtracting $\frac{dL}{dt}$ from both sides:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}v - L\right) = 0$$

So

$$H \equiv \frac{\partial L}{\partial v}v - L$$

where $H$ is a constant that does not vary over time. It turns out that this constant is called the Hamiltonian — the energy of the system, described by our Lagrangian $L$. Let us plug in our free particle Lagrangian that we found earlier, $L = \frac{1}{2}mv^2$:

$$H \equiv (mv)\,v - \frac{1}{2}mv^2$$

$$= mv^2 - \frac{1}{2}mv^2$$

$$H = \frac{1}{2}mv^2$$

If we write it in terms of momentum — $p = mv$ — we get

$$H = \frac{p^2}{2m}$$

# 7 The Schrödinger equation

The Schrödinger equation is an equation in quantum mechanics. It is defined as follows:

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = H\left|\psi\right\rangle$$

$\left|\psi\right\rangle$ is a state ket: it represents a particle's state and stores information about it: for example, its position, velocity, and sometimes which instant it's at in time (Sakurai, 1967).

We previously found our free particle Hamiltonian was in terms of momentum: $H = \frac{p^2}{2m}$. In quantum mechanics, the momentum operator is (Sakurai, 1967)

$$p = -i\hbar\frac{\partial}{\partial x}$$

To obtain the quantum free particle Hamiltonian, we can apply the momentum operator twice:

$$p^2 = \left(-i\hbar\frac{\partial}{\partial x}\right)\left(-i\hbar\frac{\partial}{\partial x}\right) = -\hbar^2\frac{\partial^2}{\partial x^2}$$

So the Hamiltonian is

$$H = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

Substituting into the Schrödinger equation, we get

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left|\psi\right\rangle$$

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\left|\psi\right\rangle = 0$$

We have a second-degree linear operator applied onto a wave function that yields zero. First, we are interested in solving for the wave function $\left|\psi\right\rangle$ in this homogeneous case. Second, we are also interested to see the properties of this linear operator being applied onto the Green's function, constituting the inhomogeneous case.

# 8 Homogeneous case

We take $\psi$ to be a function of position and time, $\psi\left(x,t\right)$. We first assume that $\psi$ is separable into independent functions of position and time, so $\psi\left(x,t\right) = \phi\left(x\right)\tau\left(t\right)$. Then we get

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\phi\left(x\right)\tau\left(t\right) = 0$$

$$\phi(x)\, i\hbar \frac{\partial}{\partial t}\tau(t) + \tau(t)\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi(x) = 0$$

$$i\hbar \frac{1}{\tau(t)}\frac{\partial}{\partial t}\tau(t) = -\frac{\hbar^2}{2m}\frac{1}{\phi(x)}\frac{\partial^2}{\partial x^2}\phi(x)$$

Since the left side depends only on $t$ and the right side depends only on $x$, both sides must equal some constant $E$. We can then solve this as two ordinary differential equations:

$$\frac{\partial}{\partial t}\tau(t) = \frac{-i}{\hbar}E\tau(t)$$

and we find that it has solution

$$\tau(t) = A\exp\left(\frac{-i}{\hbar}Et\right)$$

Next, the ODE in terms of $x$:

$$-\frac{\hbar^2}{2m}\frac{1}{\phi(x)}\frac{\partial^2}{\partial x^2}\phi(x) = E$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi(x) = E\phi(x)$$

Notice that this is the same as an eigenvalue equation with the Hamiltonian operator, $H\phi = E\phi$, so it can be said that the constant $E$ is an eigenvalue of energy. First note that $A\left|a'\right\rangle = a'\left|a'\right\rangle$ implies $f(A)\left|a'\right\rangle = f(a')\left|a'\right\rangle$ for analytic $f$:

$$f(A)\left|a'\right\rangle = \sum_{n=0}^{\infty}\frac{f^{(n)}(A)}{n!}A^n\left|a'\right\rangle$$

and applying $A$ to $\left|a'\right\rangle$ $n$ times:

$$f(A)\left|a'\right\rangle = \sum_{n=0}^{\infty}\frac{f^{(n)}(A)}{n!}(a')^n\left|a'\right\rangle = f(a')\left|a'\right\rangle$$

Therefore, as the Hamiltonian is an analytic function of momentum,

$$H\left|\phi\right\rangle = E\left|\phi\right\rangle = \frac{p^2}{2m}\left|\phi\right\rangle$$

Further separating:

$$\frac{\partial^2}{\partial x^2}\phi(x) = \frac{-2mE}{\hbar^2}\phi(x)$$

Since this is second order, there are potentially two solutions, each having the negative exponential argument of the other. The argument is clearly

$$\pm\sqrt{\frac{-2mE}{\hbar^2}}$$

and we set $k = \sqrt{\frac{2mE}{\hbar^2}}$. Quickly substituting $E = \frac{p^2}{2m}$ shows that $k = \frac{p}{\hbar}$ and $p = \hbar k$, so $k$ is the wave vector. The solutions are therefore:

$$\phi(x) = M\exp(ikx) + N\exp(-ikx)$$

We can finally combine our two functions:

$$\psi\left(x,t\right)=\phi\left(x\right)\tau\left(t\right)=A\exp\left(ikx-\frac{i}{\hbar}Et\right)$$

or

$$\psi\left(x,t\right)=B\exp\left(-ikx-\frac{i}{\hbar}Et\right)$$

and using our definition for k, we can rearrange to state that

$$\frac{E}{\hbar}=\frac{\hbar k^2}{2m}=\omega$$

As $E=\hbar\omega$, $\omega$ is clearly the particle's angular frequency. so

$$\psi_1\left(x,t\right)=\phi\left(x\right)\tau\left(t\right)=A\exp\left(ikx-i\omega t\right)$$

or

$$\psi_2\left(x,t\right)=B\exp\left(-ikx-i\omega t\right)$$

# 9 Inhomogeneous case and Green's function

The Green's function is, in two dimensions $x$ and $t$, for fixed $x_0$ and $t_0$,

$$LG\left(x,t;x',t'\right)=\delta\left(x-x'\right)\delta\left(t-t'\right)$$

where the Dirac delta is

$$\delta\left(\chi\right)=\begin{cases}0 & \chi\neq 0\\ \text{first-order }\infty & \chi=0\end{cases}$$

such that

$$\int_{-\infty}^{+\infty}\delta\left(\chi\right)d\chi=1$$

and $L$ is a second degree linear differential operator in terms of $x$ and $t$ with no cross-terms, $p$, $q$, $u$, $v$, $w$ analytic, in the form:

$$L=p\left(x,t\right)\frac{\partial}{\partial t}+q\left(x,t\right)\frac{\partial^2}{\partial t^2}+u\left(x,t\right)\frac{\partial}{\partial x}+v\left(x,t\right)\frac{\partial^2}{\partial x^2}+w\left(x,t\right)$$

Notice that our previous linear operator can easily be cast into this form.

$$L=\left(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)$$

Applying it to our above case:

$$\left(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)G=\delta\left(x-x'\right)\delta\left(t-t'\right)$$

This is the inhomogeneous case, which is is significant as here, $x_0, t_0$ is the "source". It is the exact position and time where the system is disturbed. Imagine, for example, a lake that is peaceful until you suddenly poke your finger through it, exciting the system. The Green's function represents how the lake evolves in response to that instantaneous point source. Considering our situation, it is fitting to use an outgoing Green's function, meaning that $G(x,t;x',t') = 0$ when $t < t'$. Clearly, there is no activity before our instantaneous source. In terms of space, we will assume an infinite domain from $x \in \mathbb{R}$, and that

$$\lim_{|x|\to\infty} G(x,t;x',t') = 0$$

as this inhomogeneous wave function should stay bounded at $\infty$.

## 9.1 First solution

This solution makes an educated guess that the Fourier series' variable is $R = x - x'$. The second solution does not make this guess: it instead makes a cautious deduction that the time evolution operator is included in the Green's function, which this first solution rigorously derives. So both solutions are missing something and fill each others' gaps.

First, we observe that the Green's function must include a step function as a factor. We define the step function as

$$\Theta(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

such that

$$\frac{d\Theta(t-t')}{dt} = \delta(t-t')$$

Therefore, we write

$$G(x,t;x',t') = \Theta(t-t')\,K(x,t;x',t')$$

We make an educated guess that $K$ is the Fourier transform of a function. The Fourier transform with some convenient shifts for our case is:

$$F(r) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{irt} f(t)\,dt$$

We will define our Fourier transform $K$ in terms of the wave vector $k$, which is proportional to $p$ by $p = \hbar k$. For brevity we write $\tau = t - t'$ from now.

$$K(x,t;x',t') = \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ikR} g_k(\tau)$$

$R = x - x'$. $g_k(\tau)$ is a function of both $k$ and $t$. Now, we want to extract this kernel, so we re-substitute this new expression for G back into our differential equation.

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\Theta(t-t')\,K = \delta(t-t')\,\delta(x-x')$$

$$i\hbar\delta(\tau)\,K + i\hbar\Theta(\tau)\frac{\partial K}{\partial t} + \frac{\hbar^2}{2m}\Theta(\tau)\frac{\partial^2 K}{\partial x^2} = \delta(x-x')\,\delta(\tau)$$

The second derivative of $K$ with respect to position:

$$\frac{\partial^2 K}{\partial x^2} = \frac{\partial^2}{\partial x^2}\int_{\mathbb{R}}\frac{dk}{2\pi}e^{ikR}g_k(\tau)$$

where we can differentiate under the integral sign:

$$\int_{\mathbb{R}} \frac{dk}{2\pi} g_k(\tau) \frac{\partial^2}{\partial x^2} e^{ikR} = \int_{\mathbb{R}} \frac{dk}{2\pi} g_k(\tau) (ik)^2 e^{ikR} = -\int_{\mathbb{R}} \frac{dk}{2\pi} k^2 g_k(\tau) e^{ikR}$$

Supposing $\tau > 0$ so we get a homogeneous differential equation in terms of time, we get:

$$i\hbar \frac{\partial}{\partial t} \left( \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ikR} g_k(\tau) \right) + \frac{\hbar^2}{2m} \left( -\int_{\mathbb{R}} \frac{dk}{2\pi} k^2 g_k(\tau) e^{ikR} \right) = 0$$

$$i\hbar \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ikR} \frac{\partial g_k(\tau)}{\partial t} = \frac{\hbar^2}{2m} \int_{\mathbb{R}} \frac{dk}{2\pi} k^2 g_k(\tau) e^{ikR}$$

Because the Fourier transform is one-to-one, we equate the integrands:

$$i\hbar \frac{1}{2\pi} e^{ikR} \frac{\partial g_k(\tau)}{\partial t} = \frac{\hbar^2}{2m} \frac{1}{2\pi} k^2 g_k(\tau) e^{ikR}$$

$$\frac{\partial g_k(\tau)}{\partial t} = \frac{-i\hbar}{2m} k^2 g_k(\tau)$$

Clearly, the solution is

$$g_k(\tau) = \exp\left( \frac{-i\hbar k^2}{2m} \tau \right)$$

and notably it is the time evolution operator:

$$\exp\left( \frac{-i\hbar^2 k^2}{2m\hbar} \tau \right) = \exp\left( \frac{-ip^2}{2m\hbar} \tau \right) = \exp\left( \frac{-iH}{\hbar} \tau \right)$$

Finally, we would like to solve for $K$, which now is

$$K(x,t;x',t') = \int_{\mathbb{R}} \frac{dk}{2\pi} \exp(ikR) \exp\left( \frac{-i\hbar k^2}{2m} \tau \right)$$

$$= \int_{\mathbb{R}} \frac{dk}{2\pi} \exp\left( i \left( \frac{-\hbar k^2}{2m} \tau + Rk \right) \right) = \int_{\mathbb{R}} \frac{dk}{2\pi} \exp\left( i \frac{-\hbar\tau}{2m} \left( k^2 - \frac{2mR}{\hbar\tau} k \right) \right)$$

We can use the scaled Gaussian for this:

$$\int_{\mathbb{R}} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

for $\alpha \in \mathbb{C}$.

Completing the square in $K$:

$$K(x,t;x',t') = \int_{\mathbb{R}} \frac{dk}{2\pi} \exp\left( i \frac{-\hbar\tau}{2m} \left( \left( k - \frac{mR}{\hbar\tau} \right)^2 - \left( \frac{mR}{\hbar\tau} \right)^2 \right) \right)$$

$$= \int_{\mathbb{R}} \frac{dk}{2\pi} \exp\left( -i \frac{\hbar\tau}{2m} \left( k - \frac{mR}{\hbar\tau} \right)^2 + \frac{i}{2} \left( \frac{mR^2}{\hbar\tau} \right) \right)$$

$$= \exp\left(\frac{i}{2}\frac{mR^2}{\hbar\tau}\right)\frac{1}{2\pi}\int_{\mathbb{R}}\exp\left(-i\frac{\hbar\tau}{2m}\left(k-\frac{mR}{\hbar\tau}\right)^2\right)dk$$

For $\tau \neq 0$, we can shift the integral right by $\dfrac{mR}{\hbar\tau}$ so $\alpha = \dfrac{i\hbar\tau}{2m}$ and the integral is equal to

$$\sqrt{\frac{\pi}{\alpha}} = \sqrt{\frac{\pi}{\frac{i\hbar\tau}{2m}}} = \sqrt{\frac{2\pi m}{i\hbar\tau}}$$

So finally

$$K\left(x,t;x',t'\right) = \sqrt{\frac{m}{2\pi i\hbar\tau}}\exp\left(i\frac{mR^2}{2\hbar\tau}\right)$$

And therefore our principal solution is

$$G_0\left(x,t;x',t'\right) = \sqrt{\frac{m}{2\pi i\hbar\left(t-t'\right)}}\Theta\left(t-t'\right)\exp\left(i\frac{m\left(x-x'\right)^2}{2\hbar\left(t-t'\right)}\right)$$

and with homogeneous solutions added:

$$G\left(x,t;x',t'\right) = A\psi_1\left(x,t\right) + B\psi_2\left(x,t\right) + G_0\left(x,t,x',t'\right)$$

$$= A\exp\left(ikx - i\omega t\right) + B\exp\left(-ikx - i\omega t\right)$$

$$+\sqrt{\frac{m}{2\pi i\hbar\left(t-t'\right)}}\Theta\left(t-t'\right)\exp\left(i\frac{m\left(x-x'\right)^2}{2\hbar\left(t-t'\right)}\right)$$

## 9.2 Second approach

First, we will rewrite the Hamiltonian as $H$ instead of the differential operator, for the reason that our eventual solution includes the Hamiltonian.

$$\left(i\hbar\frac{\partial}{\partial t} - H\right)G = \delta\left(x-x'\right)\delta\left(t-t'\right)$$

Our first observation is that we can construct an outgoing Green's function by having a step function as a factor. and so where $K$ is a function of possibly, but not certainly, $x$ and $t$:

$$G\left(x,t;x',t'\right) = \Theta\left(t-t'\right)K\left(x,t;x',t'\right)$$

Rearranging $HG$ to the right and substituting our new definition of $G$:

$$i\hbar\frac{\partial}{\partial t}\left(\Theta\left(t-t'\right)K\left(x,t;x',t'\right)\right) = H\left(\Theta\left(t-t'\right)K\left(x,t;x',t'\right)\right) + \delta\left(x-x'\right)\delta\left(t-t'\right)$$

$$= i\hbar\left(\delta\left(t-t'\right)K\left(x,t;x',t'\right) + \Theta\left(t-t'\right)\frac{\partial K}{\partial t}\right) = H\Theta\left(t-t'\right)K + \delta\left(x-x'\right)\delta\left(t-t'\right)$$

We are inclined to match the terms with $\Theta$ and the terms with $\delta$. If they did not correspond, then

$$\delta\left(t-t'\right)K = H\Theta\left(t-t'\right)K$$

$$\delta\left(t-t'\right)=H\Theta\left(t-t'\right)$$

which is false. So the following must be true:

$$i\hbar\delta\left(t-t'\right)K\left(x,t;x',t'\right)=\delta\left(x-x'\right)\delta\left(t-t'\right)$$

$$i\hbar\Theta\left(t-t'\right)\frac{\partial K}{\partial t}=H\Theta\left(t-t'\right)K$$

The first is clearly 0 on both sides for $t\neq t'$, but does not imply directly

$$i\hbar K\left(x,t;x',t'\right)=\delta\left(x-x'\right)$$

for $t=t'$. It may just happen that there are time-dependent components of $K$ that are equal to 1 when $t=t'$. However, we can deduce that the position-dependent component of K is equal to the position bra-ket, so $K$ is proportional to $\langle x|x'\rangle$:

$$K\propto\delta\left(x-x'\right)=\langle x|x'\rangle$$

Additionally, at least for $t\geq t'$,

$$i\hbar\frac{\partial K}{\partial t}=HK$$

which resembles the time evolution Schrodinger equation:

$$i\hbar\frac{\partial U}{\partial t}=HU$$

This allows us to also speculate that

$$K\propto U$$

and as our Hamiltonian is time-independent (after $t'$ of course), our time evolution operator is

$$U\left(t\right)=\exp\left(-\frac{i}{\hbar}H\left(t-t'\right)\right)$$

We have displaced the time origin by $t'$; this does not change any of our following operations, and we will find that this choice of origin corresponds with our final solution. With these two statements of proportionality, we can set up such that $U$ is applied to $\langle x|x'\rangle$. From this equation from earlier,

$$i\hbar\delta\left(t-t'\right)K\left(x,t;x',t'\right)=\delta\left(x-x'\right)\delta\left(t-t'\right)$$

it is clear that the constant coefficient of $K$ is $-i/\hbar$. So we have, with an educated deduction, that

$$K=\frac{-i}{\hbar}U\left\langle x|x'\right\rangle$$

Now, we know that for $p$,

$$i\hbar\frac{\partial}{\partial x}\left\langle x|\alpha\right\rangle=\left\langle x|p|\alpha\right\rangle$$

so there is a form of the momentum operator applied to the entire position wave (bra-ket), which is a differential operator; and also a form that is applied to the ket, which is a matrix operator. Right now, our time evolution operator $U$ is an analytic function of $H$ which is an analytic function of $p$. This means that $U$ is an analytic function of $p$. Next,

$$\left(-i\hbar\right)^{N}\frac{\partial^{N}}{\partial x^{N}}\left\langle x|\alpha\right\rangle=\left\langle x|p^{N}|\alpha\right\rangle$$

This is an easy induction: suppose $(-i\hbar)^m \frac{\partial^m}{\partial x^m} \langle x|\alpha\rangle = \langle x|p^m|\alpha\rangle$. Set $|\alpha\rangle = p\,|\gamma\rangle$ and therefore $(-i\hbar)^{m+1} \frac{\partial^{m+1}}{\partial x^{m+1}} \langle x|\gamma\rangle = \langle x|p^{m+1}|\gamma\rangle$.

Therefore, the time evolution operator can be written as a Taylor series, and its form outside the bra-ket is in terms of partial derivatives of $x$, while its form inside is in terms of matrix operators $p$:

$$\hat{U}\,\langle x|\alpha\rangle = \sum_{n\geq 0} \frac{f^n\left(0\right)}{n!} \left(-i\hbar\right)^n \frac{\partial^n}{\partial x^n} \langle x|\alpha\rangle = \langle x| \sum_{n\geq 0} \frac{f^n\left(0\right)}{n!} p^n |\alpha\rangle = \langle x|U|\alpha\rangle$$

where the difference between $\hat{U}$ and $U$ is a differential and matrix operator. So we can say

$$K = \frac{-i}{\hbar} \langle x|U|x'\rangle$$

and to further determine properties of $U$, we can substitute $K$ back into the differential equation.

$$= i\hbar \left( \delta\left(t-t'\right) K\left(x,t;x',t'\right) + \Theta\left(t-t'\right) \frac{\partial K}{\partial t} \right) = H\Theta\left(t-t'\right) K + \delta\left(x-x'\right)\delta\left(t-t'\right)$$

$$= \delta\left(t-t'\right) \langle x|U|x'\rangle + \Theta\left(t-t'\right) \frac{\partial}{\partial t} \langle x|U|x'\rangle = H\Theta\left(t-t'\right) \langle x|U|x'\rangle + \delta\left(x-x'\right)\delta\left(t-t'\right)$$

Given that the $\delta\left(t-t'\right)$ terms are equal to each other, we can integrate them both over all time:

$$\delta\left(t-t'\right) \langle x|U|x'\rangle = \delta\left(x-x'\right)\delta\left(t-t'\right)$$

$$\int_{-\infty}^{\infty} dt\ \delta\left(t-t'\right) \langle x|U(t)|x'\rangle = \int_{-\infty}^{\infty} dt\ \delta\left(x-x'\right)\delta\left(t-t'\right) = \delta\left(x-x'\right)$$

$\langle x|$ and $|x'\rangle$ are independent of time:

$$\langle x| \left( \int_{-\infty}^{\infty} dt\ \delta\left(t-t'\right) U\left(t\right) \right) |x'\rangle = \delta\left(x-x'\right)$$

$$\langle x|U\left(t'\right)|x'\rangle = \langle x|x'\rangle$$

Therefore, $U\left(t'\right)$ must equal 1, so

$$U\left(t\right) = \exp\left(-\frac{i}{\hbar} H\left(t-t'\right)\right)$$

Finally, to obtain actual values and not mere operators, we sandwich a complete set of momentum eigenkets to the right of $U$. This is because the momentum and time evolution operator commute as time evolution is an analytic function of momentum, so momentum eigenkets are simultaneous eigenkets of both momentum and time evolution (for a free particle with constant Hamiltonian).

$$K = -\frac{i}{\hbar} \langle x|\,U \left( \int_{\mathbb{R}} |p'\rangle\langle p'|\ dp \right) |x'\rangle = -\frac{i}{\hbar} \int_{\mathbb{R}} \langle x|U|p'\rangle \langle p'|x'\rangle\ dp'$$

$$= -\frac{i}{\hbar} \int_{\mathbb{R}} \exp\left(-\frac{i}{\hbar}\frac{p'^2}{2m}\left(t-t'\right)\right) \langle x|p'\rangle \langle p'|x'\rangle\ dp'$$

Finding the form of $\langle p|x\rangle$:

$$\langle x'|p|\alpha\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|\alpha\rangle$$

Set $|p'\rangle = |\alpha\rangle$:

$$\langle x'|p|p'\rangle = p'\langle x'|p'\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|p'\rangle$$

We can suppose $p'$ is fixed, and that $V(x') = \langle x'|p'\rangle$. Then we get an easy ODE:

$$p'V(x') = -i\hbar\frac{\partial}{\partial x'}V(x')$$

$$V(x') = \langle x'|p'\rangle = N\exp\left(\frac{i}{\hbar}p'x'\right)$$

N is:

$$\delta(x''-x') = \frac{1}{2\pi}\int_{\mathbb{R}}\exp(i(x''-x')k)dk = \langle x''|x'\rangle = \int dp'\,\langle x''|p'\rangle\langle p'|x'\rangle$$

$$\frac{1}{2\pi}\int_{\mathbb{R}}\exp\left(i\left(x''-x'\right)k\right)dk = N^2\int dp'\exp\left(\frac{i}{\hbar}p\left(x''-x'\right)\right)$$

and setting

$$k = p/\hbar$$

we find that $N^2 = \dfrac{1}{2\pi\hbar}$. Back to $K$:

$$K = -\frac{i}{\hbar}\int_{\mathbb{R}}\left(\frac{1}{2\pi\hbar}\right)\exp\left(-\frac{i}{\hbar}\frac{p'^2}{2m}\left(t-t'\right)\right)\exp\left(\frac{i}{\hbar}p'(x-x')\right)\;dp'$$

Which simplifies (omitted) to, where $\tau = t-t'$ and $R = x-x'$:

$$K = -\frac{i}{\hbar}\int_{\mathbb{R}}\left(\frac{1}{2\pi\hbar}\right)\exp\left(-\frac{i\tau}{2m\hbar}\left(p'-\frac{mR}{\tau}\right)^2+\frac{imR^2}{2\hbar\tau}\right)$$

$$= -\frac{i}{\hbar}\frac{1}{2\pi\hbar}\exp\left(\frac{imR^2}{2\hbar\tau}\right)\int_{\mathbb{R}}\exp\left(-\frac{i\tau}{2m\hbar}\left(p'-\frac{mR}{\tau}\right)^2\right)dp$$

By the same Gaussian integral formula, we get

$$K = -\frac{i}{\hbar}\frac{1}{2\pi\hbar}\exp\left(\frac{imR^2}{2\hbar\tau}\right)\sqrt{\frac{2\pi m\hbar}{i\tau}}$$

$$= \sqrt{\frac{m}{2\pi i\hbar\tau}}\exp\left(\frac{imR^2}{2\hbar\tau}\right) = \sqrt{\frac{m}{2\pi i\hbar(t-t')}}\exp\left(\frac{im(x-x')^2}{2\hbar(t-t')}\right)$$

which confirms our original solution (Economou, 2006).

## 10 Fixing the un-normalisable homogeneous wave

We would like to return to our homogeneous solution, and investigate its properties.

$$\psi = \psi_1 + \psi_2 = A_0 \exp\left(ikx - i\omega t\right) + B_0 \exp\left(-ikx - i\omega t\right)$$
$$\psi_1 + \psi_2 = \exp(-iwt)\left(A \sin(ikx) + B \cos(ikx)\right)$$

We want to know if the integral of $|\psi|^2$ over all space is bounded, in which case it could be a wave function as we can multiply it by a normalization constant. If it is not bounded, $\psi$ cannot be a particle wave.

$$\begin{aligned}\int_{\mathbb{R}} |\psi_1 + \psi_2|^2 dx &= \int_{\mathbb{R}} |\exp(-iwt)\left(A \sin(ikx) + B \cos(ikx)\right)|^2 dx \\ &= \int_{\mathbb{R}} |A \sin(kx) + B \cos(kx)|^2 dx\end{aligned}$$

As the integrand is indeterminate as $x \to \infty$, the integral is unbounded. This is because $\psi$ is a plane wave, but in reality, a particle is represented by a wave packet. To resolve this issue, we must create some boundary conditions on the particle - so that it cannot freely go to infinity.

Let us introduce a potential $V(x)$ such that

$$V(x) = \begin{cases} 0 & 0 \leq x \leq a \\ \infty & \text{elsewhere.} \end{cases}$$

This is known as the infinite square well (Griffiths, 2018; Liboff, 2003). which is a "particle in a box". We can solve this with the time-independent Schrödinger equation.

$$H\Psi = E\Psi$$
$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\Psi = \frac{p'^2}{2m}\Psi$$

As $V(x) = \infty$ outside of $[0, a]$, the particle can never go outside of that domain, so the probability of the particle being located outside is zero. Therefore, $|\Psi(x)| = 0$, which implies $\Psi = 0$ outside. This is our first condition:

$$\Psi(x) = 0, x \notin [0, a]$$

Secondly, we want $\Psi$ to be continuous, so at $x = 0$ and $x = a$, $\Psi = 0$ as well:

$$\Psi(0) = 0$$

$$\Psi(a) = 0$$

Now inside $x \in [0, a]$, the potential $V(x)$ is zero, so our time-independent Schrödinger equation is

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\Psi = \frac{p'^2}{2m}\Psi$$

Our momentum eigenvalue $p'$ can be expanded by $p' = \hbar k$. So we get

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi = \frac{\hbar^2 k^2}{2m}\Psi$$

$$\frac{d^2}{dx^2}\Psi = -k^2\Psi$$

$$\Psi_1(x) = A_0 \exp(ikx)$$

$$\Psi_2(x) = B_0 \exp(-ikx)$$

$$\Psi(x) = \Psi_1 + \Psi_2 = A_0 \exp(ikx) + B_0 \exp(-ikx) = A\sin(kx) + B\cos(kx)$$

By our first boundary condition, $\Psi(0) = 0$ so

$$\Psi(0) = A\sin(k(0)) + B\cos(k(0)) = B$$

so $B = 0$ and our wave function reduces to

$$\Psi(x) = A\sin(kx)$$

Our second boundary condition demonstrates that

$$\Psi(a) = 0 = A\sin(ka)$$

This implies that $ka = n\pi$ for integer $n$. Therefore, our wave vector $k$ is

$$k = \frac{n\pi}{a}$$

We can substitute this into our energy eigenvalues, where we can now write $E_n$ for the $n$th energy level:

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$$

Finally, we would like to see the norm of this wave function. Since the function is identically zero outside of $[0, a]$, we have:

$$\int_0^a |\Psi(x)|^2 dx = \int_0^a A^2 \sin^2(kx) dx$$

$$\begin{aligned}
&= A^2 \int_0^a \frac{1-\cos(2kx)}{2}dx = \frac{A^2 a}{2} - \left(\frac{\sin(2kx)}{4k}\right)\Bigg|_0^a \\
&= \frac{A^2 a}{2} - \frac{1}{4k}\left(\sin\left(2\frac{n\pi}{a}a\right) + \sin\left((2\frac{n\pi}{a}(0)\right)\right) \\
&= \frac{A^2 a}{2} - \frac{1}{4k}(\sin(2n\pi) - \sin(0)) = \frac{A^2 a}{2}
\end{aligned}$$

And as $a \to \infty$, this integral becomes unbounded, matching the free particle. This is unless you maintain the normalization constant as $A = \sqrt{2/a}$ (Cohen-Tannoudji et al., 1977; Merzbacher, 1998).

# 11 Discussion

This work provides a comprehensive, first-principles derivation of the Green's function for the one-dimensional time-dependent Schrödinger equation of a free particle subjected to a Dirac delta impulse, combining two distinct approaches: one via spatial Fourier transforms and another incorporating the time evolution operator. Both methods converge to the solution $G(x,t;x',t') = \Theta(t-t')\sqrt{\frac{m}{2\pi i\hbar(t-t')}}\exp\left(i\frac{m(x-x')^2}{2\hbar(t-t')}\right)$, augmented by homogeneous solutions where needed. Physically, this Green's function represents the quantum propagator, describing how a localized impulse at position $x'$ and time $t'$ evolves into a spreading Gaussian wave packet, akin to classical diffusion but with oscillatory quantum phases, highlighting wave-particle duality and the role of the Hamiltonian in time evolution.

The homogeneous free-particle solutions, while fundamental, are non-normalizable plane waves, reflecting the delocalized nature of unbound states. Introducing the infinite square well confines the particle, enforcing boundary conditions that quantize energy levels and yield normalizable standing waves, with norm $\int_0^a |\Psi_n|^2 dx = 1$. This resolves the normalization issue and illustrates quantization in bound systems, bridging free and confined quantum behaviors. The pedagogical emphasis on deductive derivations from Lagrangian mechanics to quantum solutions fills gaps in existing literature, offering insights for understanding quantum impulses in more complex systems like many-body physics or quantum field theory.

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# Solving the Green's function Schrodinger equation

Author 100114 — Submission 100107

October 22, 2025

## Strong Points

What really stands out in this paper is the thorough, first-principles approach. The paper does a very systematic job starting from classical Lagrangian mechanics and working up to the Schrödinger equation by appealing to the homogeneity of space and time. It gives students a much firmer conceptual grounding compared to treatments that just throw the quantum Hamiltonian at them.

I was genuinely impressed by the author's decision to include two different derivations of the Green's function. The Fourier transform method is done with real clarity. But the second approach is more interesting using the time-evolution operator in momentum eigenbasis.This really elevates the work. It provides deeper, operator-based insight, and seeing both methods converge to the same result creates a powerful learning moment.

The handling of the normalization problem is particularly thoughtful. Moving into the infinite square well isn't just an add-on; it's a purposeful demonstration of how boundary conditions fix the conceptual problem of non-normalizable free-particle states. Students often stumble here, and the paper tackles it head-on in a satisfying way.

The paper actually delivers on its promise to deduce the Green's function rather than just verify it. Walking through the logical steps—making educated guesses about the form, then solving for components—gives a much more authentic look at how theoretical physics actually works.

Finally, the manuscript is impressively self-contained. Deriving results like the Euler-Lagrange equation step-by-step makes the material accessible. The author guides readers through tricky arguments without skipping important logical steps, which is exactly what this audience needs.

## Weak Points and Areas for Improvement

1. **The Discussion Section Needs Major Work**

   Honestly, the current Discussion section is this paper's biggest weakness. Right now it reads more like technical notes than a proper research article. For publication, this has to be fixed.

   The author really needs to expand the discussion to:

   - Clearly state what this paper adds to quantum mechanics pedagogy.
   - Connect the Green's function to bigger concepts—it's the quantum propagator, after all. It has many interesting applications in modern cosmology, quantum field theory as well as more applied directions such as condensed matter and statistical physics.
   - Discuss the physics perspective of wave packet spreading and phase evolution
   - Point toward future applications, like:
     - Using this approach for simple potential systems
     - Extending to higher dimensions
     - Building toward Feynman diagrams (including interactions)
     - Does current computation has interesting physics applications such as tunneling?

Without some proper expansion, the paper remains incomplete.

2. **Literature Review is Too Vague** The introduction makes some sweeping claims about other papers just verifying solutions without backing them up with examples. This undermines the scholarly foundation. A better approach would be to specifically note that while Fourier methods are common, combining them with a detailed operator-based derivation in one accessible package is the real pedagogical contribution here. The author should first describe what is Greens functions and why it is relevant for Schrodinger equation. Can such methods have relativistic extensions?

3. **Mathematical Issues That Need Cleaning Up**
    - **Too Casual:** Phrases like "since we are physicists" to justify dropping terms come across as unprofessional. Better to properly justify this using variational calculus principles.
    - **Notation consistency:** The paper should have consistent notation. Why the equation numbers dont appear? its a very long computation without equation number. It is very hard for readers to track all these computations with no equation number appearing anywhere.

4. **Structure is Unbalanced** The paper spends too much time (Sections 5–7) rederiving standard classical and quantum mechanics results. While the first-principles approach is good in theory, it delays getting to the paper's actual contribution—the Green's function derivation. Trimming these sections would improve the pacing dramatically.

5. **Missing Steps in Key Derivations**
    - **Fourier Method:** Shows what $G$ looks like for $t > t'$ but doesn't verify the crucial initial condition—this needs to be addressed.
    - **Operator Method:** The jump to $K \propto \langle x|x'\rangle$ feels hand-wavy and needs more rigorous justification.

6. **References are Inconsistent** The citation style is all over the place—some authors have full first names, others just initials, publisher formatting varies. Picking one consistent style would make the paper look much more professional.

## Title suggestion

The paper does not device techniques to solve Green's function to any potential in any dimensions. Therefore "The Free-Particle Green's Function in One Dimension" or something similar seems more natural.

## Overall Assessment

With the current version I **accept with moderate revisions**. The paper is is **strong**, but the discussions section needs to be addressed before final acceptance.

# Reviewer Report

Manuscript: Solving the Green's function Schrödinger equation

## Summary

The manuscript derives the retarded Green's function (propagator) for a free particle in one spatial dimension,

$$G(x,t;x',t') = \Theta(t-t')\sqrt{\frac{m}{2\pi i\hbar(t-t')}}\,\exp\Big[\frac{im(x-x')^2}{2\hbar(t-t')}\Big],$$

using two approaches: (i) Fourier-transform methods and (ii) the evolution-operator route $U(t) = e^{-iHt/\hbar}$. The text briefly discusses normalization, plane-wave (non-)normalizability, and the trick of placing the system in a box to obtain a discrete spectrum.

## Assessment

The topic is classical and appears in standard quantum-mechanics textbooks. The presentation is generally clear and reaches the correct closed-form propagator. The paper is positioned as written by a *high school student*; given the level (distribution theory, $i\varepsilon$ prescriptions/contours, operator methods), this is surprisingly advanced. This is not a criticism, but it calls for transparency regarding mentorship and the author's independent contribution. Overall, the manuscript is pedagogical rather than original; it can be publishable as an instructional note with revisions that clarify provenance, sharpen rigor, and better situate the work.

## Major Comments

1. **Authorship & provenance (important).** Since the work is presented as authored by a high school student, please add a short disclosure: (i) mentoring/supervision received; (ii) which derivations are original versus reproductions; (iii) use of CAS/LLM or other software. This will contextualize an impressively advanced text for the stated level.

2. **Originality & positioning.** Temper novelty claims and explicitly situate the note relative to standard references. Clarify what is meant to be new: the side-by-side derivations, pedagogical framing, or specific clarifications (e.g., initial-value interpretation, normalization checks).

3. **Methodological comparison with takeaways.** Since two routes are presented, distill clear guidance on *when* each is advantageous: Fourier methods for translationally invariant PDEs and boundary-value problems; evolution operators for spectral decompositions, composition laws, and time-ordered reasoning. Summarize this in a dedicated paragraph.

4. **Rigor and distributional details.** Make the $i\varepsilon$ prescription (causality) explicit; state that $G$ is a distribution and that equal-time limits are understood distributionally. Show the limit $G \to \delta(x-x')$ as $t \to t'^{+}$ and verify the PDE $(i\hbar\partial_t + \frac{\hbar^2}{2m}\partial_x^2)G = \delta(x-x')\delta(t-t')$ in the distribution sense.

5. **Consistency checks.** Include brief, self-contained checks: (a) *composition/semigroup law* $\int dx''\, G(x,t;x'',\tau)G(x'',\tau;x',t') = G(x,t;x',t')$; (b) *unitarity* of evolution for wave packets; (c) *dimensional analysis* confirming the prefactor.

6. **Boundary conditions and boxes.** The “box” discussion is helpful; make explicit how the discrete propagator in a finite interval with Dirichlet BCs converges to the free-space propagator as $L \to \infty$ (Poisson summation or stationary-phase sketch).

## Minor Comments

- State Fourier-transform conventions up front (signs, $2\pi$ factors) and keep them consistent across both derivations.
- Clarify the role of the Heaviside function $\Theta(t-t')$ (retarded boundary condition) and mention other Green’s functions (advanced, Feynman) and how the contour choice changes the result.
- Add a short remark connecting the result to the path-integral kernel for a free particle to orient readers.
- Improve figure/text balance: add at least one illustrative figure (e.g., modulus/phase of $G$ vs. $x - x'$ for fixed $t - t' > 0$) with detailed captions; if multiple curves appear, use distinct line styles/markers for B&W print.
- Light copy-editing: standardize notation, ensure consistent $\hbar$ and $m$, and fix minor typographical issues.

## Recommendation

**Revise and resubmit.** A correct and useful pedagogical note on textbook material. With transparent authorship context, clearer positioning, explicit distributional/causality details, and a few checks/figures, the manuscript will be stronger and more credible for its intended audience.

# The free-particle Green's function in two dimensions

[redacted]
[redacted]

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## 1 Abstract

This paper will solve the one-dimensional free particle case of the Schrödinger equation, with an added Dirac delta. The purpose of the Dirac delta, which in this case is in terms of position and time, is to represent an impulse at a certain position and time; the solution represents the wave function response to this impulse. While the homogeneous case of the Schrodinger equation represents a particle peacefully existing, this particle responds to an instantaneous disturbance to a system at some point.

The solution will use a Green's function, and this is different from other solutions because firstly, it pieces together two approaches that both have a step in which an educated guess of the form of a function or component of the solution is made. This guess step is different in both approaches, and by including two, we create a more comprehensive solution. Secondly, most other papers and lecture notes only verify a solution — not solve it through deduction like we do here.

## 2 Keywords

Physics; Quantum Mechanics; Green's Functions; Partial Differential Equations; Linear Algebra.

## 3 Acknowledgements

Special thanks to [redacted], my mentor, for providing the idea for the paper, and providing guidance for the solutions. Specifically, [redacted] explained to me that the conventional solution to the Green's function schrodinger equation uses a Fourier transform.

No generative AI was used in the creation of this paper.

# 4 Introduction

The Schrödinger equation is a fundamental equation in quantum mechanics that governs the time evolution of quantum states (Shankar, 1994). We solve the one-dimensional time-dependent Schrödinger equation for a free particle subject to a Dirac delta impulse at a specific position and time, using a Green's function. This is analogous to having a particle have an instantaneous disturbance at a specific point and time (as a comparison, like dipping your finger into a still pond), but otherwise being in free space at all other moments. The primary motivation for this is to create an airtight solution to the differential equation.

There is existing literature on Green's functions for the Schrödinger equation. However, most papers and lecture notes verifying the form of the Green's function for the free particle often lack detailed deductions from basic principles, or assume it, treating it as a classical result—often rightfully, but there is a need for a full derivation. It is difficult to find accessible full derivations that are not done with an approach that seems more like verification than derivation: for example, although Fourier transform solutions are common, the phase shift $x - x'$ in is usually assumed and then verified aftewards to be true. In order to create an airtight solution, we combine two distinct methods: one based on Fourier transforms and another using the time evolution operator and deducing the components of the Green's function.

In the Fourier transform approach, we have an "educated guess" in assuming the phase to be $x - x'$. In the time evolution approach, we make an educated speculation that the Green's function kernel $K$ is proportional to the time evolution operator. By demonstrating that both methods converge to the same result, we have a more convincing bottom-up derivation that should enhance understanding of this classic result.

This paper thus contributes a thorough, accessible derivation suitable for advanced students and researchers, filling a niche in the pedagogical literature.

# 5 Methods

The Hamiltonian is quantized to form the free-particle Schrödinger equation $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$.

For the homogeneous case, separation of variables yields plane-wave solutions $\psi(x,t) = A\exp(ikx - i\omega t) + B\exp(-ikx - i\omega t)$, with $\omega = \frac{\hbar k^2}{2m}$.

The inhomogeneous case, incorporating a Dirac delta source $\delta(x - x')\delta(t - t')$, is solved using Green's functions via two complementary approaches. The first employs a Fourier transform in space, assuming a form $G(x,t;x',t') = \Theta(t - t')\int \frac{dk}{2\pi}\exp(ik(x - x'))g_k(t - t')$, leading to $g_k(\tau) = \exp\left(-i\frac{\hbar k^2}{2m}\tau\right)$ and evaluation via Gaussian integration. The second approach deduces the inclusion of the time evolution operator $U = \exp\left(-iH(t - t')/\hbar\right)$, expressing $G$ in position representation by inserting momentum eigenstates and similarly evaluating the integral.

To address non-normalizability of the free-particle wave function, an infinite square well potential $V(x) = 0$ for $0 < x < a$ and infinite elsewhere is introduced. The time-independent Schrödinger equation is solved with boundary conditions $\Psi(0) = \Psi(a) = 0$, yielding $\Psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$ and discrete energies $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$.

# 6 The Schrödinger equation

The Schrödinger equation is an equation in quantum mechanics. It is defined as follows:

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = H\left|\psi\right\rangle \tag{1}$$

$\left|\psi\right\rangle$ is a state ket: it represents a particle's state and stores information about it: for example, its position, velocity, and sometimes which instant it's at in time (Sakurai, 1967).

According to classical mechanics, the Hamiltonian of a free particle is $H = \dfrac{p^2}{2m}$. In quantum mechanics, the momentum operator is (Sakurai, 1967)

$$p = -i\hbar\frac{\partial}{\partial x} \tag{2}$$

To obtain the quantum free particle Hamiltonian, we can apply the momentum operator twice:

$$p^2 = \left(-i\hbar\frac{\partial}{\partial x}\right)\left(-i\hbar\frac{\partial}{\partial x}\right) = -\hbar^2\frac{\partial^2}{\partial x^2} \tag{3}$$

So the Hamiltonian is

$$H = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \tag{4}$$

Substituting into the Schrödinger equation, we get

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left|\psi\right\rangle \tag{5}$$

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\left|\psi\right\rangle = 0 \tag{6}$$

We have a second-degree linear operator applied onto a wave function that yields zero. First, we are interested in solving for the wave function $\left|\psi\right\rangle$ in this homogeneous case. Second, we are also interested to see the properties of this linear operator being applied onto the Green's function, constituting the inhomogeneous case.

# 7 Homogeneous case

We take $\psi$ to be a function of position and time, $\psi\left(x,t\right)$. We first assume that $\psi$ is separable into independent functions of position and time, so $\psi\left(x,t\right) = \phi\left(x\right)\tau\left(t\right)$. Then we get

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\phi\left(x\right)\tau\left(t\right) = 0 \tag{7}$$

$$\phi\left(x\right)i\hbar\frac{\partial}{\partial t}\tau\left(t\right) + \tau\left(t\right)\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi\left(x\right) = 0 \tag{8}$$

$$i\hbar\frac{1}{\tau\left(t\right)}\frac{\partial}{\partial t}\tau\left(t\right)=-\frac{\hbar^{2}}{2m}\frac{1}{\phi\left(x\right)}\frac{\partial^{2}}{\partial x^{2}}\phi\left(x\right) \tag{9}$$

Since the left side depends only on $t$ and the right side depends only on $x$, both sides must equal some constant $E$. We can then solve this as two ordinary differential equations:

$$\frac{\partial}{\partial t}\tau\left(t\right)=\frac{-i}{\hbar}E\tau\left(t\right) \tag{10}$$

and we find that it has solution

$$\tau\left(t\right)=A\exp\left(\frac{-i}{\hbar}Et\right) \tag{11}$$

Next, the ODE in terms of $x$:

$$-\frac{\hbar^{2}}{2m}\frac{1}{\phi\left(x\right)}\frac{\partial^{2}}{\partial x^{2}}\phi\left(x\right)=E \tag{12}$$

$$-\frac{\hbar^{2}}{2m}\frac{\partial^{2}}{\partial x^{2}}\phi\left(x\right)=E\phi\left(x\right) \tag{13}$$

Notice that this is the same as an eigenvalue equation with the Hamiltonian operator, $H\phi=E\phi$, so it can be said that the constant $E$ is an eigenvalue of energy. First note that $A\left|a'\right\rangle=a'\left|a'\right\rangle$ implies $f\left(A\right)\left|a'\right\rangle=f\left(a'\right)\left|a'\right\rangle$ for analytic $f$:

$$f\left(A\right)\left|a'\right\rangle=\sum_{n=0}^{\infty}\frac{f^{(n)}\left(A\right)}{n!}A^{n}\left|a'\right\rangle \tag{14}$$

and applying $A$ to $\left|a'\right\rangle$ $n$ times:

$$f\left(A\right)\left|a'\right\rangle=\sum_{n=0}^{\infty}\frac{f^{(n)}\left(A\right)}{n!}\left(a'\right)^{n}\left|a'\right\rangle=f\left(a'\right)\left|a'\right\rangle \tag{15}$$

Therefore, as the Hamiltonian is an analytic function of momentum,

$$H\left|\phi\right\rangle=E\left|\phi\right\rangle=\frac{p^{2}}{2m}\left|\phi\right\rangle \tag{16}$$

Further separating:

$$\frac{\partial^{2}}{\partial x^{2}}\phi\left(x\right)=\frac{-2mE}{\hbar^{2}}\phi\left(x\right) \tag{17}$$

Since this is second order, there are potentially two solutions, each having the negative exponential argument of the other. The argument is clearly

$$\pm\sqrt{\frac{-2mE}{\hbar^{2}}} \tag{18}$$

and we set $k=\sqrt{\frac{2mE}{\hbar^{2}}}$. Quickly substituting $E=\frac{p^{2}}{2m}$ shows that $k=\frac{p}{\hbar}$ and $p=\hbar k$, so $k$ is the wave vector. The solutions are therefore:

$$\phi\left(x\right)=M\exp\left(ikx\right)+N\exp\left(-ikx\right) \tag{19}$$

We can finally combine our two functions:

$$\psi\left(x,t\right)=\phi\left(x\right)\tau\left(t\right)=A\exp\left(ikx-\frac{i}{\hbar}Et\right) \tag{20}$$

or

$$\psi\left(x,t\right)=B\exp\left(-ikx-\frac{i}{\hbar}Et\right) \tag{21}$$

and using our definition for k, we can rearrange to state that

$$\frac{E}{\hbar}=\frac{\hbar k^2}{2m}=\omega \tag{22}$$

As $E=\hbar\omega$, $\omega$ is clearly the particle's angular frequency. so

$$\psi_1\left(x,t\right)=\phi\left(x\right)\tau\left(t\right)=A\exp\left(ikx-i\omega t\right) \tag{23}$$

or

$$\psi_2\left(x,t\right)=B\exp\left(-ikx-i\omega t\right) \tag{24}$$

## 8 Inhomogeneous case and Green's function

The Green's function is, in two dimensions $x$ and $t$, for fixed $x_0$ and $t_0$,

$$LG\left(x,t;x',t'\right)=\delta\left(x-x'\right)\delta\left(t-t'\right) \tag{25}$$

where the Dirac delta is

$$\delta\left(\chi\right)=\begin{cases}0 & \chi\neq 0\\ \text{first-order }\infty & \chi=0\end{cases} \tag{26}$$

such that

$$\int_{-\infty}^{+\infty}\delta\left(\chi\right)d\chi=1 \tag{27}$$

and $L$ is a second degree linear differential operator in terms of $x$ and $t$ with no cross-terms, $p$, $q$, $u$, $v$, $w$ analytic, in the form:

$$L=p\left(x,t\right)\frac{\partial}{\partial t}+q\left(x,t\right)\frac{\partial^2}{\partial t^2}+u\left(x,t\right)\frac{\partial}{\partial x}+v\left(x,t\right)\frac{\partial^2}{\partial x^2}+w\left(x,t\right) \tag{28}$$

Notice that our previous linear operator can easily be cast into this form.

$$L=\left(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right) \tag{29}$$

Applying it to our above case:

$$\left(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)G=\delta\left(x-x'\right)\delta\left(t-t'\right) \tag{30}$$

This is the inhomogeneous case, which is is significant as here, $x_0, t_0$ is the "source". It is the exact position and time where the system is disturbed. Imagine, for example, a lake that is peaceful until you suddenly poke your finger through it, exciting the system. The Green's function represents how the lake evolves in response to that instantaneous point source. Considering our situation, it is fitting to use an outgoing Green's function, meaning that $G\left(x,t;x',t'\right)=0$ when $t<t'$. Clearly, there is no activity before our instantaneous source impulse "excites" the system, so there is no energy prior to $t'$. Therefore, the Green's function propagator is zero prior to $t'$.

In terms of space, we will assume an infinite domain from $x\in\mathbb{R}$, and that

$$\lim_{|x|\to\infty} G\left(x,t;x',t'\right)=0 \tag{31}$$

as this inhomogeneous wave function should stay bounded at $\infty$.

## 8.1 First solution

This solution makes an educated guess that the Fourier series' variable is $R=x-x'$. The second solution does not make this guess: it instead makes a cautious deduction that the time evolution operator is included in the Green's function, which this first solution rigorously derives. So both solutions are missing something and fill each others' gaps.

First, we observe that the Green's function must include a step function as a factor, because we are working with the outgoing Green's function. The step function would ensure that firstly, the Green's function is zero prior to $t'$, and that there is a discontinuous "jump" at $t'$. We define the step function as

$$\Theta\left(t\right)=\begin{cases}0 & t<0\\ 1 & t\geq 0\end{cases} \tag{32}$$

such that

$$\frac{d\Theta\left(t-t'\right)}{dt}=\delta\left(t-t'\right) \tag{33}$$

Therefore, we write

$$G\left(x,t;x',t'\right)=\Theta\left(t-t'\right)K\left(x,t;x',t'\right) \tag{34}$$

We make an educated guess that $K$ is the Fourier transform of a function. The Fourier transform is:

$$F\left(r\right)=\frac{1}{2\pi}\int_{\mathbb{R}} e^{irt}f\left(t\right)dt \tag{35}$$

We will define our Fourier transform $K$ in terms of the wave vector $k$, which is proportional to $p$ by $p=\hbar k$. For brevity we write $\tau=t-t'$ from now.

$$K\left(x,t;x',t'\right)=\int_{\mathbb{R}}\frac{dk}{2\pi}e^{ikR}g_k\left(\tau\right) \tag{36}$$

$R=x-x'$. $g_k\left(\tau\right)$ is a function of both $k$ and $t$. Now, we want to extract this kernel, so we re-substitute this new expression for G back into our differential equation.

$$\left(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\Theta\left(t-t'\right)K=\delta\left(t-t'\right)\delta\left(x-x'\right) \tag{37}$$

$$i\hbar\delta\left(\tau\right)K+i\hbar\Theta\left(\tau\right)\frac{\partial K}{\partial t}+\frac{\hbar^{2}}{2m}\Theta\left(\tau\right)\frac{\partial^{2}K}{\partial x^{2}}=\delta\left(x-x'\right)\delta\left(\tau\right)\tag{38}$$

The second derivative of $K$ with respect to position:

$$\frac{\partial^{2}K}{\partial x^{2}}=\frac{\partial^{2}}{\partial x^{2}}\int_{\mathbb{R}}\frac{dk}{2\pi}e^{ikR}g_{k}\left(\tau\right)\tag{39}$$

where we can differentiate under the integral sign:

$$\int_{\mathbb{R}}\frac{dk}{2\pi}g_{k}\left(\tau\right)\frac{\partial^{2}}{\partial x^{2}}e^{ikR}=\int_{\mathbb{R}}\frac{dk}{2\pi}g_{k}\left(\tau\right)\left(ik\right)^{2}e^{ikR}=-\int_{\mathbb{R}}\frac{dk}{2\pi}k^{2}g_{k}\left(\tau\right)e^{ikR}\tag{40}$$

Supposing $\tau>0$ so we get a homogeneous differential equation in terms of time, we get:

$$i\hbar\frac{\partial}{\partial t}\left(\int_{\mathbb{R}}\frac{dk}{2\pi}e^{ikR}g_{k}\left(\tau\right)\right)+\frac{\hbar^{2}}{2m}\left(-\int_{\mathbb{R}}\frac{dk}{2\pi}k^{2}g_{k}\left(\tau\right)e^{ikR}\right)=0\tag{41}$$

$$i\hbar\int_{\mathbb{R}}\frac{dk}{2\pi}e^{ikR}\frac{\partial g_{k}\left(\tau\right)}{\partial t}=\frac{\hbar^{2}}{2m}\int_{\mathbb{R}}\frac{dk}{2\pi}k^{2}g_{k}\left(\tau\right)e^{ikR}\tag{42}$$

Because the Fourier transform is one-to-one, we equate the integrands:

$$i\hbar\frac{1}{2\pi}e^{ikR}\frac{\partial g_{k}\left(\tau\right)}{\partial t}=\frac{\hbar^{2}}{2m}\frac{1}{2\pi}k^{2}g_{k}\left(\tau\right)e^{ikR}\tag{43}$$

$$\frac{\partial g_{k}\left(\tau\right)}{\partial t}=\frac{-i\hbar}{2m}k^{2}g_{k}\left(\tau\right)\tag{44}$$

Clearly, the solution is

$$g_{k}\left(\tau\right)=\exp\left(\frac{-i\hbar k^{2}}{2m}\tau\right)\tag{45}$$

and notably it is the time evolution operator:

$$\exp\left(\frac{-i\hbar^{2}k^{2}}{2m\hbar}\tau\right)=\exp\left(\frac{-ip^{2}}{2m\hbar}\tau\right)=\exp\left(\frac{-iH}{\hbar}\tau\right)\tag{46}$$

Finally, we would like to solve for $K$, which now is

$$K\left(x,t;x',t'\right)=\int_{\mathbb{R}}\frac{dk}{2\pi}\exp\left(ikR\right)\exp\left(\frac{-i\hbar k^{2}}{2m}\tau\right)\tag{47}$$

$$=\int_{\mathbb{R}}\frac{dk}{2\pi}\exp\left(i\left(\frac{-\hbar k^{2}}{2m}\tau+Rk\right)\right)=\int_{\mathbb{R}}\frac{dk}{2\pi}\exp\left(i\frac{-\hbar\tau}{2m}\left(k^{2}-\frac{2mR}{\hbar\tau}k\right)\right)\tag{48}$$

We can use the scaled Gaussian for this:

$$\int_{\mathbb{R}}e^{-\alpha x^{2}}dx=\sqrt{\frac{\pi}{\alpha}}\tag{49}$$

for $\alpha\in\mathbb{C}$.

Completing the square in $K$:

$$K\left(x,t;x',t'\right)=\int_{\mathbb{R}}\frac{dk}{2\pi}\exp\left(i\frac{-\hbar\tau}{2m}\left(\left(k-\frac{mR}{\hbar\tau}\right)^{2}-\left(\frac{mR}{\hbar\tau}\right)^{2}\right)\right)\tag{50}$$

$$=\int_{\mathbb{R}}\frac{dk}{2\pi}\exp\left(-i\frac{\hbar\tau}{2m}\left(k-\frac{mR}{\hbar\tau}\right)^{2}+\frac{i}{2}\left(\frac{mR^{2}}{\hbar\tau}\right)\right)\tag{51}$$

$$=\exp\left(\frac{i}{2}\frac{mR^{2}}{\hbar\tau}\right)\frac{1}{2\pi}\int_{\mathbb{R}}\exp\left(-i\frac{\hbar\tau}{2m}\left(k-\frac{mR}{\hbar\tau}\right)^{2}\right)dk\tag{52}$$

For $\tau\neq0$, we can shift the integral right by $\dfrac{mR}{\hbar\tau}$ so $\alpha=\dfrac{i\hbar\tau}{2m}$ and the integral is equal to

$$\sqrt{\frac{\pi}{\alpha}}=\sqrt{\frac{\pi}{\frac{i\hbar\tau}{2m}}}=\sqrt{\frac{2\pi m}{i\hbar\tau}}\tag{53}$$

So finally

$$K\left(x,t;x',t'\right)=\sqrt{\frac{m}{2\pi i\hbar\tau}}\exp\left(i\frac{mR^{2}}{2\hbar\tau}\right)\tag{54}$$

And therefore our principal solution is

$$G_0\left(x,t;x',t'\right)=\sqrt{\frac{m}{2\pi i\hbar\left(t-t'\right)}}\Theta\left(t-t'\right)\exp\left(i\frac{m\left(x-x'\right)^{2}}{2\hbar\left(t-t'\right)}\right)\tag{55}$$

and with homogeneous solutions added:

$$G\left(x,t;x',t'\right)=A\psi_1\left(x,t\right)+B\psi_2\left(x,t\right)+G_0\left(x,t,x',t'\right)\tag{56}$$

$$=A\exp\left(ikx-i\omega t\right)+B\exp\left(-ikx-i\omega t\right)\tag{57}$$

$$+\sqrt{\frac{m}{2\pi i\hbar\left(t-t'\right)}}\Theta\left(t-t'\right)\exp\left(i\frac{m\left(x-x'\right)^{2}}{2\hbar\left(t-t'\right)}\right)\tag{58}$$

$K$ is the path integral kernel. This result can be obtained with Feynman's path integral formulation, according to Sakurai's Advanced Quantum Mechanics (Sakurai, 1967).

## 8.2 Second approach

First, we will rewrite the Hamiltonian as $H$ instead of the differential operator, for the reason that our eventual solution includes the Hamiltonian.

$$\left(i\hbar\frac{\partial}{\partial t}-H\right)G=\delta\left(x-x'\right)\delta\left(t-t'\right)\tag{59}$$

Our first observation is that we can construct an outgoing Green's function by having a step function as a factor. and so where $K$ is a function of possibly, but not certainly, $x$ and $t$:

$$G\left(x,t;x',t'\right)=\Theta\left(t-t'\right)K\left(x,t;x',t'\right)\tag{60}$$

Rearranging $HG$ to the right and substituting our new definition of $G$:

$$i\hbar\frac{\partial}{\partial t}\left(\Theta\left(t-t'\right)K\left(x,t;x',t'\right)\right)=H\left(\Theta\left(t-t'\right)K\left(x,t;x',t'\right)\right)+\delta\left(x-x'\right)\delta\left(t-t'\right) \tag{61}$$

$$=i\hbar\left(\delta\left(t-t'\right)K\left(x,t;x',t'\right)+\Theta\left(t-t'\right)\frac{\partial K}{\partial t}\right)=H\Theta\left(t-t'\right)K+\delta\left(x-x'\right)\delta\left(t-t'\right) \tag{62}$$

We are inclined to match the terms with $\Theta$ and the terms with $\delta$. If they did not correspond, then

$$\delta\left(t-t'\right)K=H\Theta\left(t-t'\right)K \tag{63}$$

$$\delta\left(t-t'\right)=H\Theta\left(t-t'\right) \tag{64}$$

which is false. So the following must be true:

$$i\hbar\delta\left(t-t'\right)K\left(x,t;x',t'\right)=\delta\left(x-x'\right)\delta\left(t-t'\right) \tag{65}$$

$$i\hbar\Theta\left(t-t'\right)\frac{\partial K}{\partial t}=H\Theta\left(t-t'\right)K \tag{66}$$

The first is clearly 0 on both sides for $t\neq t'$, but does not imply directly

$$i\hbar K\left(x,t;x',t'\right)=\delta\left(x-x'\right) \tag{67}$$

for $t=t'$. It may just happen that there are time-dependent components of $K$ that are equal to 1 when $t=t'$.

However, even if the entirety of $K$ is not equal to $\delta(x-x')$, we can deduce that specifically the position-dependent component of K is equal to $\delta(x-x')=\langle x|x'\rangle$, so $K$ is proportional to $\langle x|x'\rangle$:

$$K\propto\delta\left(x-x'\right)=\langle x|x'\rangle \tag{68}$$

Additionally, at least for $t\geq t'$,

$$i\hbar\frac{\partial K}{\partial t}=HK \tag{69}$$

which resembles the time evolution Schrodinger equation:

$$i\hbar\frac{\partial U}{\partial t}=HU \tag{70}$$

This allows us to also speculate that

$$K\propto U \tag{71}$$

and as our Hamiltonian is time-independent (after $t'$ of course), our time evolution operator is

$$U\left(t\right)=\exp\left(-\frac{i}{\hbar}H\left(t-t'\right)\right) \tag{72}$$

We have displaced the time origin by $t'$; this does not change any of our following operations, and we will find that this choice of origin corresponds with our final solution. With these two statements of proportionality, we can set up such that $U$ is applied to $\langle x|x'\rangle$. From this equation from earlier,

$$i\hbar\delta\left(t-t'\right)K\left(x,t;x',t'\right)=\delta\left(x-x'\right)\delta\left(t-t'\right) \tag{73}$$

it is clear that the constant coefficient of $K$ is $-i/\hbar$. So we have, with an educated deduction, that

$$K = \frac{-i}{\hbar} U \left\langle x|x' \right\rangle \tag{74}$$

Now, we know that for $p$,

$$i\hbar \frac{\partial}{\partial x} \left\langle x|\alpha \right\rangle = \left\langle x|p|\alpha \right\rangle \tag{75}$$

so there is a form of the momentum operator applied to the entire position wave (bra-ket), which is a differential operator; and also a form that is applied to the ket, which is a matrix operator. Right now, our time evolution operator $U$ is an analytic function of $H$ which is an analytic function of $p$. This means that $U$ is an analytic function of $p$. Next,

$$\left(-i\hbar\right)^N \frac{\partial^N}{\partial x^N} \left\langle x|\alpha \right\rangle = \left\langle x|p^N|\alpha \right\rangle \tag{76}$$

This is an easy induction: suppose $\left(-i\hbar\right)^m \frac{\partial^m}{\partial x^m} \left\langle x|\alpha \right\rangle = \left\langle x|p^m|\alpha \right\rangle$. Set $\left|\alpha\right\rangle = p\left|\gamma\right\rangle$ and therefore $\left(-i\hbar\right)^{m+1} \frac{\partial^{m+1}}{\partial x^{m+1}} \left\langle x|\gamma \right\rangle = \left\langle x|p^{m+1}|\gamma \right\rangle$.

Therefore, the time evolution operator can be written as a Taylor series, and its form outside the bra-ket is in terms of partial derivatives of $x$, while its form inside is in terms of matrix operators $p$:

$$\hat{U} \left\langle x|\alpha \right\rangle = \sum_{n\geq 0} \frac{f^n\left(0\right)}{n!} \left(-i\hbar\right)^n \frac{\partial^n}{\partial x^n} \left\langle x|\alpha \right\rangle = \left\langle x\right| \sum_{n\geq 0} \frac{f^n\left(0\right)}{n!} p^n \left|\alpha\right\rangle = \left\langle x|U|\alpha \right\rangle \tag{77}$$

where the difference between $\hat{U}$ and $U$ is a differential and matrix operator. So we can say

$$K = \frac{-i}{\hbar} \left\langle x|U|x' \right\rangle \tag{78}$$

and to further determine properties of $U$, we can substitute $K$ back into the differential equation.

$$= i\hbar \left( \delta\left(t-t'\right) K\left(x,t;x',t'\right) + \Theta\left(t-t'\right) \frac{\partial K}{\partial t} \right) = H\Theta\left(t-t'\right) K + \delta\left(x-x'\right)\delta\left(t-t'\right) \tag{79}$$

$$= \delta\left(t-t'\right) \left\langle x|U|x' \right\rangle + \Theta\left(t-t'\right) \frac{\partial}{\partial t} \left\langle x|U|x' \right\rangle = H\Theta\left(t-t'\right) \left\langle x|U|x' \right\rangle + \delta\left(x-x'\right)\delta\left(t-t'\right) \tag{80}$$

Given that the $\delta\left(t-t'\right)$ terms are equal to each other, we can integrate them both over all time:

$$\delta\left(t-t'\right) \left\langle x|U|x' \right\rangle = \delta\left(x-x'\right)\delta\left(t-t'\right) \tag{81}$$

$$\int_{-\infty}^{\infty} dt\ \delta\left(t-t'\right) \left\langle x|U(t)|x' \right\rangle = \int_{-\infty}^{\infty} dt\ \delta\left(x-x'\right)\delta\left(t-t'\right) = \delta\left(x-x'\right) \tag{82}$$

$\left\langle x\right|$ and $\left|x'\right\rangle$ are independent of time:

$$\left\langle x\right| \left( \int_{-\infty}^{\infty} dt\ \delta\left(t-t'\right) U\left(t\right) \right) \left|x'\right\rangle = \delta\left(x-x'\right) \tag{83}$$

$$\left\langle x|U\left(t'\right)|x' \right\rangle = \left\langle x|x' \right\rangle \tag{84}$$

Therefore, $U\left(t'\right)$ must equal 1, so

$$U\left(t\right) = \exp\left(-\frac{i}{\hbar}H\left(t-t'\right)\right) \tag{85}$$

Finally, to obtain actual values and not mere operators, we sandwich a complete set of momentum eigenkets to the right of $U$. This is because the momentum and time evolution operator commute as time evolution is an analytic function of momentum, so momentum eigenkets are simultaneous eigenkets of both momentum and time evolution (for a free particle with constant Hamiltonian).

$$K = -\frac{i}{\hbar}\left\langle x\right| U\left(\int_{\mathbb{R}}\left|p'\right\rangle\left\langle p'\right|\ dp\right)\left|x'\right\rangle = -\frac{i}{\hbar}\int_{\mathbb{R}}\left\langle x\middle|U\middle|p'\right\rangle\left\langle p'\middle|x'\right\rangle\ dp' \tag{86}$$

$$= -\frac{i}{\hbar}\int_{\mathbb{R}}\exp\left(-\frac{i}{\hbar}\frac{p'^2}{2m}\left(t-t'\right)\right)\left\langle x\middle|p'\right\rangle\left\langle p'\middle|x'\right\rangle\ dp' \tag{87}$$

Finding the form of $\left\langle p\middle|x\right\rangle$:

$$\left\langle x'\middle|p\middle|\alpha\right\rangle = -i\hbar\frac{\partial}{\partial x'}\left\langle x'\middle|\alpha\right\rangle \tag{88}$$

Set $\left|p'\right\rangle = \left|\alpha\right\rangle$:

$$\left\langle x'\middle|p\middle|p'\right\rangle = p'\left\langle x'\middle|p'\right\rangle = -i\hbar\frac{\partial}{\partial x'}\left\langle x'\middle|p'\right\rangle \tag{89}$$

We can suppose $p'$ is fixed, and that $V(x') = \left\langle x'\middle|p'\right\rangle$. Then we get an easy ODE:

$$p'V(x') = -i\hbar\frac{\partial}{\partial x'}V(x') \tag{90}$$

$$V(x') = \left\langle x'\middle|p'\right\rangle = N\exp\left(\frac{i}{\hbar}p'x'\right) \tag{91}$$

N is:

$$\delta(x''-x') = \frac{1}{2\pi}\int_{\mathbb{R}}\exp(i(x''-x')k)dk = \left\langle x''\middle|x'\right\rangle = \int dp'\left\langle x''\middle|p'\right\rangle\left\langle p'\middle|x'\right\rangle \tag{92}$$

and setting

$$k = p/\hbar$$

we find that $N^2 = \dfrac{1}{2\pi\hbar}$. Back to $K$:

$$K = -\frac{i}{\hbar}\int_{\mathbb{R}}\left(\frac{1}{2\pi\hbar}\right)\exp\left(-\frac{i}{\hbar}\frac{p'^2}{2m}\left(t-t'\right)\right)\exp\left(\frac{i}{\hbar}p'(x-x')\right)\ dp' \tag{93}$$

Which simplifies (omitted) to, where $\tau = t-t'$ and $R = x-x'$:

$$K = -\frac{i}{\hbar}\int_{\mathbb{R}}\left(\frac{1}{2\pi\hbar}\right)\exp\left(-\frac{i\tau}{2m\hbar}\left(p'-\frac{mR}{\tau}\right)^2+\frac{imR^2}{2\hbar\tau}\right) \tag{94}$$

$$= -\frac{i}{\hbar}\frac{1}{2\pi\hbar}\exp\left(\frac{imR^2}{2\hbar\tau}\right)\int_{\mathbb{R}}\exp\left(-\frac{i\tau}{2m\hbar}\left(p'-\frac{mR}{\tau}\right)^2\right)dp \tag{95}$$

By the same Gaussian integral formula, we get

$$K = -\frac{i}{\hbar}\frac{1}{2\pi\hbar}\exp\left(\frac{imR^2}{2\hbar\tau}\right)\sqrt{\frac{2\pi m\hbar}{i\tau}} \tag{96}$$

$$= \sqrt{\frac{m}{2\pi i\hbar\tau}}\exp\left(\frac{imR^2}{2\hbar\tau}\right) = \sqrt{\frac{m}{2\pi i\hbar(t-t')}}\exp\left(\frac{im(x-x')^2}{2\hbar(t-t')}\right) \tag{97}$$

which confirms our original solution (Economou, 2006).

# 9 Discussion

Previous derivations and verifications of this Green's function solution often feel incomplete, as if you had to know the result beforehand to make those derivations. For example, the lecture notes for Physics 221B at UC Berkeley state a solution $\langle x|U|x'\rangle$, then verify the solution works afterwards (Littlejohn, 2020). This one is more robust. Even though the individual solutions involve guesses—the Fourier transform approach guesses $R = x - x'$ and the time evolution approach guesses, that the time evolution operator is included based on some proportionality conditions, the combination of both fill in each other's gaps. Additionally, the Fourier transform solution always assumes that the solution is continuous, so having the alternative solution further makes the derivation airtight.

This result is a foundational part of quantum mechanics and is mostly used as a concept in teaching. Therefore, it's often accepted to take this solution as a known result. But in the same vein, it's important to have a complete derivation to ensure people's foundational quantum mechanics knowledge is robust.

Beyond this, we can begin looking at the free particle in higher spatial dimensions (which is mostly similar to this), and even potential systems, where we add a potential term $V(x)$ to the Hamiltonian. For example, we could look at the infintite square well with a Green's function, by including the Dirac delta impulse. We can investigate things like quantum tunnelling, by defining a potential region with $V(x) = 0$, and another with $V(x) = 1$.

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## Reviewer 1

"Clearly state what this paper adds to quantum mechanics pedagogy."

- Added in Discussion section. Explained the importance of these side-by-side derivations

"Connect the Green's function to bigger concepts—it's the quantum propagator, after all. It has many interesting applications in modern cosmology, quantum field theory as well as more applied directions such as condensed matter and statistical physics."

- This feels inappropriate to include/explain in the scope of a high school paper, and if I alternatively just state these names, it seems too vague and not adding any value. To remain focus on the importance of this specific paper, I will stick to explaining the derivations.

**Discuss the physics perspective of wave packet spreading and phase evolution**

- I think this is too much for a high school journal if this is discussed wrt. the Green's function
- On the other hand, I deleted the Infinite Square Well part because it doesn't add anything new to quantum mechanics pedagogy/literature, and makes the paper very long.

**"Using this approach for simple potential systems/Extending to higher dimensions/Does current computation has interesting physics applications such as tunneling?"**

- Added in the final paragraph of the Discussion

**"Literature Review is Too Vague"**

- The specific contribution of the accessible double derivation package is explained in both the Introduction and Discussion
- Example given in Discussion section of UC Berkeley's lecture notes which only derive the result

**"Mathematical Issues That Need Cleaning Up"**

- "Too Casual" — Lagrangian derivation part removed already
- "Notation consistency" — Added equation numbers

**"Structure is Unbalanced"**

- Removed the Lagrangian section because it doesn't contribute anything.
- In a similar vein, removed the infinite square well section.

**"Missing Steps in Key Derivations"**

- The initial condition is explained in Section 8.1 in the second paragraph, and in Section 8 at the end (start of Page 6).
- The explanation for why K must be proportional to <x|x’> is done more clearly now.

**“References are Inconsistent”**

- I can’t find these issues. If you observe them please let me know.

| Change | Page number |
|---|---|
| The introduction is more specific in terms of how the derivation is different/more in-depth<br>• It is explained that although Fourier transform solutions are common, putting two solutions side-by-side is now | 2 |
| The discussion emphasizes more on what this paper adds to the pedagogical literature | 12 |
| Explained the pedagogical applications of this Green’s function derivation in the Discussion section | 12 |
| Deleted the Lagrangian mechanics Hamiltonian derivation because it takes up too much space<br><br>Also deleted the Infinite Square Well derivation at the end<br><br>In response to “4. Structure is imbalanced”. This will help prioritize the Green’s function derivations. | / |
| Explained how K is proportional to <x\|x’> more clearly. | 9 |
| Changed title to “The free-particle Green’s function in two dimensions” | 1 |
| Added an Acknowledgements section explaining that no AI was used and acknowledging my mentor | 1 |
| Explained why there is a step function in the Green’s function. | 6, Section 8.1 |
| “Add a short remark connecting the result to the path-integral kernel for a free particle to orient readers.”<br><br>Explained that K is the path integral kernel and can be obtained with Feynman’s path integral formulation. | 8 |
| Explained at the end why the Green’s function is zero prior to t’ | 6, Section 8 |

| | |
|---|---|
| Explained further problems to build upon at the end of Discussion (potential systems, multiple dimensions, tunnelling) | 12 |

# Reviewer 2

"Authorship & provenance (important)."

- Short disclosure has been added in the Acknowledgements section.

"Originality & positioning."

- The novel part—the side by side derivations—has been explained.

"Methodological comparison with takeaways."

- The scope of this paper is looking at the classic one dimensional free particle case. Though there are advantages and disadvantages that exist in various other problems, including them here seems out of the scope and confusing especially to a high school audience—if I don't get to elaborate fully on them.
- As I have not studied these topics, I feel that it is inappropriate to explain them; however, I have explained the mathematical merits of each solution and where their individual assumptions lie.

**4. Rigor and distributional details.**

In the realm of physics, I feel that it is sufficient to deal with the delta function as a function: as this is a high school journal paper, it would be too complex (and tangential to the topic) to delve into distribution theory.

**5. Consistency checks.**

I also believe this is too inappropriate for a high school journal and wold stray away from the main point.

**6. Boundary conditions and boxes.**

This section has been deleted already.

**"Light copy-editing: standardize notation, ensure consistent ħ and m, and fix minor typo-graphical issues."**

- I cannot find any of these issues. If you observe one, please let me know.

**"Add a short remark connecting the result to the path-integral kernel for a free particle to orient readers."**

- Explained that K is the path integral kernal and can be obtained with Feynman's path integral formulation.

**Improve figure/text balance: add at least one illustrative figure**

- I don't think it makes sense to add a figure for this given the scope and the topic of this paper. This is a Green's function after all (and the focus is on the derivation).

**Clarify the role of the Heaviside function**

- This is added, in Section 8.1 (First solution)

**"State Fourier-transform conventions up front (signs, 2π factors) and keep them consistent across both derivations."**

- The conventions are consistent across both derivations (the other derivation does not even have a Fourier transform), and since people use different forms of the Fourier transform often, with different shifts and signs, I think it is sufficient to stick to one without explaining the difference to "other" Fourier Transforms.

| Change | Page number |
|---|---|
| The introduction is more specific in terms of how the derivation is different/more in-depth<br>• It is explained that although Fourier transform solutions are common, putting two solutions side-by-side is now | 2 |
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| | |
|---|---|
| “Add a short remark connecting the result to the path-integral kernel for a free particle to orient readers.”<br>Explained that K is the path integral kernel and can be obtained with Feynman’s path integral formulation. | 8 |
| Explained at the end why the Green’s function is zero prior to t’ | 6, Section 8 |
| Explained further problems to build upon at the end of Discussion (potential systems, multiple dimensions, tunnelling) | 12 |

**My comments to the Author:**

1. In the Abstract, it is mentioned: "This paper will solve the one-dimensional free particle case of the Schrödinger equation," but the Title says Green's function in two dimensions. In quantum mechanics (QM), time is not treated as a dimension
(In a sense, it is not manifestly Lorentz invariance-meaning space and time directions are treated on equal footing).
The current paper deals with Standard QM. So I am not sure why the title reads two dimensions and the abstract says the opposite.
Please fix this here and in the other parts of the paper.

2. Similar confusion arises in the introduction, section 8

2. I still think the paper should discuss some further broad connections and applications of the methods you introduce.

3. In the conclusion, it is written: “We can begin looking…”
“We could look at....”
“We can investigate things like…”

It is much better if you write in a nicer and sincere fashion.
such as->
“We plan to investigate…”
“We aim to generalize such techniques....”

My recommendation is: ***accept pending minor revisions***.

The author has significantly improved the clarity and focus of the manuscript, and the main derivations now form a coherent pedagogical narrative. The key concerns from the first round have been addressed to a level I consider satisfactory for publication in this venue. However, a few minor but essential issues remain (mainly regarding the title, dimensionality, acknowledgements, and consistency of presentation) that should be corrected in a final version.

Below I include my post-revision comments intended for the author.

**Comments to the author:**

The revised manuscript is notably clearer and more focused, and it is now close to publishable form. The decision to concentrate on two complementary derivations of the retarded free-particle Green's function is pedagogically effective, and the main result you present is standard and correct. The structure is more coherent than in the original version, and your discussion of the role of the step function and causality is an improvement.

That said, there are several specific points that you should still address to strengthen the paper.

(1) Title and dimensionality: Your analysis treats a free particle in one spatial dimension with time as the second variable. The current title and phrasing may be read as implying two spatial dimensions. Please either adjust the title to something like "The retarded free-particle Green's function in one dimension" or make it explicit early in the text that the "two dimensions" under discussion are (x, t), and that only one spatial dimension is considered. As it stands, this is potentially confusing for readers familiar with Green's functions in higher dimensions.

(2) Clarify what is new (or not): Please state explicitly that the Green's function / propagator you derive is a standard textbook result, and that the contribution of the paper is expository and pedagogical: presenting two parallel derivations, emphasizing the retarded boundary condition and the appearance of the step function, and giving a transparent route that a strong student can follow. Making this explicit will prevent any impression that the work is claiming novelty where it is primarily clarifying known material

(3) Avoid phrases such as "clearly" or "it must be" where the justification has not been shown; instead, add one or two explicit intermediate steps. This will make the argument more convincing without making it significantly longer.

(4)Notation, terminology, and minor language issues:

(4a) Use "Schrödinger" (or consistently "Schrodinger") throughout; be uniform with notation for $\hbar$, m, and other symbols.

(4b) Maintain consistency in switching between operator notation and wavefunction notation; try not to mix them within a single key step without explanation.