Infinite Disk Well

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1 Abstract

This paper investigates the quantum mechanical behavior of a particle confined in two novel potential systems: the infinite disk well and the infinite ring/annulus well. By solving the time-independent Schrödinger equation in polar coordinates, we derive the wave functions and energy eigenvalues for a particle in these two potentials. For the infinite disk well, the solutions are Bessel functions of the first kind, with energy levels determined by the Bessel function roots. In the infinite annulus well, the solutions are the Bessel functions of the first and second kind. As the annulus gets infinitesimally thin, the particle's energy approaches infinity. These findings extend the traditional infinite square well problem to radially symmetric systems (Zettili, 2020).

2 Introduction

Special thanks to [redacted] who gave me the idea for this problem, and for the invaluable advice. The term "Infinite Disk Well" is also credited to him.

The infinite square well is a cornerstone of quantum mechanics (Griffiths and Schroeter, 2018). It is a simplified model to study a confined particle in one dimension (Shankar, 2012). The solutions of the infinite square well yield quantized energy levels and define the wave function behavior (Cohen-Tannoudji et al., 1977). However, some real-world systems are not square (Liboff, 2003). Some may be radially symmetric (Merzbacher, 1998). We coin the terms "infinite disk well" and "infinite ring well", which are as their name suggests: a particle in a circular disk, and a particle in an annulus/ring (Krieger et al., 1999; Meister, 2016). These extend the square well to two-dimensional polar coordinates (Zettili, 2020).

This investigation is done by solving the time-independent Schrödinger equation (TISE) for both cases (Griffiths and Schroeter, 2018). The infinite disk well confines a particle within a circular region of radius R, while the infinite ring well confines the particle to an annulus centered at radius R with a set width (Watson, 1995). These geometries are clearly more conveniently solved with polar coordinates, which yields Bessel functions for solutions (Bailey et al., 2008; Abramowitz and Stegun, 1972). We aim to derive the wave functions, energy eigenvalues, and probability distributions, and compare them to the hallmark infinite square well (Shankar, 2012).

This paper aims to fill a gap in quantum mechanics literature by providing detailed solutions for these radial systems (Merzbacher, 1998). The results may have potential applications in understanding quantum dots, nanoscale rings, and other systems in condensed matter physics (Kittel, 2005).

3 Methods

To analyze the infinite disk well and infinite ring well, we solve the TISE in two-dimensional polar coordinates (Arfken et al., 2013).

We use separation of variables, $\Psi(r, \theta) = f(r)g(\theta)$, to simplify the TISE into two separate ODEs (Griffiths and Schroeter, 2018). The angular component is simply $g(\theta) = A\exp(i\mu\theta)$, with $\mu \in \mathbb{Z}$ (Shankar, 2012). The radial ODE is a Bessel equation solved using Bessel functions of the first kind (J_μ) for the disk well and both first and second kinds (J_μ, Y_μ) for the ring well (Bailey et al., 2008; Watson, 1995). The conditions that the wave is 0 at the boundary determines the value of the wave vectors are proportional to the first Bessel function roots (Abramowitz and Stegun, 1972). Finally, we normalize the wave for the infinite disk well (Zettili, 2020).

The ring well, which is the annulus well as $\Delta r \to 0$, is solved using a first-degree Taylor expansion (Arfken et al., 2013).

4 Particle on a disk

4.1 Time-independent Schrodinger equation

We will make use of the time independent Schrodinger equation, which is

$$H\Psi = E\Psi$$

where E is an energy eigenvalue and Ψ is a function of two-dimensional position - independent of time (Griffiths and Schroeter, 2018). Our Hamiltonian operator H is the sum of the kinetic energy and potential energy (Shankar, 2012). The kinetic energy is $p^2/2m$ while the potential is a function of two-dimensional position (Cohen-Tannoudji et al., 1977).

$$H = \frac{p^2}{2m} + V$$

The momentum operator in two-dimensions is

$$p = -i\hbar\nabla$$

where

$$\nabla \equiv \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

So

$$p^2 = -\hbar^2\nabla^2$$

and the kinetic energy operator is

$$-\frac{\hbar^2}{2m}\nabla^2$$

(Zettili, 2020)

Meanwhile, we will take the potential operator in polar coordinates, as we are dealing with a disk that is symmetrical across all angles (Arfken et al., 2013). We will take a disk of radius R, therefore we define the potential as:

$$V(r,\theta) = \begin{cases} 0 & r \leq R \\ \infty & r > R \end{cases}$$

so our Hamiltonian is

$$H = -\hbar^2\nabla^2 + V(r,\theta)$$

(Griffiths and Schroeter, 2018)

Next, we want the 2D Laplacian ∇^2 in polar coordinates, which is:

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

(Arfken et al., 2013)

This allows us to obtain our Schrodinger equation as

$$\left(-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) + V(r,\theta)\right)\Psi(r,\theta) = E\Psi(r,\theta)$$

(Shankar, 2012)

4.2 Solving this equation

First, consider the case $r > R$ where the particle is outside of the disk. This means the particle is in the infinite potential area, which is impossible (Griffiths and Schroeter, 2018). Therefore, the probability of the particle being outside the disk is 0:

$$|\Psi(r,\theta)|^2 = 0, \quad r > R$$

This clearly implies that Ψ is 0 outside of $r = R$. We impose a Dirichlet boundary condition here:

$$\Psi(r,\theta) = 0, \quad r > R$$

(Zettili, 2020)

Next, we consider inside the disk. In this region, the potential entirely disappears, so we have

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\Psi(r,\theta)=E\Psi(r,\theta)$$

(Cohen-Tannoudji et al., 1977)

Since there is no potential, our energy eigenvalue is just the kinetic energy eigenvalue, which is $E = p'^2/2m$. We will use the wave vector, in $p' = \hbar k$.

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\Psi(r,\theta)=\frac{\hbar^2 k^2}{2m}\Psi(r,\theta)$$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\Psi(r,\theta)=-k^2\Psi(r,\theta)$$

(Shankar, 2012)

We initially assume our PDE is separable, so that we can define

$$\Psi(r,\theta)=f(r)g(\theta)$$

(Arfken et al., 2013) Substituting:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)f(r)g(\theta)=-k^2 f(r)g(\theta)$$

$$g(\theta)\frac{\partial^2 f}{\partial r^2}+g(\theta)\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}f(r)\frac{\partial^2 g}{\partial \theta^2}=-k^2 f(r)g(\theta)$$

Dividing both sides by $f(r)g(\theta)$:

$$\frac{1}{f(r)}\frac{\partial^2 f}{\partial r^2}+\frac{1}{f(r)}\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{1}{g(\theta)}\frac{\partial^2 g}{\partial \theta^2}=-k^2$$

Multiplying both sides by r^2:

$$\frac{r^2}{f(r)}\frac{\partial^2 f}{\partial r^2}+\frac{r}{f(r)}\frac{\partial f}{\partial r}+\frac{1}{g(\theta)}\frac{\partial^2 g}{\partial \theta^2}=-r^2k^2$$

$$=-\frac{1}{g(\theta)}\frac{\partial^2 g}{\partial \theta^2}$$

Now, we can separate the PDE into two parts. As one side depends on r and the other on θ, we set them both equal to some constant μ^2. First, the θ part:

4.2.1 Solving for $g(\theta)$

$$-\frac{1}{g(\theta)}\frac{\partial^2 g}{\partial\theta^2} = \mu^2$$

$$\frac{\partial^2 g}{\partial\theta^2} = -\mu^2 g(\theta)$$

The two solutions are

$$g(\theta) = A\exp(i\mu\theta) + B\exp(-i\mu\theta)$$

(Arfken et al., 2013) However, the ring and therefore system is cylindrically symmetrical, so the probability of a particle being at a position must be invariant under rotation (Merzbacher, 1998). In other words, $|\Psi|^2 =$ **constant** for fixed θ. Therefore, $|g(\theta)|$ must be identically constant, so either A or B must be zero.

$$g(\theta) = A\exp(\pm i\mu\theta)$$

Since 0 and 2π are at the same place, we need $g(0) = g(2\pi)$, so $\mu \in \mathbb{Z}$ and

$$g(\theta) = A\exp(i\mu\theta)$$

or

$$g(\theta) = B\exp(-i\mu\theta)$$

(Shankar, 2012)

4.2.2 Solving for $f(r)$

Next, the difficult part - the r part.

$$\frac{r^2}{f(r)}\frac{\partial^2 f}{\partial r^2} + \frac{r}{f(r)}\frac{\partial f}{\partial r} + r^2k^2 = \mu^2$$

$$r^2\frac{\partial^2 f}{\partial r^2} + r\frac{\partial f}{\partial r} + r^2k^2 f(r) - \mu^2 f(r) = 0$$

We do a transformation, $x = kr$. Then $f(r) = f(x/k)$ and we can set a new function for now $y(x) = f(x/k) = f(r)$. This means

$$f'(r) = \frac{df}{dr} = \frac{dy}{dr} = y'(x)\cdot\frac{dx}{dr} = ky'(x)$$

$$f''(r) = \frac{d}{dr}(ky'(x)) = k^2y''(x)$$

Substituting back, we get:

$$r^2k^2y''(x) + rky(x) + (r^2k^2 - \mu^2)y(x) = 0$$

Using $x = kr$:

$$x^2y''(x) + xy(x) + (x^2 - \mu^2)y(x) = 0$$

This is the Bessel differential equation (Bailey et al., 2008; Abramowitz and Stegun, 1972), with two solutions which are the Bessel functions: $J_\mu(x)$ and $Y_\mu(x)$. We can write

$$y(x) = c_1 J_\mu(x) + c_2 Y_\mu(x)$$

and

$$f(r) = c_1 J_\mu(kr) + c_2 Y_\mu(kr)$$

(Watson, 1995)

4.2.3 Boundary conditions on r

The form of Y is complicated - it is defined as

$$Y_\mu(x) = \frac{J_\mu(x)\cos(\mu\pi) - J_{-\mu}(x)}{\sin(\mu\pi)}$$

(Abramowitz and Stegun, 1972)

where the form of J_μ is

$$J_\mu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(\mu+m+1)} \left(\frac{x}{2}\right)^{\mu+2m}$$

(Watson, 1995)

Near $x = 0$, the first $m = 0$ term clearly dominates, so we have an approximation

$$J_\mu(x) \approx \frac{1}{\Gamma(\mu+1)} \left(\frac{x}{2}\right)^{\mu}$$

(Abramowitz and Stegun, 1972)

We can show that Y_μ has a singularity at $x = 0$ for all μ. Consider first the case of non-integer μ, so the denominator $\sin(\mu\pi) \neq 0$. Using the approximation:

$$Y_\mu(x) \approx \frac{\frac{1}{\Gamma(\mu+1)}\left(\frac{x}{2}\right)^{\mu}\cos(\mu\pi) - \frac{1}{\Gamma(-\mu+1)}\left(\frac{x}{2}\right)^{-\mu}}{\sin(\mu\pi)}$$

(Watson, 1995)

The right term, which comes from $J_{-\mu}(x)$, dominates because as $x \to 0^+$, $x^{-\mu} \to \infty$. If μ was negative, the left term would dominate. This demonstrates that for non-integer μ, $Y_\mu(x)$ has a singularity at $x = 0$ of order μ (Olver et al., 2010).

Second, in the case of $\mu = 0$, it is well defined that near $x = 0$

$$Y_0(x) \approx \frac{2\gamma}{\pi} + \frac{2}{\pi}\log\left(\frac{x}{2}\right) + \text{higher order terms}$$

(Abramowitz and Stegun, 1972)

which shows that at $x = 0$, there is a logarithmic singularity (Watson, 1995).

Third, for integer $\mu \geq 1$, we write the approximate expansion in more detail:

$$Y_\mu(x) \approx \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + \gamma\right] J_\mu(x) - \frac{1}{\pi}\sum_{k=0}^{\mu-1} \frac{(\mu-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-\mu} + \text{other terms}$$

(Olver et al., 2010)

The first term of the sum is

$$-\frac{(\mu-1)!}{\pi}\left(\frac{x}{2}\right)^{-\mu}$$

which is a singularity of order μ (Watson, 1995).

Therefore, if we want our function to be continuous inside the disk, and especially want the wave to be defined at the origin, $c_2 = 0$, then

$$f(r) = c_1 J_\mu(kr)$$

(Bailey et al., 2008)

And $J_\mu(kr)$ is

$$J_\mu(kr) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(\mu+m+1)}\left(\frac{kr}{2}\right)^{\mu+2m}$$

(Abramowitz and Stegun, 1972)

The graph of this function represents a damped sine wave: it oscillates with decreasing amplitude, and converges for all finite kr. This means it has infinite roots (Watson, 1995).

Our next boundary condition is that we want J to approach zero as $r \to R$. Let the nth root of $J_\mu(x)$ be $j_{\mu,n}$. Then to make sure the wave function stays continuous, we must set $kR = j_{\mu,n}$ for any n, and $k = j_{\mu,n}/R$.

So we get an infinite number of solutions for each μ:

$$f_{\mu,n}(r) = c_1 J_\mu\left(\frac{j_{\mu,n}}{R} r\right)$$

(Bailey et al., 2008)

Combining our two separable solutions, we finally get

$$\Psi_{\mu,n}(r,\theta) = f_{\mu,n}(r)g(\theta)$$
$$\Psi_{\mu,n}(r,\theta) = c_3 J_\mu\left(\frac{j_{\mu,n}}{R}r\right)\exp(i\mu\theta)$$

where $c_3 = c_1c_2$ (Shankar, 2012).

Or fully expanded:

$$\Psi_{\mu,n}(r,\theta) = c_3 \exp(i\mu\theta)\sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma(m+1)\Gamma(\mu+m+1)}\left(\frac{j_{\mu,n}}{2R}\right)^{\mu+2m} r^{\mu+2m}$$

(Abramowitz and Stegun, 1972)

4.2.4 Normalizing

We have the following condition:

$$\int_0^{2\pi}\int_0^R |\Psi(r,\theta)|^2 r dr d\theta = 1$$

(Griffiths and Schroeter, 2018)

$$\int_0^{2\pi}\int_0^R \left|c_3\exp(i\mu\theta)J_\mu\left(\frac{j_{\mu,n}}{R}r\right)\right|^2 r dr d\theta = 1$$

$$= |c_3|^2\int_0^{2\pi}|\exp(i\mu\theta)|^2 d\theta\int_0^R\left|J_\mu\left(\frac{j_{\mu,n}}{R}r\right)\right|^2 r dr$$

The θ integral is just:

$$\int_0^{2\pi}|\exp(i\mu\theta)|^2 d\theta = \int_0^{2\pi} d\theta = 2\pi$$

(Arfken et al., 2013)

$$|c_3|^2 2\pi\int_0^R\left|J_\mu\left(\frac{j_{\mu,n}}{R}r\right)\right|^2 r dr = 1$$

Since the Bessel function is real, we can also remove the absolute value.

$$|c_3|^2 2\pi\int_0^R\left(J_\mu\left(\frac{j_{\mu,n}}{R}r\right)\right)^2 r dr = 1$$

Substitute $u = \frac{j_{\mu,n}}{R}r$, so $r = \frac{R}{j_{\mu,n}}u$, $dr = \frac{R}{j_{\mu,n}}du$:

$$|c_3|^2 2\pi\frac{R^2}{j_{\mu,n}^2}\int_0^{j_{\mu,n}}[J_\mu(u)]^2 u\, du = 1$$

(Olver et al., 2010)

There is a standard identity for Bessel functions, obtained from Olver et al. (2010) which is

$$\int_0^{j_{\nu,n}} [J_\nu(u)]^2 u \, du = \frac{1}{2} j_{\nu,n}^2 [J_{\nu+1}(j_{\nu,n})]^2$$

Using this in our original expression:

$$|c_3|^2 2\pi \frac{R^2}{j_{\mu,n}^2} \frac{1}{2} j_{\mu,n}^2 \left(J_{\mu+1}(j_{\mu,n})\right)^2 = 1$$

$$|c_3|^2 \pi R^2 \left(J_{\mu+1}(j_{\mu,n})\right)^2 = 1$$

$$|c_3| = \frac{1}{\sqrt{\pi} R |J_{\mu+1}(j_{\mu,n})|}$$

and without loss of generality we can remove the absolute value. This makes our final wave

$$\Psi_{\mu,n}(r,\theta) = \frac{1}{\sqrt{\pi} R |J_{\mu+1}(j_{\mu,n})|} J_\mu \left(\frac{j_{\mu,n}}{R} r \right)$$

(Zettili, 2020)

For example, for $\mu = 0$, $n = 1$, $j_{0,1} \approx 2.4048$, $J_1(j_{0,1}) \approx 0.5191$. Therefore

$$|c_3| \approx \frac{1}{0.5191\sqrt{\pi} R}$$

So for this case, without loss of generality,

$$c_3 \approx \frac{1.087}{R}$$

(Abramowitz and Stegun, 1972)

And the normalized wave, for the specific case $\mu = 0$ and $n = 1$, is

$$\Psi_{0,1}(r,\theta) \approx \frac{1.087}{R} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{j_{0,1}}{2R} \right)^{2m} r^{2m}$$

Or simply:

$$\Psi_{0,1}(r) \approx \frac{1.087}{R} J_0 \left(\frac{j_{0,1}}{R} r \right)$$

(Bailey et al., 2008)

4.3 Analyzing the wave

We must recall that $k_{\mu,n} = j_{\mu,n}/R$. This means that there are infinite energy levels, with each successive Bessel function root corresponding to a higher energy level (Watson, 1995). Each energy level will therefore correspond to a different particle probability distribution/wave function (Krieger et al., 1999).

First, we can observe the solutions with the $\mu = 0$ Bessel function of the first kind J_0, with the first four roots:

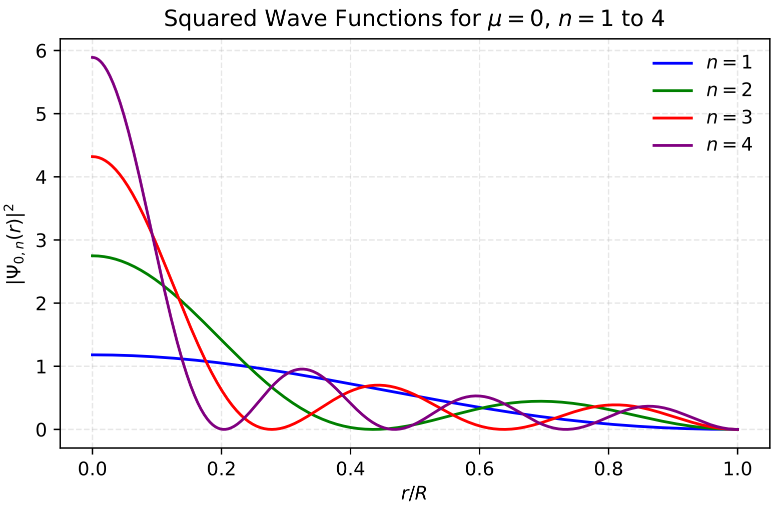


Figure 1: Squared wave functions (probability distributions) for $\mu = 0$ and $n = 1$ to 4

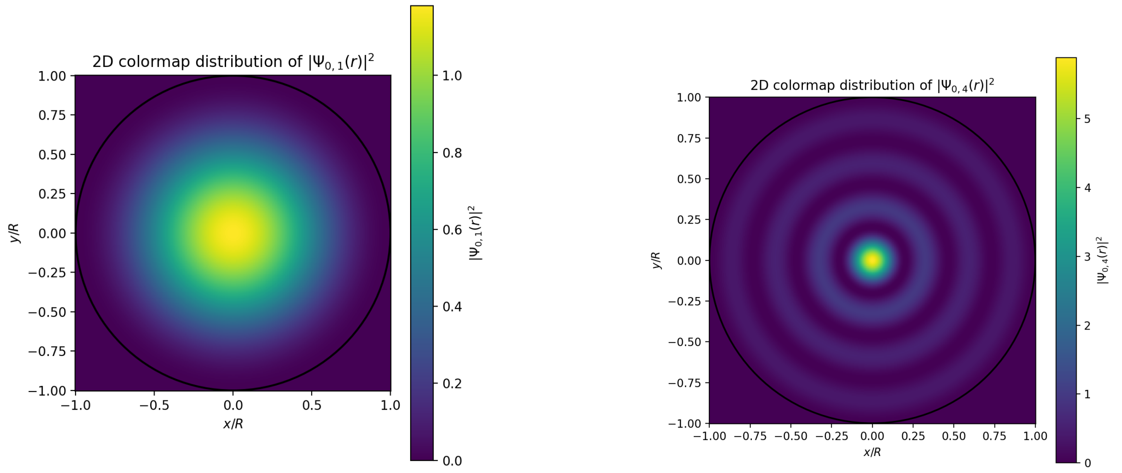
While the ground state probability distribution decreases towards the boundary, the others oscillate with a decreasing envelope, and have probability equal to zero at some radii (Griffiths and Schroeter, 2018). The number of points where the probability is zero is equal to the index of the root n (Shankar, 2012).

We can look at two cases, $n = 1$ and $n = 4$, on a colormap:

5 Infinite Ring Well

For this situation, we just need to modify our potential function to describe an annulus (Meister, 2016).

Suppose half of the width of the ring is Δr, with $\Delta r \leq R$ where R is the midpoint of the inner and



(a) Colormap of $|\Psi_{0,1}(r,\theta)|^2$

(b) Colormap of $|\Psi_{0,4}(r,\theta)|^2$

Figure 2: Colormaps of wavefunction distributions

outer radius. Then the ring can be defined with the potential function

$$V(r,\theta) = \begin{cases} 0 & |r-R| \leq \Delta r \\ \infty & |r-R| > \Delta r \end{cases}$$

(Zettili, 2020)

Restating our Schrodinger equation:

$$\left(-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) + V(r,\theta)\right)\Psi(r,\theta) = E\Psi(r,\theta)$$

(Griffiths and Schroeter, 2018)

Once again, the particle cannot escape the ring, so $\Psi = 0$ outside the ring; and we first consider the inside where the particle is free (Shankar, 2012).

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\Psi(r,\theta) = -k^2\Psi(r,\theta)$$

(Cohen-Tannoudji et al., 1977)

Since this is the same equation as earlier but just with the same boundary conditions, the initial solution formation will be the same, with

$$\Psi(r,\theta) = f_1(r)g_1(\theta)$$

(Arfken et al., 2013)

and since the constraint $g(0) = g(2\pi)$ still applies here, along with cylindrical symmetry of $|g_1(\theta)|$, we have

$$g_1(\theta) = A\exp(i\mu\theta)$$

where $\mu \in \mathbb{Z}$ (Shankar, 2012).

Since the domain is now an annulus, there is no need for continuity at $r = 0$, so we can keep the Y_μ term:

$$f_1(r) = c_1 J_\mu(kr) + c_2 Y_\mu(kr)$$

(Watson, 1995)

5.1 The ring boundary conditions

Now, we need

$$f_1(R - \Delta r) = f_1(R + \Delta r) = 0$$

(Zettili, 2020) Now, we need two roots $2\Delta r$ apart. Let $r_1 = R - \Delta r$ and $r_2 = R + \Delta r$. Then,

$$f_1(r_1) = f_1(r_2) = 0$$

$$c_1 J_\mu(kr_1) + c_2 Y_\mu(kr_1) = c_1 J_\mu(kr_2) + c_2 Y_\mu(kr_2) = 0$$

Isolating the constant c_1 to eliminate it:

$$c_1 J_\mu(kr_1) = -c_2 Y_\mu(kr_1) \to c_1 = -c_2 \frac{Y_\mu(kr_1)}{J_\mu(kr_1)}$$

Substituting this into the previous:

$$\left(-c_2 \frac{Y_\mu(kr_1)}{J_\mu(kr_1)}\right) J_\mu(kr_2) + c_2 Y_\mu(kr_2) = 0$$

$$-c_2 Y_\mu(kr_1) J_\mu(kr_2) + c_2 Y_\mu(kr_2) J_\mu(kr_1) = 0$$

$$Y_\mu(kr_1) J_\mu(kr_2) - Y_\mu(kr_2) J_\mu(kr_1) = 0$$

$$Y_\mu(kr_1) J_\mu(kr_2) = Y_\mu(kr_2) J_\mu(kr_1)$$

(Bailey et al., 2008) To determine the values of k that satisfy this equation, we first note that the zeroes of Y and J are interleaving (Bailey et al., 2008; Abramowitz and Stegun, 1972), which means that between every two adjacent zero of J, there is a zero of Y, and vice versa (Watson, 1995).

Therefore for each μ there are infinite k that satisfy the equation. We can denote the root as $k_{\mu,n}$, where $\mu \in \mathbb{Z}$ and $n \in \mathbb{Z}$. Then, the infinite energy levels are

$$E_{\mu,n} = \frac{\hbar^2 k_{\mu,n}^2}{2m}$$

(Griffiths and Schroeter, 2018)

5.2 The limiting case

We are interested to see what happens to the particle as the width of the ring limits to zero. It is possible that at $\Delta r = 0$, we should have the free particle case; but the particle may also be trapped in an infinitesimally thin ring (Meister, 2016). Either hypothesis may be true, so we must investigate.

We will pick $\mu = 0$ and $n = 1$ to evaluate this limit, for simplicity. And we need to satisfy the condition $Y_\mu(kr_1)J_\mu(kr_2) = Y_\mu(kr_2)J_\mu(kr_1)$ (Watson, 1995).

First, we write $kr_1 = kR - k\Delta r$ and $kr_2 = kR + k\Delta r$. Since Δr is small in the limit, we can use a Taylor expansion only up to the first order:

$$J_0(kr_2) = J_0(kR + k\Delta r) \approx J_0(kR) + J_0'(kR)(k\Delta r)$$

$$Y_0(kr_2) = Y_0(kR + k\Delta r) \approx Y_0(kR) + Y_0'(kR)(k\Delta r)$$

$$J_0(kr_1) = J_0(kR - k\Delta r) \approx J_0(kR) - J_0'(kR)(k\Delta r)$$

$$Y_0(kr_1) = Y_0(kR - k\Delta r) \approx Y_0(kR) - Y_0'(kR)(k\Delta r)$$

(Arfken et al., 2013)

Also, the derivatives of the Bessel functions are

$$J_0'(x) = -J_1(x) \quad \text{and} \quad Y_0'(x) = -Y_1(x)$$

(Abramowitz and Stegun, 1972)

So we can substitute this into our boundary condition $Y_0(kr_1)J_0(kr_2) = Y_0(kr_2)J_0(kr_1)$. We get on the left side:

$$Y_0(kr_1)J_0(kr_2) \approx [Y_0(kR) + Y_1(kR)(k\Delta r)]\,[J_0(kR) - J_1(kR)(k\Delta r)]$$

$$= Y_0(kR)J_0(kR) - Y_0(kR)J_1(kR)(k\Delta r) + Y_1(kR)J_0(kR)(k\Delta r) - Y_1(kR)J_1(kR)(k\Delta r)^2$$

Right side:

$$Y_0(kr_2)J_0(kr_1) \approx [Y_0(kR) - Y_1(kR)(k\Delta r)]\,[J_0(kR) + J_1(kR)(k\Delta r)]$$

$$= Y_0(kR)J_0(kR) + Y_0(kR)J_1(kR)(k\Delta r) - Y_1(kR)J_0(kR)(k\Delta r) + Y_1(kR)J_1(kR)(k\Delta r)^2$$

(Olver et al., 2010)

We can eliminate the common term $Y_0(kR)J_0(kR)$.

$$-Y_0(kR)J_1(kR)(k\Delta r) + Y_1(kR)J_0(kR)(k\Delta r)$$

$$= Y_0(kR)J_1(kR)(k\Delta r) - Y_1(kR)J_0(kR)(k\Delta r) + 2Y_1(kR)J_1(kR)(k\Delta r)^2$$

$$Y_1(kR)J_0(kR) - Y_0(kR)J_1(kR) = 2Y_1(kR)J_1(kR)(k\Delta r)^2$$

Notice the left side is the same as the Wronskian of Y_0 and J_0, $J_0(x)Y_0'(x) - Y_0(x)J_0'(x)$. It is known that it is equal to $2/\pi x$ (Olver et al., 2010). So we have:

$$\frac{2}{\pi k R} = 2Y_1(kR)J_1(kR)(k\Delta r)^2$$

$$\frac{1}{\pi k^3 R Y_1(kR) J_1(kR)} = (\Delta r)^2$$

Because Y_1 and J_1 have zeros, we cannot consider this as a continuous limit. We can consider that the envelope of Y_1 and J_1 approaches zero as kR approaches infinity (Watson, 1995). If the order of the zeros at infinity of Y_1 and J_1 add up to less than 3, then k approaching infinity is a valid solution as it would allow the left side to approach zero (Abramowitz and Stegun, 1972).

For Y_1, we can use the approximation for large arguments:

$$Y_1(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{3\pi}{4}\right)$$

(Olver et al., 2010) To consider the zero at ∞, we can simply consider the zero of $Y_1(1/z)$ at $z = 0$. We have:

$$Y_1\left(\frac{1}{z}\right) \sim \sqrt{\frac{2z}{\pi}} \sin\left(\frac{1}{z} - \frac{3\pi}{4}\right)$$

(Watson, 1995)

Since the sine function is always bounded between -1 and 1 the order of the zero is $1/2$ (Abramowitz and Stegun, 1972).

Similarly, for large x, $J_1(x)$ is

$$J_1(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{3\pi}{4}\right)$$

(Olver et al., 2010)

And similarly the zero at infinity is order $1/2$. Therefore, their zeros combined are order 1, meaning that as k^3 limits to infinity, it still dominates (Watson, 1995).

Therefore the wave vector k approaching infinity is a solution. This means that as our disk gets infinitely thin, the energy of our particle becomes unbounded inside the tiny region (Meister, 2016).

6 Discussion

First, the infinite disk well's solutions are all Bessel functions of the first kind J (Bailey et al., 2008). For each value of μ defining a different Bessel function J_μ, there are infinite solutions due to the infinite roots (Abramowitz and Stegun, 1972). As the coefficient of r is scaled so that when $r = R$, the argument of J is a root, we have infinite coefficients $j_{\mu,n}/R$ (Watson, 1995).

The roots are significant as in our solution, we set $k = j_{\mu,n}/R$. This means the roots are proportional to the wave vector, so each successive root corresponds to a quantized higher energy level, growing quadratically (Griffiths and Schroeter, 2018). Notably, the wave function is normalizable using an integral identity, and we have

$$\Psi_{\mu,n}(r,\theta) = \frac{1}{\sqrt{\pi}R|J_{\mu+1}(j_{\mu,n})|} J_\mu \left(\frac{j_{\mu,n}}{R} r \right)$$

(Zettili, 2020)

It is interesting to see what this solution means and what it looks like. It oscillates as the position varies radially—of course we have isotropic symmetry—and has rings where the probability of finding the particle is zero, equal to the order of the root n. So as the energy level increases, the oscillations get more frequent, and the amplitude at the center decreases. More energy leads to a more scattered distribution (Merzbacher, 1998).

This is similar to the infinite square well, where the solution is a sine function . However, for our infinite disk well, as we get closer to the boundary ring, the amplitude of the oscillation decreases, while it doesn't in the infinite square well (Cohen-Tannoudji et al., 1977).

In the infinite ring well/infinite annulus well, the solution involves both the Bessel function of the first kind and the second kind, J and Y (Watson, 1995). They must be of the same μ value, and for each μ, there are infinite wave vectors k that satisfy the equation. There are a lot of energy levels (Zettili, 2020).

But the most important finding is in taking the limit as the width $\Delta r \to 0$. We found that k must approach infinity in this case, so the infinitesimally thin ring does not actually approach a free particle. This is likely because even if the annulus "disappears" in the limit, the particle is still trapped inside. The energy approaching infinity is an interesting finding2.

The results support theoretical expectations for quantum systems with cylindrical symmetry, such as quantum dots or nanorings, and may possibly suggest potential applications in nanotechnology and quantum computing (Kittel, 2005; Merzbacher, 1998).

A limitation is the idealized infinite potential barrier outside of the ring/disk, which is not accurate in real life. Future work could explore finite potential wells—especially to explore quantum tunnelling—or use external fields to model more realistic scenarios. Additionally, more numerical methods can be employed to look at the probability distribution in the annulus (Arfken et al., 2013).

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Review Report: Author 100116 - Submission 100109

Infinite Disk Well

October 22, 2025

Strong Points

This paper delivers a well-executed exploration that makes a solid contribution to quantum mechanics pedagogy. The author's examination of the "infinite disk well" and "infinite ring well" proves both novel and insightful, offering a fresh take on the classic infinite square well problem through extension into two-dimensional, radially symmetric systems. This approach not only challenges students but also builds a clear bridge between textbook examples and more complex applications.

The methodological rigor stands out consistently. The step-by-step derivation in polar coordinates is thorough yet accessible, particularly in handling Bessel functions. The author correctly identifies why the Y_μ function must be excluded from the disk solution due to its singularity at the origin, presenting this justification with clear reasoning. This mathematical attention ensures solutions remain both physically meaningful and mathematically sound.

The analysis of the ring well in the infinitesimally thin annulus limit particularly impressed me. Using a first-order Taylor expansion alongside the Wronskian to show the wave vector k must diverge—leading to unbounded energy—represents sophisticated work. This non-intuitive result is derived convincingly and adds real depth to the findings.

The normalization of the disk well wave function marks another strength. Applying the Bessel function integral identity correctly, while including a numerical example for the $\mu = 0, n = 1$ state, grounds the theoretical work in concrete terms. This approach meaningfully enhances the paper's accessibility and practical utility.

The discussion effectively connects mathematical results to physical interpretations, clearly explaining how quantum numbers μ and n influence probability distributions. Linking these findings to applications in quantum dots and nanoscale rings is well-executed and underscores the research's relevance beyond pure theory. The references are well-chosen and authoritative, providing solid support for the derivations and methodologies.

Overall, this paper successfully delivers a clear, rigorous, and original investigation of these novel potential systems, making it a valuable resource for advanced students and field researchers.

Weak Points and Areas for Improvement

The paper demonstrates strong technical derivations but requires significant revisions before being suitable for publication. The most critical issue is the underdeveloped discussion section, which currently reads more as a summary than a substantive analysis. The author must substantially expand this section to connect the mathematical results with broader research contexts in condensed matter and theoretical physics. Specifically, the discussion should explore how these novel potential systems relate to active research areas such as quantum dots, topological insulators, or mesoscopic rings, and should suggest concrete applications in nanofabricated systems or device design. The discussion section should highlight potential applications of such computations in experimental physics.

The physical interpretation of results remains underdeveloped, with insufficient analysis of what the quantum numbers µ and n represent physically in these geometries or how energy level spacing differs from standard systems. Structurally, the methods section is quite brief compared to the extensive derivations, and the transition between disk and ring well analyses feels abrupt. The paper would benefit from a comparative table of key features across different well geometries and clearer explanatory text around the mathematical steps, particularly in the ring well limit analysis and normalization sections. Finally, there appears to have no equation numbers appearing in this paper.

Title suggestion

The paper does not only discuss "Infinite Disk Well" but also "infinite ring/annulus well". Therefore I think a name like this "Quantum Mechanical Behavior in Novel Potential Systems: The Infinite Disk Well and Infinite Ring Well" or something similar seems more appropraite.

Overall Assessment

With the current version I **accept with moderate revisions**. The paper is **strong** but the discussion section needs attention before I can give the decision for acceptance.

Reviewer Report

Manuscript: Infinite Disk Well

Summary

This manuscript tackles the time-independent Schrödinger equation in polar coordinates for two idealized 2D confinements with infinite walls: a circular disk and an annulus. For the disk the authors recover the familiar $R(r) \propto J_\mu(kr)$ with quantization $k = j_{\mu,n}/R$ and provide a neat closed-form normalization. For the annulus they consider $R(r) = aJ_\mu(kr) + bY_\mu(kr)$ and impose Dirichlet conditions at the inner and outer radii, yielding the standard transcendental condition that fixes k. The discussion culminates in the thin-ring limit $\Delta r \to 0$, where they argue that the allowed k (hence energy) diverges.

Assessment

The presentation is generally clear and most derivations are correct. The material, however, is largely classical. The value of the paper seems primarily pedagogical rather than conceptual: a careful walk-through of known results with emphasis on boundary conditions and limiting behavior. To strengthen the contribution, the manuscript would benefit from a more precise placement within the existing literature and from a quantitative treatment of the thin-ring asymptotics rather than a purely qualitative statement.

A striking point is authorship. The work is presented as written by a *high school student*, yet it relies on nontrivial tools (Bessel functions, Wronskians, asymptotic reasoning) and sustained algebraic manipulation. This is impressive, but also unusual for that level and warrants transparent clarification of mentorship and provenance.

Major Comments

Authorship and provenance (important). Given the stated author level, please include a brief statement describing: (i) the scope of mentorship or supervision; (ii) which parts reflect the student's original derivations versus standard results reproduced from references; (iii) what computational tools were used (CAS/LLM/code), and (iv) availability of scripts or notebooks for any numerical checks or plots. This will not diminish the work; it will contextualize it and enhance credibility.

Positioning and originality. The disk and annulus problems, including the Bessel-function machinery and the boundary-matching for the annulus, are standard. Please temper claims of novelty and add a compact "Related work" paragraph that explains exactly what is new here (e.g., a compact normalization formula, a streamlined derivation, or a pedagogical synthesis). Explicit comparisons will help readers understand the incremental contribution.

Thin-ring limit: provide a quantitative asymptotic. The conclusion that $k \to \infty$ as $\Delta r \to 0$ is physically sensible, but readers will expect a leading-order estimate. Derive an explicit scaling for the lowest radial mode (e.g., $k_{\mu,1} \sim \pi/\Delta r$ by analogy to a one-dimensional Dirichlet slab in the radial coordinate, or a uniform approximation obtained directly from the Bessel framework).

A small table or plot of numerically computed roots for decreasing Δr would convincingly support the asymptotic.

Angular sector and degeneracy. Since the geometry is rotationally symmetric, states with $\pm\mu$ are degenerate (for $\mu \neq 0$). Please add a short discussion noting this two-fold degeneracy and the fact that real superpositions yield angular standing waves. This is a useful physical point for readers.

Radial node count. Where you discuss zeros of the wavefunction in the disk, clarify that $J_\mu(j_{\mu,n}r/R)$ exhibits $n-1$ *interior* nodes; the nth zero lies at the boundary $r = R$ by construction. A sentence and a corrected caption/annotation in the relevant figure will remove potential confusion.

Figures, style, and accessibility

At present there is a pronounced imbalance between the sheer number of formulas and the very small number of figures (only two). For a pedagogical paper, more visual support will help. I recommend adding: (i) a clear schematic of the geometries (disk, annulus) with labels r_1, r_2, Δr; (ii) representative mode shapes for a few (μ, n) pairs; (iii) radial probability densities annotated with nodes; and (iv) a plot of the lowest radial wavenumber $k_{\mu,1}$ versus Δr to illustrate the requested thin-ring scaling.

Please also improve figure accessibility. In Fig. 1, use distinct line styles and markers (solid, dashed, dot–dashed, dotted; circles/squares/triangles) so curves remain distinguishable in black-and-white printing, and add a clear legend. Captions should be more detailed, guiding the reader through what is plotted, how it was computed, and what the key takeaways are.

Presentation notes

Number and consistently reference the key equations (polar Schrödinger equation, radial Bessel ODE, annulus quantization condition). Define $k_{\mu,n}$ at first use and maintain consistent subscripts thereafter. A light language and typography pass (including standardized spelling of "Schrödinger") will further polish the manuscript.

Recommendation

Revise and resubmit. The paper is a solid and careful treatment of classical material, with genuine pedagogical potential. To reach publishable form, it needs transparent authorship context, a quantitative thin-ring asymptotic supported by a small numerical check, clearer positioning relative to standard references, and stronger, more numerous figures with accessible design and explanatory captions.

The Infinite Disk & Ring Well

[redacted]

September 2025

1 Abstract

This paper investigates the quantum mechanical behavior of a particle confined in two non-conventional potential systems: the infinite disk well and the infinite ring/annulus well. By solving the time-independent Schrödinger equation in polar coordinates, we derive the wave functions and energy eigenvalues for a particle in these two potentials. For the infinite disk well, the solutions are Bessel functions of the first kind, with energy levels determined by the roots of the Bessel functions. In the infinite annulus well, the solutions are the Bessel functions of the first and second kind.

Finally, the paper crucially argues that as the annulus gets infinitesimally thin, the wave number eigenvalue approaches infinity. These findings are pedagogically valuable as they extend the traditional infinite square well problem to radially symmetric systems, and constructs an inversely proportional relationship between the annulus width and the energy eigenvalues.

2 Acknowledgements

Special thanks to [redacted] who gave me the idea for this problem, and for the invaluable advice.

No AI tools were used in the creation of this paper. Matplotlib, a python package, was used to render graphs. There is a built-in function in matplotlib to render Bessel functions and their roots.

3 Introduction

The infinite square well is a cornerstone of quantum mechanics. It is a simplified model to study a confined particle in one dimension (Shankar, 2012). The solutions of the infinite square well yield quantized energy levels and define the wave function behavior. However, it is also interesting to investigate radially symmetric systems, like a disk or an annulus., as they are quite prevalent in nature. We investigate the "infinite disk well" and "infinite ring well", which are as their name suggests: a particle in a circular disk, and a particle in an annulus/ring. These extend the square well to two-dimensional polar coordinates (Zettili, 2020).

This investigation is done by solving the time-independent Schrödinger equation (TISE) for both cases. The infinite disk well confines a particle within a circular region of radius R, while the infinite ring well confines the particle to an annulus centered at radius R with a set width. We aim to derive the wave functions, energy eigenvalues, and probability distributions, and compare them to the hallmark infinite square well.

This paper aims to fill a gap in the quantum mechanics pedagogical literature by providing detailed solutions for these radial systems (Merzbacher, 1998). The mathematical construction of the disk, annulus and ring are all synthesized into one paper.

In particular, an argument that a particle in an infintesimally thin ring has infinite energy is made. The results may have potential applications in understanding quantum dots, nanoscale rings, and other systems in condensed matter physics (Kittel, 2005).

4 Methods

To analyze the infinite disk well and infinite ring well, we solve the TISE in two-dimensional polar coordinates.

We use separation of variables, $\Psi(r,\theta) = f(r)g(\theta)$, to simplify the TISE into two separate ODEs. The angular component is simply $g(\theta) = A\exp(i\mu\theta)$, with $\mu \in \mathbb{Z}$. The radial ODE is a Bessel equation solved using Bessel functions of the first kind (J_μ) for the disk well and both first and second kinds (J_μ, Y_μ) for the ring well. The conditions that the wave is 0 at the boundary determines the value of the wave vectors are proportional to the first Bessel function roots (Abramowitz and Stegun, 1972). Finally, we normalize the wave for the infinite disk well using Bessel function identities. We find a result that the particle wave oscillates in a damped manner as it varies radially, so we use technology to model the probability distribution of the wave, finding that higher energy particles are more concentrated at the middle.

We also observe the relationships between μ and angular momentum, and the infinite energy levels defined by n, which are the indices of the Bessel function roots.

The ring well, which is the annulus well as $\Delta r \to 0$, is solved using a first-degree Taylor expansion and the Wronskian for Bessel functions (Arfken et al., 2013). Finally, by analyzing the asymptotic behavior of the relation, we find an inverse proportionality between the annulus width and energy eigenvalue, and therefore that a particle in the infinite ring well has infinite energy.

5 Particle on a disk

5.1 Time-independent Schrödinger equation

We will make use of the time independent Schrödinger equation, which is

$$H\Psi = E\Psi$$

where E is an energy eigenvalue and Ψ is a function of two-dimensional position - independent of time (Griffiths and Schroeter, 2018). Our Hamiltonian operator H is the sum of the kinetic energy and potential energy (Shankar, 2012). The kinetic energy is $p^2/2m$ while the potential is a function of two-dimensional position (Cohen-Tannoudji et al., 1977).

$$H = \frac{p^2}{2m} + V$$

The momentum operator in two-dimensions is

$$p = -i\hbar\nabla$$

where

$$\nabla \equiv \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}$$

So

$$p^2 = -\hbar^2\nabla^2$$

and the kinetic energy operator is

$$-\frac{\hbar^2}{2m}\nabla^2$$

(Zettili, 2020)

Meanwhile, we will take the potential operator in polar coordinates, as we are dealing with a disk that is symmetrical across all angles (Arfken et al., 2013). We will take a disk of radius R, therefore we define the potential as:

$$V(r,\theta) = \begin{cases} 0 & r \leq R \\ \infty & r > R \end{cases}$$

so our Hamiltonian is

$$H = -\hbar^2\nabla^2 + V(r,\theta)$$

(Griffiths and Schroeter, 2018)

Next, we want the 2D Laplacian ∇^2 in polar coordinates, which is:

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

(Arfken et al., 2013)

This allows us to obtain our Schrödinger equation as

$$\left(-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) + V(r,\theta)\right)\Psi(r,\theta) = E\Psi(r,\theta)$$

5.2 Solving this equation

First, consider the case $r > R$ where the particle is outside of the disk. This means the particle is in the infinite potential area, which is impossible (Griffiths and Schroeter, 2018). Therefore, the probability of the particle being outside the disk is 0:

$$|\Psi(r,\theta)|^2 = 0, \quad r > R$$

This clearly implies that Ψ is 0 outside of $r = R$. We impose a Dirichlet boundary condition here:

$$\Psi(r,\theta) = 0, \quad r > R$$

Next, we consider inside the disk. In this region, the potential entirely disappears, so we have

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\Psi(r,\theta) = E\Psi(r,\theta)$$

(Cohen-Tannoudji et al., 1977)

Since there is no potential, our energy eigenvalue is just the kinetic energy eigenvalue, which is $E = p'^2/2m$. We will use the wave vector, in $p' = \hbar k$.

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\Psi(r,\theta) = \frac{\hbar^2 k^2}{2m}\Psi(r,\theta)$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\Psi(r,\theta) = -k^2\Psi(r,\theta)$$

We initially assume our PDE is separable, so that we can define

$$\Psi(r,\theta) = f(r)g(\theta)$$

Substituting:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)f(r)g(\theta) = -k^2 f(r)g(\theta)$$

$$g(\theta)\frac{\partial^2 f}{\partial r^2} + g(\theta)\frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}f(r)\frac{\partial^2 g}{\partial \theta^2} = -k^2 f(r)g(\theta)$$

Dividing both sides by $f(r)g(\theta)$:

$$\frac{1}{f(r)}\frac{\partial^2 f}{\partial r^2} + \frac{1}{f(r)}\frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{1}{g(\theta)}\frac{\partial^2 g}{\partial \theta^2} = -k^2$$

Multiplying both sides by r^2:

$$\frac{r^2}{f(r)}\frac{\partial^2 f}{\partial r^2} + \frac{r}{f(r)}\frac{\partial f}{\partial r} + \frac{1}{g(\theta)}\frac{\partial^2 g}{\partial \theta^2} = -r^2 k^2$$

$$= -\frac{1}{g(\theta)}\frac{\partial^2 g}{\partial \theta^2}$$

Now, we can separate the PDE into two parts. As one side depends on r and the other on θ, we set them both equal to some constant μ^2. First, the θ part:

5.2.1 Solving for $g(\theta)$

$$-\frac{1}{g(\theta)}\frac{\partial^2 g}{\partial \theta^2} = \mu^2$$

$$\frac{\partial^2 g}{\partial \theta^2} = -\mu^2 g(\theta)$$

The two solutions are

$$g(\theta) = A\exp(i\mu\theta) + B\exp(-i\mu\theta)$$

However, the ring and therefore system is cylindrically symmetrical, so the probability of a particle being at a position must be invariant under rotation (Merzbacher, 1998). In other words, $|\Psi|^2 =$ **constant** for fixed θ. Therefore, $|g(\theta)|$ must be identically constant, so either A or B must be zero.

$$g(\theta) = A\exp(\pm i\mu\theta)$$

Since 0 and 2π are at the same place, we need $g(0) = g(2\pi)$, so $\mu \in \mathbb{Z}$ and

$$g(\theta) = A\exp(i\mu\theta)$$

or

$$g(\theta) = B\exp(-i\mu\theta)$$

5.2.2 Solving for $f(r)$

Next, the difficult part - the r part.

$$\frac{r^2}{f(r)}\frac{\partial^2 f}{\partial r^2} + \frac{r}{f(r)}\frac{\partial f}{\partial r} + r^2k^2 = \mu^2$$

$$r^2\frac{\partial^2 f}{\partial r^2} + r\frac{\partial f}{\partial r} + r^2k^2 f(r) - \mu^2 f(r) = 0$$

We do a transformation, $x = kr$. Then $f(r) = f(x/k)$ and we can set a new function for now $y(x) = f(x/k) = f(r)$. This means

$$f'(r) = \frac{df}{dr} = \frac{dy}{dr} = y'(x)\cdot\frac{dx}{dr} = ky'(x)$$

$$f''(r) = \frac{d}{dr}(ky'(x)) = k^2y''(x)$$

Substituting back, we get:

$$r^2k^2y''(x) + rky'(x) + (r^2k^2 - \mu^2)y(x) = 0$$

Using $x = kr$:

$$x^2y''(x) + xy'(x) + (x^2 - \mu^2)y(x) = 0 \tag{5.1}$$

This is the Bessel differential equation (Bailey et al., 2008; Abramowitz and Stegun, 1972), with two solutions which are the Bessel functions: $J_\mu(x)$ and $Y_\mu(x)$. We can write

$$y(x) = c_1J_\mu(x) + c_2Y_\mu(x)$$

and

$$f(r) = c_1J_\mu(kr) + c_2Y_\mu(kr)$$

5.2.3 Boundary conditions on r

The form of Y is complicated - it is defined as

$$Y_\mu(x) = \frac{J_\mu(x)\cos(\mu\pi) - J_{-\mu}(x)}{\sin(\mu\pi)}$$

(Abramowitz and Stegun, 1972)

where the form of J_μ is

$$J_\mu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(\mu+m+1)} \left(\frac{x}{2}\right)^{\mu+2m}$$

(Watson, 1995)

Near $x = 0$, the first $m = 0$ term clearly dominates, so we have an approximation

$$J_\mu(x) \approx \frac{1}{\Gamma(\mu+1)} \left(\frac{x}{2}\right)^{\mu}$$

We can show that Y_μ has a singularity at $x = 0$ for all μ. Consider first the case of non-integer μ, so the denominator $\sin(\mu\pi) \neq 0$. Using the approximation:

$$Y_\mu(x) \approx \frac{\frac{1}{\Gamma(\mu+1)}\left(\frac{x}{2}\right)^{\mu}\cos(\mu\pi) - \frac{1}{\Gamma(-\mu+1)}\left(\frac{x}{2}\right)^{-\mu}}{\sin(\mu\pi)}$$

(Watson, 1995)

The right term, which comes from $J_{-\mu}(x)$, dominates because as $x \to 0^+$, $x^{-\mu} \to \infty$. If μ was negative, the left term would dominate. This demonstrates that for non-integer μ, $Y_\mu(x)$ has a singularity at $x = 0$ of order μ (Olver et al., 2010).

Second, in the case of $\mu = 0$, it is well defined that near $x = 0$

$$Y_0(x) \approx \frac{2\gamma}{\pi} + \frac{2}{\pi} \log\left(\frac{x}{2}\right) + \text{higher order terms}$$

(Abramowitz and Stegun, 1972)

which shows that at $x = 0$, there is a logarithmic singularity (Watson, 1995).

Third, for integer $\mu \geq 1$, we write the approximate expansion in more detail:

$$Y_\mu(x) \approx \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + \gamma\right] J_\mu(x) - \frac{1}{\pi}\sum_{k=0}^{\mu-1} \frac{(\mu-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-\mu} + \text{other terms}$$

(Olver et al., 2010)

The first term of the sum is

$$-\frac{(\mu-1)!}{\pi}\left(\frac{x}{2}\right)^{-\mu}$$

which is a singularity of order μ.

Therefore, if we want our function to be continuous inside the disk, and especially want the wave to be defined at the origin, $c_2 = 0$, then

$$f(r) = c_1 J_\mu(kr)$$

(Bailey et al., 2008)

And $J_\mu(kr)$ is

$$J_\mu(kr) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(\mu+m+1)}\left(\frac{kr}{2}\right)^{\mu+2m}$$

The graph of this function represents a damped sine wave: it oscillates with decreasing amplitude, and converges for all finite kr. This means it has infinite roots.

Our next boundary condition is that we want J to approach zero as $r \to R$. Let the nth root of $J_\mu(x)$ be $j_{\mu,n}$. Then to make sure the wave function stays continuous, we must set $kR = j_{\mu,n}$ for any n, and $k = j_{\mu,n}/R$.

So we get an infinite number of solutions for each μ:

$$f_{\mu,n}(r) = c_1 J_\mu\left(\frac{j_{\mu,n}}{R} r\right)$$

Combining our two separable solutions, we finally get

$$\Psi_{\mu,n}(r,\theta) = f_{\mu,n}(r)g(\theta)$$
$$\Psi_{\mu,n}(r,\theta) = c_3 J_\mu\left(\frac{j_{\mu,n}}{R}r\right)\exp(i\mu\theta)$$

where $c_3 = c_1c_2$ (Shankar, 2012).

Or fully expanded:

$$\Psi_{\mu,n}(r,\theta) = c_3\exp(i\mu\theta)\sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma(m+1)\Gamma(\mu+m+1)}\left(\frac{j_{\mu,n}}{2R}\right)^{\mu+2m} r^{\mu+2m}$$

(Abramowitz and Stegun, 1972)

5.2.4 Normalizing

We have the following condition:

$$\int_0^{2\pi}\int_0^R |\Psi(r,\theta)|^2 r dr d\theta = 1$$

(Griffiths and Schroeter, 2018)

$$\int_0^{2\pi}\int_0^R \left|c_3\exp(i\mu\theta)J_\mu\left(\frac{j_{\mu,n}}{R}r\right)\right|^2 r dr d\theta = 1$$
$$= |c_3|^2\int_0^{2\pi}|\exp(i\mu\theta)|^2 d\theta\int_0^R\left|J_\mu\left(\frac{j_{\mu,n}}{R}r\right)\right|^2 r dr$$

The θ integral is just:

$$\int_0^{2\pi}|\exp(i\mu\theta)|^2 d\theta = \int_0^{2\pi} d\theta = 2\pi$$

(Arfken et al., 2013)

$$|c_3|^2 2\pi\int_0^R\left|J_\mu\left(\frac{j_{\mu,n}}{R}r\right)\right|^2 r dr = 1$$

Since the Bessel function is real, we can also remove the absolute value.

$$|c_3|^2 2\pi\int_0^R\left(J_\mu\left(\frac{j_{\mu,n}}{R}r\right)\right)^2 r dr = 1$$

Substitute $u = \frac{j_{\mu,n}}{R}r$, so $r = \frac{R}{j_{\mu,n}}u$, $dr = \frac{R}{j_{\mu,n}}du$:

$$|c_3|^2 2\pi\frac{R^2}{j_{\mu,n}^2}\int_0^{j_{\mu,n}}[J_\mu(u)]^2 u\,du = 1$$

(Olver et al., 2010)

There is a standard identity for Bessel functions, obtained from Olver et al. (2010) which is

$$\int_0^{j_{\nu,n}} [J_\nu(u)]^2 u \, du = \frac{1}{2} j_{\nu,n}^2 [J_{\nu+1}(j_{\nu,n})]^2$$

Using this in our original expression:

$$|c_3|^2 2\pi \frac{R^2}{j_{\mu,n}^2} \frac{1}{2} j_{\mu,n}^2 \left(J_{\mu+1}(j_{\mu,n})\right)^2 = 1$$

$$|c_3|^2 \pi R^2 \left(J_{\mu+1}(j_{\mu,n})\right)^2 = 1$$

$$|c_3| = \frac{1}{\sqrt{\pi} R |J_{\mu+1}(j_{\mu,n})|}$$

and without loss of generality we can remove the absolute value. This makes our final wave

$$\Psi_{\mu,n}(r,\theta) = e^{i\mu\theta} \frac{1}{\sqrt{\pi} R |J_{\mu+1}(j_{\mu,n})|} J_\mu\left(\frac{j_{\mu,n}}{R} r\right)$$

(Zettili, 2020)

For example, for $\mu = 0$, $n = 1$, $j_{0,1} \approx 2.4048$, $J_1(j_{0,1}) \approx 0.5191$. Therefore

$$|c_3| \approx \frac{1}{0.5191\sqrt{\pi} R}$$

So for this case, without loss of generality,

$$c_3 \approx \frac{1.087}{R}$$

And the normalized wave, for the specific case $\mu = 0$ and $n = 1$, is

$$\Psi_{0,1}(r,\theta) \approx \frac{1.087}{R} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{j_{0,1}}{2R}\right)^{2m} r^{2m}$$

Or simply:

$$\Psi_{0,1}(r) \approx \frac{1.087}{R} J_0\left(\frac{j_{0,1}}{R} r\right)$$

5.3 Analyzing the wave

We must recall that $k_{\mu,n} = j_{\mu,n}/R$. This means that there are infinite energy levels, with each successive Bessel function root corresponding to a higher energy level. Each energy level will therefore correspond to a different particle probability distribution/wave function.

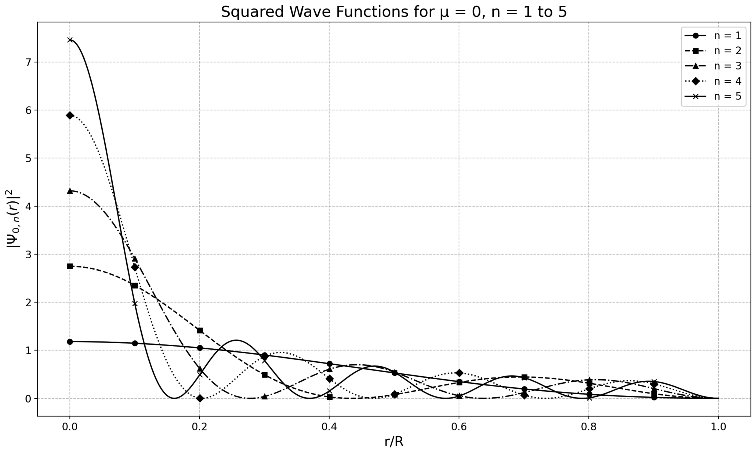
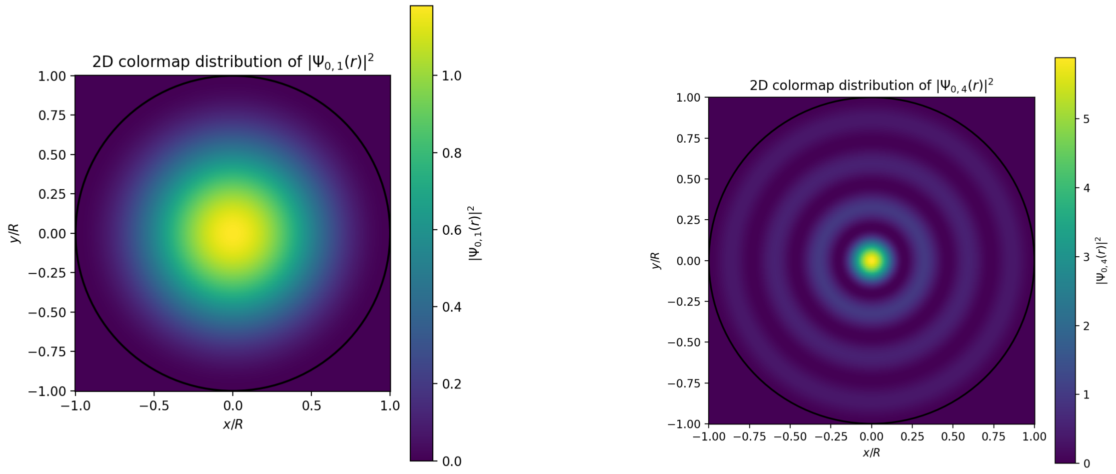


Figure 1: Squared wave functions (probability distributions) for $\mu = 0$ and $n = 1$ to 4

First, we can observe the solutions with the $\mu = 0$ Bessel function of the first kind J_0, with the first four roots:

While the ground state probability distribution decreases towards the boundary, the others oscillate with a decreasing envelope, and have probability equal to zero at some radii. The number of points where the probability is zero is equal to the index of the root n. By construction, we place the nth root at the boundary.

We can look at two cases, $n = 1$ and $n = 4$, on a colormap:



(a) Colormap of $|\Psi_{0,1}(r, \theta)|^2$

(b) Colormap of $|\Psi_{0,4}(r, \theta)|^2$

Figure 2: Colormaps of wavefunction distributions

5.4 Quantum angular momentum

In our wave function, we have a θ-dependent term.

$$\Psi_{\mu,n}(r, \theta) = e^{i\mu\theta} \frac{1}{\sqrt{\pi} R |J_{\mu+1}(j_{\mu,n})|} J_\mu \left(\frac{j_{\mu,n}}{R} r \right)$$

In quantum mechanics, the angular momentum operator L_z is defined as

$$L_z \equiv -i\hbar \frac{\partial}{\partial \theta}$$

(Griffiths and Schroeter, 2018) We can apply this to our wave function noting that we have separate θ-dependent and r-dependent factors.

$$L_z \Psi = L_z(f(r)g(\theta)) = -i\hbar f(r) g'(\theta)$$

$$L_z \Psi = (-i\hbar)(i\mu)\Psi(r, \theta) = \hbar\mu\Psi$$

So the constant μ is interpreted as proportional to the quantum angular momentum. The angular momentum eigenvalue is $\hbar\mu$.

Because our potential is invariant under rotational displacements, this causes μ to be a conserved quantity by Noether's theorem through the continuous symmetries of our wave function. This is why μ must be a constant.

6 Infinite Ring Well

For this situation, we just need to modify our potential function to describe an annulus (Meister, 2016).

Suppose half of the width of the ring is Δr, with $\Delta r \leq R$ where R is the midpoint of the inner and outer radius. Then the ring can be defined with the potential function

$$V(r,\theta) = \begin{cases} 0 & |r-R| \leq \Delta r \\ \infty & |r-R| > \Delta r \end{cases}$$

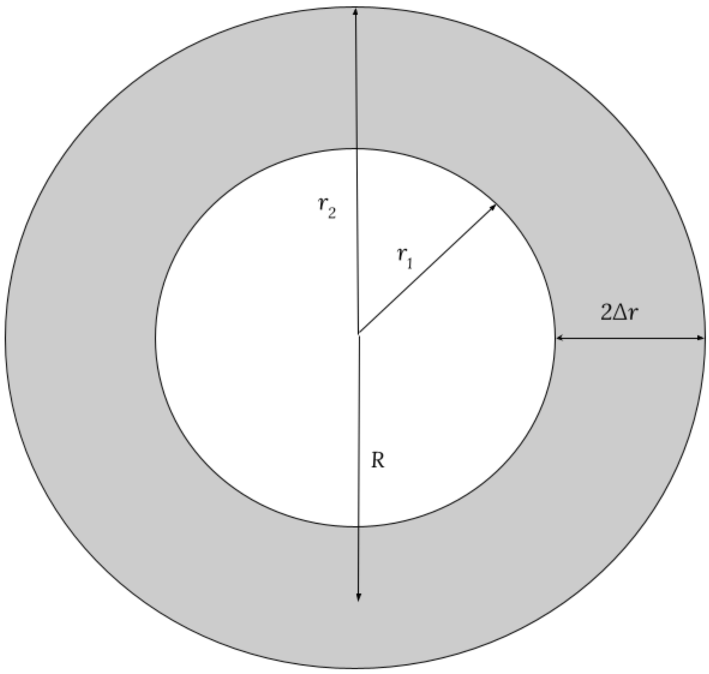


Figure 3: Diagram of the ring

Restating our Schrödinger equation:

$$\left(-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) + V(r,\theta)\right)\Psi(r,\theta) = E\Psi(r,\theta) \tag{6.1}$$

(Griffiths and Schroeter, 2018)

Once again, the particle cannot escape the ring, so $\Psi = 0$ outside the ring; and we first consider the inside where the particle is free (Shankar, 2012).

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\Psi(r,\theta) = -k^2\Psi(r,\theta)$$

Since this is the same equation as earlier but just with the same boundary conditions, the initial solution formation will be the same, with

$$\Psi(r,\theta) = f_1(r)g_1(\theta)$$

and since the constraint $g(0) = g(2\pi)$ still applies here, along with cylindrical symmetry of $|g_1(\theta)|$, we have

$$g_1(\theta) = A\exp(i\mu\theta)$$

where $\mu \in \mathbb{Z}$.

Since the domain is now an annulus, there is no need for continuity at $r = 0$, so we can keep the Y_μ term:

$$f_1(r) = c_1 J_\mu(kr) + c_2 Y_\mu(kr)$$

6.1 The ring boundary conditions

Now, we need

$$f_1(R - \Delta r) = f_1(R + \Delta r) = 0$$

Now, we need two roots $2\Delta r$ apart. Let $r_1 = R - \Delta r$ and $r_2 = R + \Delta r$. Then,

$$f_1(r_1) = f_1(r_2) = 0$$

$$c_1 J_\mu(kr_1) + c_2 Y_\mu(kr_1) = c_1 J_\mu(kr_2) + c_2 Y_\mu(kr_2) = 0$$

Isolating the constant c_1 to eliminate it:

$$c_1 J_\mu(kr_1) = -c_2 Y_\mu(kr_1) \rightarrow c_1 = -c_2\frac{Y_\mu(kr_1)}{J_\mu(kr_1)}$$

Substituting this into the previous:

$$\left(-c_2\frac{Y_\mu(kr_1)}{J_\mu(kr_1)}\right) J_\mu(kr_2) + c_2 Y_\mu(kr_2) = 0$$

$$-c_2 Y_\mu(kr_1)J_\mu(kr_2) + c_2 Y_\mu(kr_2)J_\mu(kr_1) = 0$$

$$Y_\mu(kr_1)J_\mu(kr_2) - Y_\mu(kr_2)J_\mu(kr_1) = 0$$

$$Y_\mu(kr_1)J_\mu(kr_2) = Y_\mu(kr_2)J_\mu(kr_1) \tag{6.2}$$

To determine the values of k that satisfy this equation, we first note that the zeroes of Y and J are interleaving (Bailey et al., 2008; Abramowitz and Stegun, 1972), which means that between every two adjacent zero of J, there is a zero of Y, and vice versa.

Therefore for each μ there are infinite k that satisfy the equation. We can denote the root as $k_{\mu,n}$, where $\mu \in \mathbb{Z}$ and $n \in \mathbb{Z}$. Then, the infinite energy levels are

$$E_{\mu,n} = \frac{\hbar^2 k_{\mu,n}^2}{2m}$$

We have shown an analytic quantized condition for k for an annulus.

6.2 The limiting case

We are interested to see what happens to the particle as the width of the ring limits to zero. It is possible that at $\Delta r = 0$, we should have the free particle case; but the particle may also be trapped in an infinitesimally thin ring. Either hypothesis may be true, so we must investigate.

We will pick $\mu = 0$ and $n = 1$ to evaluate this limit, for simplicity. And we need to satisfy the condition $Y_\mu(kr_1)J_\mu(kr_2) = Y_\mu(kr_2)J_\mu(kr_1)$.

First, we write $kr_1 = kR - k\Delta r$ and $kr_2 = kR + k\Delta r$. Since Δr is small in the limit, we can use a Taylor expansion only up to the first order:

$$J_0(kr_2) = J_0(kR + k\Delta r) \approx J_0(kR) + J_0'(kR)(k\Delta r)$$

$$Y_0(kr_2) = Y_0(kR + k\Delta r) \approx Y_0(kR) + Y_0'(kR)(k\Delta r)$$

$$J_0(kr_1) = J_0(kR - k\Delta r) \approx J_0(kR) - J_0'(kR)(k\Delta r)$$

$$Y_0(kr_1) = Y_0(kR - k\Delta r) \approx Y_0(kR) - Y_0'(kR)(k\Delta r)$$

(Arfken et al., 2013)

Also, the derivatives of the Bessel functions are

$$J_0'(x) = -J_1(x) \quad \text{and} \quad Y_0'(x) = -Y_1(x)$$

(Abramowitz and Stegun, 1972)

So we can substitute this into our boundary condition $Y_0(kr_1)J_0(kr_2) = Y_0(kr_2)J_0(kr_1)$. We get on the left side:

$$Y_0(kr_1)J_0(kr_2) \approx [Y_0(kR) + Y_1(kR)(k\Delta r)]\,[J_0(kR) - J_1(kR)(k\Delta r)]$$

$$= Y_0(kR)J_0(kR) - Y_0(kR)J_1(kR)(k\Delta r) + Y_1(kR)J_0(kR)(k\Delta r) - Y_1(kR)J_1(kR)(k\Delta r)^2$$

Right side:

$$Y_0(kr_2)J_0(kr_1) \approx [Y_0(kR) - Y_1(kR)(k\Delta r)]\,[J_0(kR) + J_1(kR)(k\Delta r)]$$

$$= Y_0(kR)J_0(kR) + Y_0(kR)J_1(kR)(k\Delta r) - Y_1(kR)J_0(kR)(k\Delta r) + Y_1(kR)J_1(kR)(k\Delta r)^2$$

We can eliminate the common term $Y_0(kR)J_0(kR)$.

$$-Y_0(kR)J_1(kR)(k\Delta r) + Y_1(kR)J_0(kR)(k\Delta r)$$
$$= Y_0(kR)J_1(kR)(k\Delta r) - Y_1(kR)J_0(kR)(k\Delta r) + 2Y_1(kR)J_1(kR)(k\Delta r)^2$$

$$Y_1(kR)J_0(kR) - Y_0(kR)J_1(kR) = 2Y_1(kR)J_1(kR)(k\Delta r)^2$$

Notice the left side is the same as the Wronskian of Y_0 and J_0, $J_0(x)Y_0'(x) - Y_0(x)J_0'(x)$. It is known that it is equal to $2/\pi x$ (Olver et al., 2010). So we have:

$$\frac{2}{\pi kR} = 2Y_1(kR)J_1(kR)(k\Delta r)^2$$
$$\frac{1}{\pi k^3 RY_1(kR)J_1(kR)} = (\Delta r)^2$$

Because Y_1 and J_1 have zeros, we cannot consider this as a continuous limit. We can consider that the envelope of Y_1 and J_1 approaches zero as kR approaches infinity (Watson, 1995). If the order of the zeros at infinity of Y_1 and J_1 add up to less than 3, then k approaching infinity is a valid solution as it would allow the left side to approach zero (Abramowitz and Stegun, 1972).

For Y_1, we can use the approximation for large arguments:

$$Y_1(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{3\pi}{4}\right)$$

(Olver et al., 2010) To consider the zero at ∞, we can simply consider the zero of $Y_1(1/z)$ at $z = 0$. We have:

$$Y_1\left(\frac{1}{z}\right) \sim \sqrt{\frac{2z}{\pi}} \sin\left(\frac{1}{z} - \frac{3\pi}{4}\right)$$

Since the sine function is always bounded between -1 and 1, the order of the zero is $1/2$.

Similarly, for large x, $J_1(x)$ is

$$J_1(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{3\pi}{4}\right)$$

(Olver et al., 2010)

And similarly the zero at infinity is order $1/2$. Therefore, their zeros combined are order 1, meaning that as k^3 limits to infinity, it still dominates.

Therefore the wave vector k approaching infinity is a solution. This means that as our disk gets infinitely thin, the energy of our particle becomes unbounded inside the tiny region (Meister, 2016).

7 Discussion

First, the infinite disk well's solutions are all Bessel functions of the first kind J (Bailey et al., 2008). μ is proportional to the quantum angular momentum of the wave function: $L_z\Psi = \hbar\mu\Psi$. For each value of μ, defining a different Bessel function J_μ, there are infinite solutions due to the infinite roots (Abramowitz and Stegun, 1972). As the coefficient of r is scaled so that when $r = R$, the argument of J is a root, we have infinite coefficients $j_{\mu,n}/R$.

The roots are significant as in our solution, we set $k = j_{\mu,n}/R$. This means the roots are proportional to the wave vector, so each successive root corresponds to a quantized higher energy level, growing quadratically. Notably, the wave function is normalizable using an integral identity, and we have

$$\Psi_{\mu,n}(r,\theta) = \frac{1}{\sqrt{\pi}R|J_{\mu+1}(j_{\mu,n})|} J_\mu\left(\frac{j_{\mu,n}}{R}r\right)$$

In particular, because the disk is rotationally symmetric, there is a two-fold degeneracy in the angular momentum for $\pm\mu$.

It is interesting to see what this solution means and what it looks like. It oscillates as the position varies radially—of course we have isotropic symmetry—and has rings where the probability of finding the particle is zero, equal to the order of the root n. By construction, the nth root is at the boundary $r = R$, so there are $n - 1$ roots inside the circular disk. So as the energy level increases, the oscillations get more frequent, and the amplitude at the center increases. More energy leads to a more concentrated distribution (Merzbacher, 1998).

This is similar to the infinite square well, where the solution is a sine function. However, for our infinite disk well, as we get closer to the boundary ring, the amplitude of the oscillation decreases, while it doesn't in the infinite square well (Cohen-Tannoudji et al., 1977).

In the infinite ring well/infinite annulus well, the solution involves both the Bessel function of the first kind and the second kind, J and Y (Watson, 1995). They must be of the same μ value, and for each μ, there are infinite wave vectors k that satisfy the equation. There are a lot of energy levels. The derivations here are pedagogically valuable for a better understanding.

But the most important finding is in taking the limit as the width $\Delta r \to 0$. We found that k must approach infinity in this case, so the infinitesimally thin ring does not actually approach a free particle as hypothesized. This is likely because even if the annulus "disappears" in the limit, the particle is still trapped inside. The energy approaching infinity is an interesting finding.

The results may possibly suggest potential applications in nanotechnology (Kittel, 2005; Merzbacher, 1998). Specifically, this infinite disk well is the same as the radial confinement in 2D quantum dots, where electrons are trapped in tiny structures in things like semiconductors. Importantly, the explanatory power of this paper in illustrating concepts such as the discrete energy levels helps to understand work in this field.

A limitation is the idealized infinite potential barrier outside of the ring/disk, which is not accurate in real life. Future work could explore finite potential wells—especially to explore quantum tunnelling—or use external fields to model more realistic scenarios. Additionally, more numerical methods can be employed to look at the probability distribution in the annulus (Arfken et al., 2013).

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Reviewer 1

"clearer explanatory text around the mathematical steps, particularly in the ring well limit analysis and normalization sections."

- The explanations are more clarified now in both these sections.

"The discussion section should highlight potential applications of such computations in experimental physics."

- The applications to quantum dots are explained now.

"there appears to have no equation numbers appearing in this paper."

- The radial Schrodinger equation, Bessel function condition, and quantization condition on k for the annulus are numbered now.

"The physical interpretation of results remains underdeveloped, with insufficient analysis of what the quantum numbers μ and n represent physically in these geometries or how energy level spacing differs from standard systems."

- μ's relationship to the angular momentum is explained, and n's relationship to the energy level is explained. In terms of differing from standard systems, the roots' even spacing is made clear in the diagrams and text; the specific quantitative analyses of the envelope of the wave would likely distract from the point.

Change	Page number
The title is changed to be more specific	1
The radial Schrodinger equation, Bessel function condition, and quantization condition on k for the annulus are numbered now.	6, section 5.1; 12, section 6.1; 14, section 6.2
In the discussion, it is explained that the function is constructed such that the nth root is on the boundary. This is also explained next to the diagrams now.	10, 16
The explanations are more clarified now in both the ring well limit analysis and the normalization section.	14, 9
Schematic of ring geometry has been added. See Figure 3	12
Spelling of Schrodinger corrected to Schrödinger throughout	/
The applications to quantum dots are explained now in the Discussion.	17

Novelty claims are now tamed in the introduction. The emphasis is on making a detailed pedagogical solution. The introduction now explains what exactly is new, in the last paragraph.	2
The degeneracy of angular momentum eigenvalues is now explained in the discussion.	16.

Reviewer 2

"Positioning and originality"

- Novelty claims are now tamed in the introduction. The emphasis is on making a detailed pedagogical solution. The introduction now explains what exactly is new, in the last paragraph.

"Thin-ring limit: provide a qualitative asymptotic"

- A simple approximation was done to demonstrate that k approaches infinity, making a rigorous analysis for a high school audience beyond the scope of this paper. This is especially because non-infinitesimal rings were not quantitatively explored — only the analytic condition on k was included — so it would be distracting and likely confusing.

"Radial node count"

- In the discussion, it is explained that the function is constructed such that the nth root is on the boundary. This is also explained next to the diagrams now.

"Angular sector and degeneracy"

- The degeneracy is now explained in the discussion.
- In section 4.2.1, it is explained that the rotational symmetry prevents any superposition from happening, because the absolute value of the wave must be invariant under rotation. This means that I cannot claim superposition.

"Figures, style, and accessibility"

- Schematic of ring geometry has been added. See Figure 3
- There is a representative mode shape in Figure one on 4.3
- The plot for the annulus would require analyzing more about the ring well apart from the limit as the width goes to zero. For clarity purposes, I have not done as much analysis for the ring to keep focus.

"Presentation notes"

- The radial Schrodinger equation, Bessel function condition, and quantization condition on k for the annulus are numbered now.

Change	Page number
The title is changed to be more specific	1
The radial Schrodinger equation, Bessel function condition, and quantization condition on k for the annulus are numbered now.	6, section 5.1; 12, section 6.1; 14, section

	6.2
In the discussion, it is explained that the function is constructed such that the nth root is on the boundary. This is also explained next to the diagrams now.	10, 16
The explanations are more clarified now in both the ring well limit analysis and the normalization section.	14, 9
Schematic of ring geometry has been added. See Figure 3	12
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The applications to quantum dots are explained now in the Discussion.	17
Novelty claims are now tamed in the introduction. The emphasis is on making a detailed pedagogical solution. The introduction now explains what exactly is new, in the last paragraph.	2
The degeneracy of angular momentum eigenvalues is now explained in the discussion.	16.

Reviewer 1

The paper needs attention. The paper does not contain any equation numbers. This makes the paper extremely difficult to read.
The computational details near the end of page 14 and the beginning of page 15 are super difficult for the readers.
The author should describe Jμ(x) and Yμ(x): What kind of Bessel functions are they? I am happy with the newly added discussions.
In usual physics papers, the abstract should be in one paragraph, so please correct it.

<u>Verdict:</u> Accept with these (minor) changes.

Reviewer 2

<u>Proposed decision:</u> Accept with minor revisions.

The revision resolves the core clarity issues and adds the needed schematic and example mode plots; what remains are polish and small consistency items plus a light quantitative check for the thin-ring discussion.

Post-revision feedback:

(1) Thin-ring limit: Add a quick numerical sanity check to make the statement about the thin-ring limit verifiable. Fix R and choose a few ring widths Δr that shrink toward zero. For a simple case such as $\mu = 0$, solve the annulus quantization condition to obtain the lowest admissible k for each width, and plot k versus Δr. A small log–log inset or a concise trend plot that shows k increasing as Δr decreases would make the narrative concrete without adding heavy analysis. This will reassure readers that the qualitative reasoning in the text is borne out by an explicit computation.

(2) Angular sector and superposition: The revision suggests that rotational symmetry prevents superpositions. That is not quite accurate. Energy and angular momentum eigenstates form a basis, and superpositions are always allowed in principle. What symmetry gives you is conservation of the quantum number μ, not a prohibition on superpositions. I suggest rephrasing to: We present eigenstates with fixed μ; arbitrary superpositions are possible, but for energy eigenvalues and degeneracy we analyze fixed-μ states. Keep the concise explanation of the two-fold degeneracy between $+\mu$ and $-\mu$.

(3) Nodes and captions: You now state that the n-th root is at $r = R$ and there are $n - 1$ interior nodes. Please mirror this explicitly in the caption of the first figure and, for one representative plot, add vertical guide rings at the node radii. This will eliminate a common point of confusion for readers new to Bessel zeros.

(4) Figures: For all plots, ensure axis labels with units where applicable, add legends, and use line styles or markers that are distinguishable in grayscale. Expand captions by one or two sentences so that a reader can understand what was computed and the main takeaway without scanning the main text.

(5) Equations and symbols: Number the core equations consistently and cross-reference them in text. Define R, Δr, k, $j_{\mu,n}$, and all symbols at first use. Keep notation for indices and subscripts uniform.

(6) Context: Add one concrete sentence connecting the infinite disk well to pedagogical models for quantum dots, and one caveat that the infinite barrier is idealized, with a pointer to finite-well and tunneling effects as natural extensions.