Electromagnetic Wave Equ^{n}s

In vacuum
In a conducting medium

EM wave is a transverse wave in which the electric component is ⊥ar to the mag. component & both of them are ⊥ar to the dir^{n} of propagation of waves.

Electromagnetic Wave eq^{n} in free space

In free space, $\rho = 0$ and $\vec{J} = 0$

so the Maxwell's e m equ^{n}s in free space are

$$\vec{\nabla} \cdot \vec{D} = 0 \quad \text{--- (1)} \qquad , \; D = \epsilon_0 \vec{E}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{--- (2)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{--- (3)}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t} \quad \text{--- (4)}$$

$$(4) \Rightarrow \vec{\nabla} \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \text{--- (4a)}$$

$$(3) \Rightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Taking curl on both sides

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) \quad \text{--- (5)}$$

On applying the theorems of vector calculus on LHS of eq^{n} (5) we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\cancel{\vec{\nabla} \cdot \vec{E}}^{\,0}) - (\vec{\nabla} \cdot \vec{\nabla}) \vec{E}$$

Here, $\vec{\nabla} \cdot \vec{D} = 0 \Rightarrow \vec{\nabla} \cdot \vec{E} = 0$

$$\vec{\nabla} \cdot \vec{\nabla} = \nabla^2$$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\nabla^2 \vec{E} \quad \text{--- (6a)}$$

$$\text{RHS (5)} \Rightarrow -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) = -\frac{\partial}{\partial t} \left[\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right] \quad \text{using (4)}$$

$$= -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{--- (6b)}$$

Using eq^{n}s (6a) & (6b) in eq^n (5), we get

$$-\nabla^2\vec{E} = -\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

or $$\boxed{\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}} \longrightarrow (7)$$ e-m wave eq^n for electric component

Similarly we can find the e-m wave eq^n for mag component as

$$\boxed{\nabla^2\vec{B} = \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}} \longrightarrow (8)$$ e-m wave eq^n for magnetic component

Q. Derive eq^n (8)

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - (\vec{\nabla}\cdot\vec{\nabla})\vec{E}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{B})}_{\to 0} - (\vec{\nabla}\cdot\vec{\nabla})\vec{B}$$

* Special Case:

It is known that the general wave eq^n is given by

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

ψ = Wave function = $\psi(\vec{r}, t)$

v = Velocity of wave

We obtained, $\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$ and $\nabla^2\vec{B} = \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$

On comparing these eq^{n}s, $\frac{1}{v^2} = \mu_0\varepsilon_0 \Rightarrow \boxed{v = \frac{1}{\sqrt{\mu_0\varepsilon_0}}}$

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Electromagnetic Wave Eqn in a Charge Free Conducting Medium

Charge free $\rho = 0$ conducting medium $\therefore \epsilon_0 \to \epsilon$

$\mu_0 \to \mu$

If σ = conductivity
$\vec{J} = \sigma \vec{E}$

The Maxwell's equns for this case are

$$\vec{\nabla}.\vec{D} = 0 \quad \text{——} (1) \quad \Rightarrow \vec{\nabla}.\vec{E} = 0$$

$$\vec{\nabla}.\vec{B} = 0 \quad \text{——} (2)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{——} (3)$$

$$\vec{\nabla} \times \vec{B} = \mu\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$$

$$= \mu\left(\sigma\vec{E} + \frac{\partial \vec{D}}{\partial t}\right) \quad \text{——} (4)$$

$$= \mu\left(\sigma\vec{E} + \varepsilon\frac{\partial \vec{E}}{\partial t}\right) \qquad D = \epsilon E$$

(3) $\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$ —— (3)

Taking curl on both sides

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

but $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\underbrace{\cancel{\vec{\nabla}.\vec{E}}}_{0}) - (\underbrace{\vec{\nabla}.\vec{\nabla}}_{\nabla^2})\vec{E}$

$$= -\nabla^2\vec{E} \quad \text{——} (5a)$$

RHS $\Rightarrow -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -\frac{\partial}{\partial t}\left[\mu\left(\sigma\vec{E} + \epsilon\frac{\partial \vec{E}}{\partial t}\right)\right]$

$$= -\frac{\partial}{\partial t}\left(\mu\sigma\vec{E} + \mu\epsilon\frac{\partial \vec{E}}{\partial t}\right)$$

$$= -\mu\sigma\frac{\partial \vec{E}}{\partial t} - \mu\epsilon\frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{——} (5b)$$

On using equns (5a) & (5b) in equn (3), we get

$$\cancel{-}\nabla^2\vec{E} = \cancel{-}\left[\mu\sigma\frac{\partial \vec{E}}{\partial t} + \mu\epsilon\frac{\partial^2 \vec{E}}{\partial t^2}\right]$$

or $\boxed{\nabla^2\vec{E} = \mu\sigma\frac{\partial \vec{E}}{\partial t} + \mu\epsilon\frac{\partial^2 \vec{E}}{\partial t^2}}$ (6) for electric component

Similarly for magnetic component the e-m eqⁿ for charge free conducting medium is given by

$$\boxed{\nabla^2 \vec{B} = \mu\sigma \frac{\partial \vec{B}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}} \quad \text{——(7)}$$ for magnetic comp

Q. Derive eqⁿ (7).

$$\vec{\nabla} \times \vec{B} = \mu\left(\sigma\vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t}\right)$$

Taking curl on both sides

$$\underbrace{(\vec{\nabla}\cdot\vec{B})}_{0}\vec{\nabla} - \nabla^2 \vec{B} = \mu\sigma\left(-\frac{\partial \vec{B}}{\partial t}\right) + \mu\epsilon\left(-\frac{\partial^2 \vec{B}}{\partial t^2}\right)$$

6/11/19 **Transverse Nature of Electro-magnetic Wave**

An e-m wave consist of both electric & magnetic field components.

The plane wave solⁿ of electric field intensity vector $\vec{E}$ & mag. field intensity vector $\vec{H}$ can be written as

$$\vec{E} = \vec{E}(\vec{r}, t) = E_0 e^{i(\vec{K}\cdot\vec{r} - \omega t)} = \hat{e} E_0 e^{i(\vec{K}\cdot\vec{r} - \omega t)}$$

$$\vec{H} = \vec{H}(\vec{r}, t) = H_0 e^{i(\vec{K}\cdot\vec{r} - \omega t)} = \hat{b} H_0 e^{i(\vec{K}\cdot\vec{r} - \omega t)}$$

$\vec{E}$ = Electric field intensity
$\vec{H}$ = Magnetic " "
E_0 = Peak value of $\vec{E}$
H_0 = Peak value of $\vec{H}$
$\vec{r}$ = Space coordinate
t = time coordinate
ω = angular frequency
$\vec{K}$ = wave propagation vector $= \hat{i}K_x + \hat{j}K_y + \hat{k}K_z$
$i = \sqrt{-1}$
$\hat{e}, \hat{b}$ = unit vectors associated with $\vec{E}$ and $\vec{H}$

For a charge free region, Maxwell's first eq^n can be expressed as

$$\vec{\nabla}\cdot\vec{E} = 0$$
$$\vec{\nabla}\cdot\left[\hat{e}E_0 e^{i(\vec{K}\cdot\vec{r}-\omega t)}\right] = 0$$

From vector calculus, it is known that

$$\vec{\nabla}\cdot(\phi\vec{A}) = \vec{\nabla}\phi\cdot\vec{A} + \phi(\vec{\nabla}\cdot\vec{A})$$

Here ϕ = Scalar field
$\vec{A}$ = Vector field
$\vec{\nabla}\phi$ = grad ϕ
$\vec{\nabla}\cdot\vec{A}$ = div $\vec{A}$

Using this identity in the above eq^n we find

$$\vec{\nabla}\left[E_0 e^{i(\vec{K}\cdot\vec{r}-\omega t)}\right]\cdot\hat{e} + E_0 e^{i(\vec{K}\cdot\vec{r}-\omega t)}\underbrace{(\vec{\nabla}\cdot\hat{e})}_{0} = 0$$

Further it is known that the divergence of a constant unit vector is 0, i.e. $\vec{\nabla}\cdot\hat{e} = 0$

So $\vec{\nabla}\left[E_0 e^{i(\vec{K}\cdot\vec{r}-\omega t)}\right]\cdot\hat{e} = 0$

Since, $\vec{K} = \hat{i}K_x + \hat{j}K_y + \hat{k}K_z$

Hence $\left[(i\vec{K})e^{i(\vec{K}\cdot\vec{r}-\omega t)}\right]\cdot\hat{e} = 0$

or $i(\vec{K}\cdot\hat{e}) = 0$ since $e^{i(\vec{K}\cdot\vec{r}-\omega t)} \neq 0$
or $(\vec{K}\cdot\hat{e}) = 0$

$\Rightarrow$ Transverse nature of electric field
because $\hat{K}$ → wave propagation vector
$\hat{e}$ → unit vector of $\vec{E}$

Similarly on considering the Maxwell's 2nd eq^n
$$\vec{\nabla}\cdot\vec{B} = 0$$

and following the same procedure as above we can prove that
$\vec{K}\cdot\hat{b} = 0 \Rightarrow$ tr. nature of $\vec{B}$ or $\vec{H}$.

Hence from these two observations we can explain the electric and magnetic components of an e-m w are ⊥ar to the dirⁿ of propagation of e-m wave. Hence e-m wave is transverse in nature.

Q1. Define gradient, divergence and curl and hence explain their significance.

Q2 Derive the Maxwell's electr eqᵘⁿˢ of e-m induction differential form and in integral form.

Q3. Distinguish betⁿ displ. current & conduction current.

Q4. Derive the eqⁿ of continuity.

Q5. Define Faraday's law of electromagnetic induction.

Q6. Prove that magnetic monopoles does not exist.

Q7. Give the interpretation of Maxwell's eqn.

Q8. Derive the e-m wave eqⁿ in a free space in vacuum

Q9. Derive the e-m wave eqns in a conducting medium.

Q10 Prove that $\vec{J} = \sigma\vec{E}$

Q11. Prove the transverse nature of e-m wave.

Numericals

Q1. ~~Show that~~ Examine the vector $\vec{A} = (x+3y)\hat{i} + (y+az)\hat{j} + (x+az)\hat{k}$ is solenoidal it or not.

Q2. Show that the vector field $\vec{A} = (x^2+xy^2)\hat{i} + (y^2+x^2y)\hat{j}$ is irrotational.

Q3. If $\vec{A} = (x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}$ then prove that $\vec{A}\cdot(\vec{\nabla}\times\vec{A}) = 0$

Q4. If the potential $V(x,y,z) = (4x^2+2y^2+z^2)^{1/2}$, find $\vec{E}$ at $(1,1,1)$
Hint:- $E = -\nabla V$

Q5. If $\phi = 3x^2y - y^3z^2$, find $\vec{\nabla}\phi$ at $(1,-2,-1)$