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The Infinite Disk & Ring Wells

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Abstract

This paper investigates the quantum mechanical behavior of a particle confined in two non-conventional potential systems: the infinite disk well and the infinite ring/annulus well. By solving the time-independent Schrödinger equation in polar coordinates, we derive the wave functions and energy eigenvalues for a particle in these two potentials. For the infinite disk well, the solutions are Bessel functions of the first kind, with energy levels determined by the roots of the Bessel functions. In the infinite annulus well, the solutions are the Bessel functions of the first and second kind.

This paper crucially argues that as the annulus gets infinitesimally thin, the wave number eigenvalue approaches infinity. These findings are pedagogically valuable as they extend the traditional infinite square well problem to radially symmetric systems and construct an inversely proportional relationship between the annulus width and the energy eigenvalues.

Keywords: quantum mechanics, Schrödinger equation, infinite potential well, Bessel functions, polar coordinates, quantum confinement, partial differential equations (PDEs), ordinary differential equations (ODEs)

1. Introduction

The infinite square well is a cornerstone of quantum mechanics. It is a simplified model to study a confined particle in one dimension (Shankar, 2012). The solutions of the infinite square well yield quantized energy levels and define the wave function behavior. However, it is also interesting to investigate radially symmetric systems, like a disk or an annulus, as they are quite prevalent in nature. We investigate the "infinite disk well" and "infinite ring well," which are, as their names suggest, a particle in a circular disk and a particle in an annulus/ring, respectively. These extend the square well to two-dimensional polar coordinates (Zettili, 2020).

This investigation is done by solving the time-independent Schrödinger equation (TISE) for both cases. The infinite disk well confines a particle within a circular region of radius R, while the infinite ring well confines the particle to an annulus centered at radius R with a set width. We aim to derive the wave functions, energy eigenvalues, and probability distributions, and compare them to the hallmark infinite square well.

This paper aims to fill a gap in the quantum mechanics pedagogical literature by providing detailed solutions for these radial systems (Merzbacher, 1998). The mathematical constructions of the disk, annulus, and ring are all synthesized into one paper.

In particular, an argument that a particle in an infinitesimally thin ring has infinite energy is made. The results may have potential applications in understanding quantum dots, nanoscale rings, and other systems in condensed matter physics (Kittel, 2005).

2. Methods

To analyze the infinite disk well and the infinite ring well, we solve the TISE in two-dimensional polar coordinates.

We use separation of variables, $\Psi(r,\theta) = f(r)g(\theta)$, to simplify the TISE into two separate ODEs. The angular component is simply $g(\theta)$ = A$\exp(i\mu\theta)$, with $\mu \in \mathbb{Z}$. The radial ODE is a Bessel equation solved using Bessel functions of the first kind (J_μ) for the disk well and both first and second kinds (J_μ,Y_μ) for the ring well. The conditions that the wave is 0 at the boundary determine that the value of the wave vectors is proportional to the first Bessel function roots (Abramowitz and Stegun, 1972). Finally, we normalize the wave for the infinite disk well using Bessel function identities. We find a result that the particle wave oscillates in a damped manner as it varies radially, so we use technology to model the probability distribution of the wave, finding that higher energy particles are more concentrated at the middle.

We also observe the relationships between μ and angular momentum, and the infinite energy levels defined by n, which are the indices of the Bessel function roots.

The ring well, which is the annulus well as $\Delta r \to 0$, is solved using a first-degree Taylor expansion and the Wronskian for Bessel functions (Arfken et al., 2013). Finally, by analyzing the asymptotic behavior of the relation, we find an inverse proportionality between the annulus width and energy eigenvalue, and therefore that a particle in the infinite ring well has infinite energy.

3. Particle on a disk

3.1. Time-independent Schrödinger equation

We will make use of the time-independent Schrödinger equation, which is

$$H\Psi = E\Psi\,, \tag{1}$$

where E is an energy eigenvalue, and Ψ is a function of two-dimensional position independent of time (Griffiths and Schroeter, 2018). Our Hamiltonian operator H is the sum of the kinetic energy and potential energy (Shankar, 2012). The kinetic energy is $p^2/2m$ while the potential is a function of two-dimensional position (Cohen-Tannoudji et al., 1977).

$$H = \frac{p^2}{2m} + V\,. \tag{2}$$

The momentum operator in two dimensions is

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$$p = -i\hbar\nabla\,, \tag{3}$$

where

$$\nabla \equiv \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}\,. \tag{4}$$

So,

$$p^2 = -\hbar^2\nabla^2\,, \tag{5}$$

and the kinetic energy operator is

$$-\frac{\hbar^2}{2m}\nabla^2 \tag{6}$$

(Zettili, 2020).

Meanwhile, we will take the potential operator in polar coordinates, as we are dealing with a disk that is symmetrical across all angles (Arfken et al., 2013). We will take a disk of radius R. Therefore, we define the potential as

$$V(r,\theta) = \begin{cases} 0 & r \le R \\ \infty & r > R \end{cases} \tag{7}$$

So, our Hamiltonian is

$$H = -\hbar^2\nabla^2 + V(r,\theta) \tag{8}$$

(Griffiths and Schroeter, 2018).

Next, we want the 2D Laplacian ∇^2 in polar coordinates, which is:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{9}$$

(Arfken et al., 2013).

This allows us to obtain our Schrödinger equation as

$$\left(-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) + V(r,\theta)\right)\Psi(r,\theta) = E\Psi(r,\theta) \tag{10}$$

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3.2. Solving this equation

First, consider the case r > R where the particle is outside of the disk. This means the particle is in the infinite potential area, which is impossible (Griffiths and Schroeter, 2018). Therefore, the probability of the particle being outside the disk is 0:

$$|\Psi(r,\theta)|^2 = 0, \quad r > R\,. \tag{11}$$

This clearly implies that Ψ is 0 outside of r = R. We impose a Dirichlet boundary condition here:

$$\Psi(r,\theta) = 0, \quad r > R\,. \tag{12}$$

Next, we consider the inside of the disk. In this region, the potential entirely disappears, so we have

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\Psi(r,\theta) = E\Psi(r,\theta) \tag{13}$$

(Cohen-Tannoudji et al., 1977).

Since there is no potential, our energy eigenvalue is just the kinetic energy eigenvalue, which is $E = p^2/2m$. We will use the wave vector, in $p' = \hbar k$.

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\Psi(r,\theta) = \frac{\hbar^2 k^2}{2m}\Psi(r,\theta)\,. \tag{14}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\Psi(r,\theta) = -k^2\Psi(r,\theta)\,. \tag{15}$$

We initially assume our PDE is separable, so that we can define

$$\Psi(r,\theta) = f(r)g(\theta)\,. \tag{16}$$

Substituting:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)f(r)g(\theta) = -k^2 f(r)g(\theta)\,. \tag{17}$$

$$g(\theta)\frac{\partial^2 f}{\partial r^2} + g(\theta)\frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}f(r)\frac{\partial^2 g}{\partial \theta^2} = -k^2 f(r)g(\theta)\,. \tag{18}$$

Dividing both sides by $f(r)g(\theta)$:

$$\frac{1}{f(r)}\frac{\partial^2 f}{\partial r^2} + \frac{1}{f(r)}\frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{1}{g(\theta)}\frac{\partial^2 g}{\partial \theta^2} = -k^2\,. \tag{19}$$

Multiplying both sides by r^2:

$$\frac{r^2}{f(r)}\frac{\partial^2 f}{\partial r^2} + \frac{r}{f(r)}\frac{\partial f}{\partial r} + \frac{1}{g(\theta)}\frac{\partial^2 g}{\partial \theta^2} = - r^2 k^2$$

$$= - \frac{1}{g(\theta)}\frac{\partial^2 g}{\partial \theta^2} . \tag{20}$$

Now, we can separate the PDE into two parts. As one side depends on r and the other on θ, we set them both equal to some constant μ^2. First, the θ part:

Solving for $g(\theta)$

$$- \frac{1}{g(\theta)}\frac{\partial^2 g}{\partial \theta^2} = \mu^2 . \tag{21}$$

$$\frac{\partial^2 g}{\partial \theta^2} = - \mu^2 g(\theta) . \tag{22}$$

The two solutions are

$$g(\theta) = A \exp(i\mu\theta) + B \exp(- i\mu\theta). \tag{23}$$

However, the ring and therefore system is cylindrically symmetrical, so the probability of a particle being at a position must be invariant under rotation (Merzbacher, 1998). In other words, $|\Psi|^2$ = constant for fixed θ. Therefore, $|g(\theta)|$ must be identically constant, so either A or B must be zero.

$$g(\theta) = A \exp(\pm i\mu\theta). \tag{24}$$

Since 0 and 2π are at the same place, we need $g(0) = g(2\pi)$, so $\mu \in Z$ and

$$g(\theta) = A \exp(i\mu\theta), \tag{25}$$

or

$$g(\theta) = B \exp(- i\mu\theta). \tag{26}$$

Solving for $f(r)$

Next, the difficult part - the r part.

$$\frac{r^2}{f(r)}\frac{\partial^2 f}{\partial r^2} + \frac{r}{f(r)}\frac{\partial f}{\partial r} + r^2 k^2 = \mu^2 \tag{27}$$

$$r^2 \frac{\partial^2 f}{\partial r^2} + r\frac{\partial f}{\partial r} + r^2 k^2 f(r) - \mu^2 f(r) = 0 \tag{28}$$

We do a transformation, $x = kr$. Then $f(r) = f(x/k)$, and we can set a new function for now: $y(x) = f(x/k) = f(r)$. This means that

$$f'(r) = \frac{df}{dr} = \frac{dy}{dr} = y'(x) \cdot \frac{dx}{dr} = ky'(x) . \tag{29}$$

$$f''(r) = \frac{d}{dr}(ky'(x)) = k^2 y''(x)\ . \tag{30}$$

Substituting back, we get:

$$r^2 k^2 y''(x) + rky'(x) + \left(r^2k^2 - \mu^2\right)y(x) = 0\ . \tag{31}$$

Using $x = kr$:

$$x^2 y''(x) + xy'(x) + \left(x^2 - \mu^2\right)y(x) = 0\ . \tag{32}$$

This is the Bessel differential equation (Bailey et al., 2008; Abramowitz and Stegun, 1972), with two solutions which are the Bessel functions: $J_\mu(x)$ and $Y_\mu(x)$. We can write

$$y(x) = c_1 J_\mu(x) + c_2 Y_\mu(x) \tag{33}$$

and

$$f(r) = c_1 J_\mu(kr) + c_2 Y_\mu(kr)\ . \tag{34}$$

Boundary conditions on *r*

The form of Y is complicated; it is defined as

$$Y_\mu(x) = \frac{J_\mu(x)\cos(\mu\pi) - J_{-\mu}(x)}{\sin(\mu\pi)} \tag{35}$$

(Abramowitz and Stegun, 1972)

where the form of J_μ is

$$J_\mu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(\mu+m+1)}\left(\frac{x}{2}\right)^{\mu+2m} \tag{36}$$

(Watson, 1995).

Near $x = 0$, the first $m = 0$ term clearly dominates, so we have an approximation:

$$J_\mu(x) \approx \frac{1}{\Gamma(\mu+1)}\left(\frac{x}{2}\right)^{\mu}. \tag{37}$$

We can show that Y_μ has a singularity at $x = 0$ for all μ. Consider first the case of non-integer μ, so the denominator $\sin(\mu\pi) \neq 0$. Using the approximation:

$$Y_\mu(x) \approx \frac{\frac{1}{\Gamma(\mu+1)}\left(\frac{x}{2}\right)^{\mu}\cos(\mu\pi) - \frac{1}{\Gamma(-\mu+1)}\left(\frac{x}{2}\right)^{-\mu}}{\sin(\mu\pi)} \tag{38}$$

(Watson, 1995).

The right term, which comes from $J_{-\mu}(x)$, dominates because as $x \to 0+$, $x^{-\mu} \to \infty$. If μ were negative, the left term would dominate. This demonstrates that for non-integer μ, $Y_\mu(x)$ has a singularity at $x = 0$ of order μ (Olver et al., 2010).

Second, in the case of $\mu = 0$, it is well defined that near $x = 0$,

$$Y_0(x) \approx \frac{2\gamma}{\pi} + \frac{2}{\pi}\log\left(\frac{x}{2}\right) + higher - order\ terms \tag{39}$$

(Abramowitz and Stegun, 1972), which shows that at $x = 0$, there is a logarithmic singularity (Watson, 1995).

Third, for integer $\mu \geq 1$, we write the approximate expansion in more detail:

$$Y_\mu(x) \approx \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + \gamma\right]J_\mu(x) - \frac{1}{\pi}\sum_{k=0}^{\mu-1}\frac{(\mu-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-\mu} + higher - order\ terms \tag{40}$$

(Olver et al., 2010). The first term of the sum is

$$-\frac{(\mu-1)!}{\pi}\left(\frac{x}{2}\right)^{-\mu} \tag{41}$$

which is a singularity of order μ.

Therefore, if we want our function to be continuous inside the disk, and especially want the wave to be defined at the origin, $c_2 = 0$, then $f(r) = c_{1J_\mu}(kr)$ (Bailey et al., 2008), and $J_\mu(kr)$ is

$$J_\mu(kr) = \sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma(m+1)\Gamma(\mu+m+1)}\left(\frac{kr}{2}\right)^{\mu+2m}. \tag{42}$$

The graph of this function represents a damped sine wave: it oscillates with decreasing amplitude, and converges for all finite kr. This means it has infinite roots.

Our next boundary condition is that we want J to approach zero as $r \to R$. Let the nth root of $J_\mu(x)$ be $j_{\mu,n}$. Then to make sure the wave function stays continuous, we must set $kR = j_{\mu,n}$ for any n, and $k = j_{\mu,n}/R$.

So, we get an infinite number of solutions for each μ:

$$f_{\mu,n}(r) = c_1 J_\mu\left(\frac{j_{\mu,n}}{R}r\right). \tag{43}$$

Combining our two separable solutions, we finally get

$$\Psi_{\mu,n}(r,\theta) = f_{\mu,n}(r)g(\theta). \tag{44}$$

$$\Psi_{\mu,n}(r,\theta) = c_3 J_\mu\left(\frac{j_{\mu,n}}{R}r\right)\exp(i\mu\theta) \tag{45}$$

where $c_3 = c_1 c_2$ (Shankar, 2012).

Or fully expanded:

$$\Psi_{\mu,n}(r,\theta) = c_3 \exp(i\mu\theta) \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(\mu+m+1)}\left(\frac{j_{\mu,n}}{2R}\right)^{\mu+2m} r^{\mu+2m} \tag{46}$$

(Abramowitz and Stegun, 1972).

Normalizing

We have the following condition:

$$\int_0^{2\pi}\int_0^{R} |\Psi(r,\theta)|^2 r\, dr\, d\theta = 1 \tag{47}$$

(Griffiths and Schroeter, 2018) and

$$\int_0^{2\pi}\int_0^{R} \left|c_3 \exp(i\mu\theta)\, J_\mu\left(\frac{j_{\mu,n}}{R}r\right)\right|^2 r\, dr\, d\theta = 1\,. \tag{48}$$

The θ integral is just:

$$\int_0^{2\pi} |\exp(i\mu\theta)|^2 d\theta = \int_0^{2\pi} d\theta = 2\pi \tag{49}$$

(Arfken et al., 2013).

$$|c_3|^2 2\pi \int_0^R \left|J_\mu\left(\frac{j_{\mu,n}}{R}r\right)\right|^2 r dr = 1\,. \tag{50}$$

Since the Bessel function is real, we can also remove the absolute value:

$$|c_3|^2 2\pi \int_0^R \left(J_\mu\left(\frac{j_{\mu,n}}{R}r\right)\right)^2 r\, dr = 1\,. \tag{51}$$

Substitute $u = \frac{j_{\mu,n}}{R}r$:

$$r = \frac{R}{j_{\mu,n}}u,\ dr = \frac{R}{j_{\mu,n}}du\,. \tag{52}$$

$$\left|c_3\right|^2 2\pi \frac{R^2}{j_{\mu,n}^2} \int_0^{j_{\mu,n}} \left[J_\mu(u)\right]^2 u\, du = 1 \tag{53}$$

(Olver et al., 2010).

There is a standard identity for Bessel functions, obtained from Olver et al. (2010), which is

$$\int_0^{j_{\nu,n}} \left[J_\nu(u)\right]^2 u\, du = \frac{1}{2} j_{\nu,n}^2 \left[J_{\nu+1}\left(j_{\nu,n}\right)\right]^2 . \tag{54}$$

Using this in our original expression:

$$\left|c_3\right|^2 2\pi \frac{R^2}{j_{\mu,n}^2} \frac{1}{2} j_{\mu,n}^2 \left(J_{\mu+1}\left(j_{\mu,n}\right)\right)^2 = 1 . \tag{55}$$

$$\left|c_3\right|^2 \pi R^2 \left(J_{\mu+1}\left(j_{\mu,n}\right)\right)^2 = 1 . \tag{56}$$

$$\left|c_3\right| = \frac{1}{\sqrt{\pi} R \left|J_{\mu+1}\left(j_{\mu,n}\right)\right|} , \tag{57}$$

and without loss of generality, we can remove the absolute value. This makes our final wave:

$$\Psi_{\mu,n}(r,\theta) = e^{i\mu\theta} \frac{1}{\sqrt{\pi} R \left|J_{\mu+1}\left(j_{\mu,n}\right)\right|} J_\mu\left(\frac{j_{\mu,n}}{R} r\right) \tag{58}$$

(Zettili, 2020).

For example, for $\mu = 0$, $n = 1$, $j_{0,1} \approx 2.4048$, $J_1(j_{0,1}) \approx 0.5191$. Therefore,

$$\left|c_3\right| \approx \frac{1}{0.5191\sqrt{\pi} R} . \tag{59}$$

So for this case, without loss of generality,

$$c_3 \approx \frac{1.087}{R} . \tag{60}$$

Tthe normalized wave, for the specific case $\mu = 0$ and $n = 1$, is

$$\Psi_{0,1}(r,\theta) \approx \frac{1.087}{R} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{j_{0,1}}{2R}\right)^{2m} r^{2m} , \tag{61}$$

or simply

$$\Psi_{0,1}(r) \approx \frac{1.087}{R} J_0\left(\frac{j_{0,1}}{R} r\right) . \tag{62}$$

3.3. Analyzing the wave

We must recall that $k_{\mu n} = J_{\mu n}/R$, which means that there are an infinite number of energy levels, with each successive Bessel function root corresponding to a higher energy level. Each energy level will therefore correspond to a different particle probability distribution/wave function.

First, we can observe the solutions with the μ = 0 Bessel function of the first kind J_0, with the first four roots.

While the ground state probability distribution decreases towards the boundary, the others oscillate with a decreasing envelope and have probability equal to zero at some radii. The number of points where the probability is zero is equal to the index of the root n. By construction, we place the nth root at the boundary.

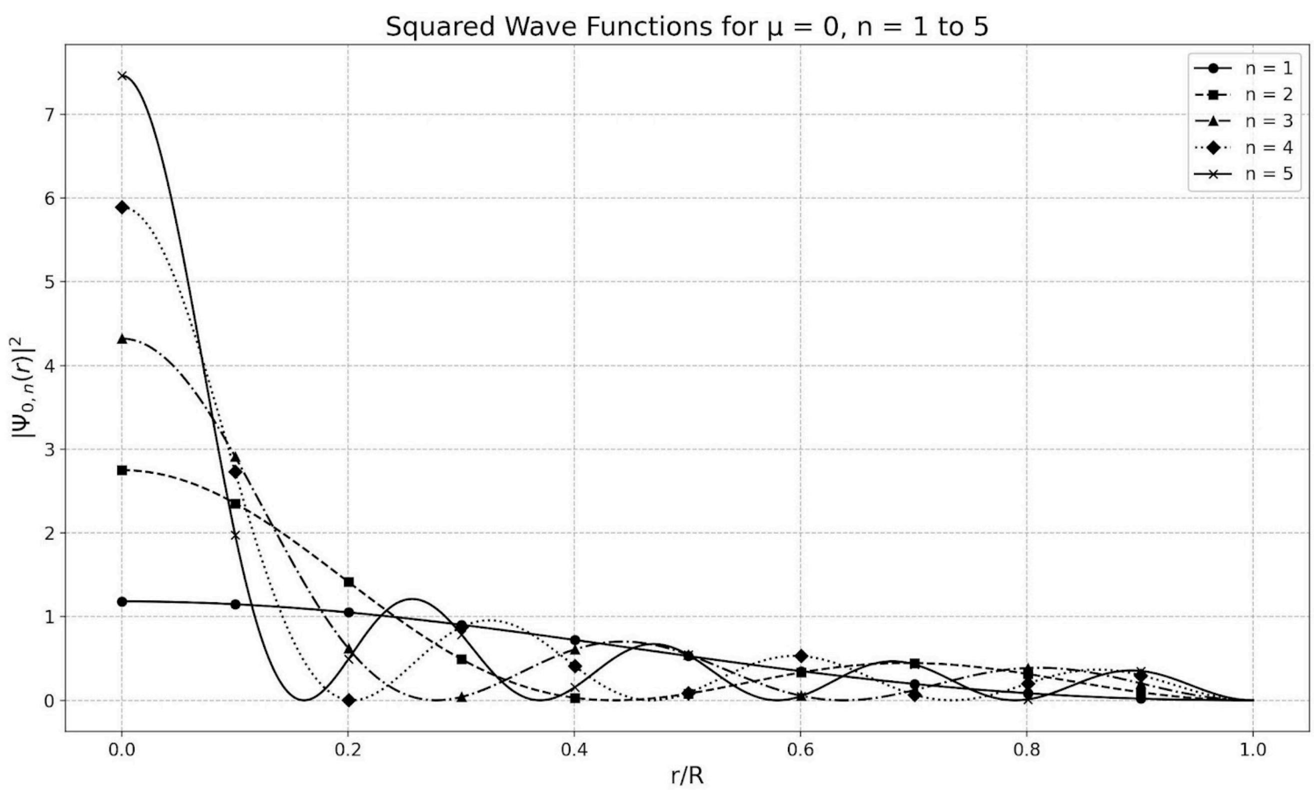
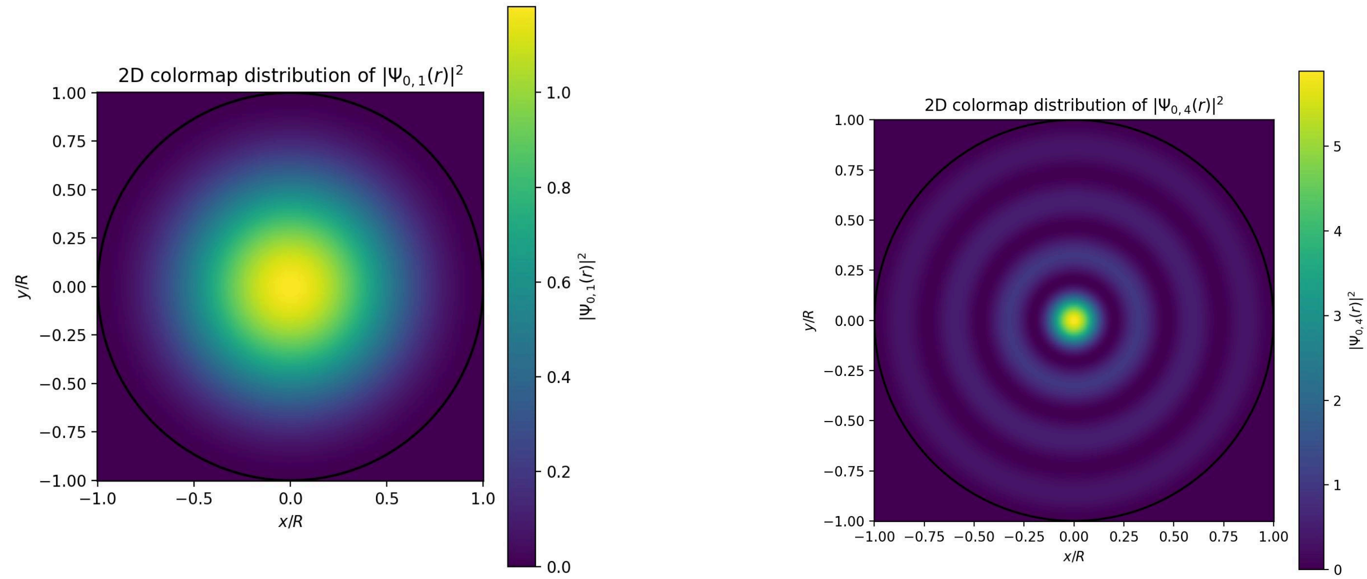


Figure 1: Squared wave functions (probability distributions) for μ = 0 and n = 1 to 4.

We can look at two cases, n = 1 and n = 4, on a colormap:



(a) Colormap of $|\Psi_{0,1}(r,\theta)|^2$

(b) Colormap of $|\Psi_{0,4}(r,\theta)|^2$

Figure 2: Colormaps of wavefunction distributions

3.4. Quantum angular momentum

In our wave function, we have a θ-dependent term:

$$\Psi_{\mu,n}(r,\theta) = e^{i\mu\theta}\frac{1}{\sqrt{\pi R}\left|J_{\mu+1}\left(j_{\mu,n}\right)\right|}J_{\mu}\left(\frac{j_{\mu,n}}{R}r\right). \tag{63}$$

In quantum mechanics, the angular momentum operator L_z is defined as

$$L_z \equiv - i\hbar\frac{\partial}{\partial\theta} \tag{64}$$

(Griffiths and Schroeter, 2018). We can apply this to our wave function, noting that we have separate θ-dependent and r-dependent factors:

$$L_z\Psi = L_z(f(r)g(\theta)) = - i\hbar f(r)g'(\theta). \tag{65}$$

$$L_z\Psi = (- i\hbar)(i\mu)\Psi(r,\theta) = \hbar\mu\Psi. \tag{66}$$

So, the constant μ is interpreted as proportional to the quantum angular momentum. The angular momentum eigenvalue is $\hbar_{\mu}$.

Because our potential is invariant under rotational displacements, this causes μ to be a conserved quantity by Noether's theorem through the continuous symmetries of our wave function. This is why μ must be a constant.

4. Infinite Ring Well

For this situation, we just need to modify our potential function to describe an annulus (Meister, 2016).

Suppose half of the width of the ring is Δr, with $\Delta r \le R$, where R is the midpoint of the inner and outer radius. Then the ring can be defined with the potential function

$$V(r,\theta) = \begin{cases} 0 & |r-R| \le \Delta r \\ \infty & |r-R| > \Delta r \end{cases} \tag{67}$$

Restating our Schrödinger equation:

$$\left(-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) + V(r,\theta)\right)\Psi(r,\theta) = E\Psi(r,\theta) \tag{68}$$

(Griffiths and Schroeter, 2018).

Once again, the particle cannot escape the ring, so $\Psi = 0$ outside the ring, and we first consider the inside where the particle is free (Shankar, 2012):

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\Psi(r,\theta) =- k^2\Psi(r,\theta)\,. \tag{69}$$

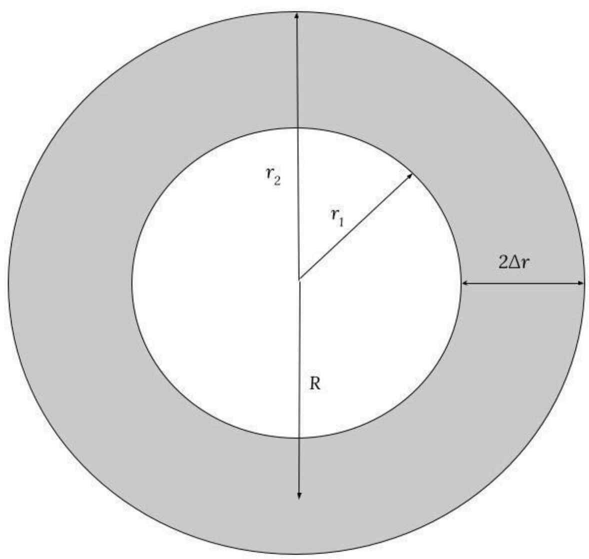


Figure 3: Diagram of the ring

Since this is the same equation as earlier, but just with the same boundary conditions, the initial solution formation will be the same, with

$$\Psi(r, \theta) = f_1(r) g_1(\theta), \tag{70}$$

and since the constraint $g(0) = g(2\pi)$ still applies here, along with cylindrical symmetry of $|g_1(\theta)|$, we have

$$g_1(\theta) = A\exp(i\mu\theta), \tag{71}$$

where $\mu \in \mathbb{Z}$.

Since the domain is now an annulus, there is no need for continuity at $r = 0$, so we can keep the Y_μ term:

$$f_1(r) = c_1 J_\mu(kr) + c_2 Y_\mu(kr). \tag{72}$$

4.1. The ring boundary conditions

Now, we need $f_1(R - \Delta r) = f_1(R + \Delta r) = 0$.

We also need two roots $2\Delta r$ apart. Let $r_1 = R - \Delta r$ and $r_2 = R + \Delta r$. Then,

$$f_1(r_1) = f_1(r_2) = 0\,. \tag{73}$$

$$c_1 J_\mu(kr_1) + c_2 Y_\mu(kr_1) = c_1 J_\mu(kr_2) + c_2 Y_\mu(kr_2) = 0\,. \tag{74}$$

Isolating the constant c_1 to eliminate it:

$$c_1 J_\mu(kr_1) = -\, c_2 Y_\mu(kr_1) \rightarrow c_1 = -\, c_2 \frac{Y_\mu(kr_1)}{J_\mu(kr_1)}\,. \tag{75}$$

Substituting this into the previous:

$$\left(-\, c_2 \frac{Y_\mu(kr_1)}{J_\mu(kr_1)}\right) J_\mu(kr_2) + c_2 Y_\mu(kr_2) = 0\,, \tag{76}$$

$$-\, c_2 Y_\mu(kr_1) J_\mu(kr_2) + c_2 Y_\mu(kr_2) J_\mu(kr_1) = 0\,. \tag{77}$$

$$Y_\mu(kr_1) J_\mu(kr_2) - Y_\mu(kr_2) J_\mu(kr_1) = 0\,. \tag{78}$$

$$Y_\mu(kr_1) J_\mu(kr_2) = Y_\mu(kr_2) J_\mu(kr_1)\,. \tag{79}$$

To determine the values of k that satisfy this equation, we first note that the zeroes of Y and J are interleaving (Bailey et al., 2008; Abramowitz and Stegun, 1972), which means that between every two adjacent zeroes of J, there is a zero of Y, and vice versa.

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Therefore, for each μ, there are infinite k that satisfy the equation. We can denote the root as $k_{\mu,n}$, where $\mu \in Z$ and $n \in Z$. Then, the infinite energy levels are

$$E_{\mu,n} = \frac{\hbar^2 k_{\mu,n}^2}{2m} . \tag{80}$$

We have shown an analytic quantized condition for k for an annulus.

4.2. The limiting case

We are interested in seeing what happens to the particle as the width of the ring approaches zero. It is possible that at $\Delta r = 0$, we should have the free particle case, but the particle may also be trapped in an infinitesimally thin ring. Either hypothesis may be true, so we must investigate.

We will pick $\mu = 0$ and $n = 1$ to evaluate this limit, for simplicity. We also need to satisfy the condition $Y_\mu(kr_1)J_\mu(kr_2) = Y_\mu(kr_2)J_\mu(kr_1)$.

First, we write $kr_1 = kR - k\Delta r$ and $kr_2 = kR + k\Delta r$. Since Δr is small in the limit, we can use a Taylor expansion only up to the first order:

$$J_0(kr_2) = J_0(kR + k\Delta r) \approx J_0(kR) + J_0'(kR)(k\Delta r) . \tag{81}$$

$$Y_0(kr_2) = Y_0(kR + k\Delta r) \approx Y_0(kR) + Y_0'(kR)(k\Delta r) . \tag{82}$$

$$J_0(kr_1) = J_0(kR - k\Delta r) \approx J_0(kR) - J_0'(kR)(k\Delta r) . \tag{83}$$

$$Y_0(kr_1) = Y_0(kR - k\Delta r) \approx Y_0(kR) - Y_0'(kR)(k\Delta r) \tag{84}$$

(Arfken et al., 2013).

Also, the derivatives of the Bessel functions are

$$J_0'(x) = - J_1(x) \quad and \quad Y_0'(x) = - Y_1(x) \tag{85}$$

(Abramowitz and Stegun, 1972).

So, we can substitute this into our boundary condition $Y_0(kr_1)J_0(kr_2) = Y_0(kr_2)J_0(kr_1)$. We get on the left side:

$$Y_0(kr_1)J_0(kr_2) \approx \left[Y_0(kR) + Y_1(kR)(k\Delta r)\right]\left[J_0(kR) - J_1(kR)(k\Delta r)\right]$$

$$= Y_0(kR)J_0(kR) - Y_0(kR)J_1(kR)(k\Delta r) + Y_1(kR)J_0(kR)(k\Delta r) - Y_1(kR)J_1(kR)(k\Delta r)^2. \tag{86}$$

On the right side:

$$Y_0\left(kr_2\right)J_0\left(kr_1\right) \approx \left[Y_0(kR) - Y_1(kR)(k\Delta r)\right]\left[J_0(kR) + J_1(kR)(k\Delta r)\right]$$

$$= Y_0(kR)J_0(kR) + Y_0(kR)J_1(kR)(k\Delta r) - Y_1(kR)J_0(kR)(k\Delta r) + Y_1(kR)J_1(kR)(k\Delta r)^2 \tag{87}$$

We can eliminate the common term $Y_0(kR)J_0(kR)$:

$$-Y_0(kR)J_1(kR)(k\Delta r) + Y_1(kR)J_0(kR)(k\Delta r)$$

$$= Y_0(kR)J_1(kR)(k\Delta r) - Y_1(kR)J_0(kR)(k\Delta r) + 2Y_1(kR)J_1(kR)(k\Delta r)^2 . \tag{88}$$

$$Y_1(kR)J_0(kR) - Y_0(kR)J_1(kR) = 2Y_1(kR)J_1(kR)(k\Delta r)^2 . \tag{89}$$

Note the left side is the same as the Wronskian of Y_0 and J_0, $J_0(x)Y_0'(x) - Y_0(x)J_0'(x)$. It is known that it is equal to $2/\pi x$ (Olver et al., 2010). So we have:

$$\frac{2}{\pi kR} = 2Y_1(kR)J_1(kR)(k\Delta r)^2 . \tag{90}$$

$$\frac{1}{\pi k^3 R Y_1(kR)J_1(kR)} = (\Delta r)^2 . \tag{91}$$

Because Y_1 and J_1 have zeros, we cannot consider this as a continuous limit. We can consider that the envelope of Y_1 and J_1 approaches zero as kR approaches infinity (Watson, 1995). If the order of the zeros at infinity of Y_1 and J_1 add up to less than 3, then k approaching infinity is a valid solution, as it would allow the left side to approach zero (Abramowitz and Stegun, 1972).

For Y_1, we can use the approximation for large arguments:

$$Y_1(x) \sim \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{3\pi}{4}\right) \tag{92}$$

(Olver et al., 2010). To consider the zero at ∞, we can simply consider the zero of $Y_1(1/z)$ at $z = 0$. We have:

$$Y_1\left(\frac{1}{z}\right) \sim \sqrt{\frac{2z}{\pi}}\sin\left(\frac{1}{z} - \frac{3\pi}{4}\right) . \tag{93}$$

Since the sine function is always bounded between −1 and 1, the order of the zero is 1/2.

Similarly, for large x, $J_1(x)$ is

$$J_1(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{3\pi}{4}\right) \tag{94}$$

(Olver et al., 2010).

Similarly, the zero at infinity is of order 1/2. Therefore, their zeros combined are order 1, meaning that as k^3 limits to infinity, it still dominates.

Therefore, the wave vector k approaching infinity is a solution. This means that as our disk gets infinitely thin, the energy of our particle becomes unbounded inside the tiny region (Meister, 2016).

5. Discussion

First, the infinite disk well's solutions are all Bessel functions of the first kind J (Bailey et al., 2008). μ is proportional to the quantum angular momentum of the wave function: $L_z\Psi = \hbar_\mu \Psi$. For each value of μ, defining a different Bessel function J_μ, there are infinite solutions due to the infinite roots (Abramowitz and Stegun, 1972). As the coefficient of r is scaled so that when r = R, the argument of J is a root, we have infinite coefficients $j_{\mu,n}/R$.

The roots are significant, as in our solution, we set $k = j_{\mu,n}/R$. This means the roots are proportional to the wave vector, so each successive root corresponds to a quantized higher energy level, growing quadratically. Notably, the wave function is normalizable using an integral identity, and we have:

$$\Psi_{\mu,n}(r,\theta) = \frac{1}{\sqrt{\pi R}\left|J_{\mu+1}\left(j_{\mu,n}\right)\right|} J_\mu\left(\frac{j_{\mu,n}}{R} r\right). \tag{94}$$

In particular, because the disk is rotationally symmetric, there is a two-fold degeneracy in the angular momentum for $\pm\mu$.

It is interesting to see what this solution means and what it looks like. It oscillates as the position varies radially—of course, we have isotropic symmetry—and has rings where the probability of finding the particle is zero, equal to the order of the root n. By construction, the nth root is at the boundary r = R, so there are $n - 1$ roots inside the circular disk. So as the energy level increases, the oscillations get more frequent, and the amplitude at the center increases. More energy leads to a more concentrated distribution (Merzbacher, 1998).

This is similar to the infinite square well, where the solution is a sine function. However, for our infinite disk well, as we get closer to the boundary ring, the amplitude of the oscillation decreases, while it doesn't in the infinite square well (Cohen-Tannoudji et al., 1977).

In the infinite ring well/infinite annulus well, the solution involves both the Bessel function of the first kind and the second kind, J and Y (Watson, 1995). They must be of the same μ value, and for each μ, there are infinite wave vectors k that satisfy the equation. There are a lot of energy levels. The derivations here are pedagogically valuable for a better understanding.

But the most important finding is in taking the limit as the width $\Delta r \to 0$. We found that k must approach infinity in this case, so the infinitesimally thin ring does not actually approach a free particle as hypothesized. This is likely because even if the annulus "disappears" in the limit, the particle is still trapped inside. The energy approaching infinity is an interesting finding.

The results may suggest potential applications in nanotechnology (Kittel, 2005; Merzbacher, 1998). Specifically, this infinite disk well is the same as the radial confinement in 2D quantum dots, where electrons are trapped in tiny structures in things like semiconductors. Importantly, the explanatory power of this paper in illustrating concepts such as the discrete energy levels helps to understand work in this field.

A limitation is the idealized infinite potential barrier outside of the ring/disk, which is not accurate in real life. Future work could explore finite potential wells—especially to explore quantum tunnelling—or use external fields to model more realistic scenarios. Additionally, more numerical methods can be employed to look at the probability distribution in the annulus (Arfken et al., 2013).

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No AI tools were used in the creation of this paper. Matplotlib, a Python package, was used to render graphs. There is a built-in function in matplotlib to render Bessel functions and their roots.

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Author Biography

Ryan Ng is a senior high school student at Renaissance College with a strong interest in many diverse fields, such as mechanics, general relativity, complex analysis, abstract algebra, critical theory, intersectionality, feminist peace theory, and evolutionary economics.

His work in "The Infinite Disk & Ring Well" investigates radially symmetric quantum mechanical potential wells, derives the wave functions and quantized energy levels for the infinite disk and annulus systems, and presents an argument for the divergence of energy in the infinitesimally thin ring limit. He has also completed a separate paper on the free-particle Green's function in one spatial and one time dimension, where he constructs an alternative pedagogical derivation using two parallel methods.

In high school, Ryan has written reports on theoretical fluid dynamics, experimentally investigating the relationship between a rigid body's pivot point and its rotational properties. He has also conducted a study on intersectional feminism in Hong Kong.

Ryan hopes to continue pursuing physics research at university while exploring many fields in academia.

Mentor Contribution Statement

The student author conducted all aspects of the research and manuscript preparation independently, without the involvement of a research mentor.