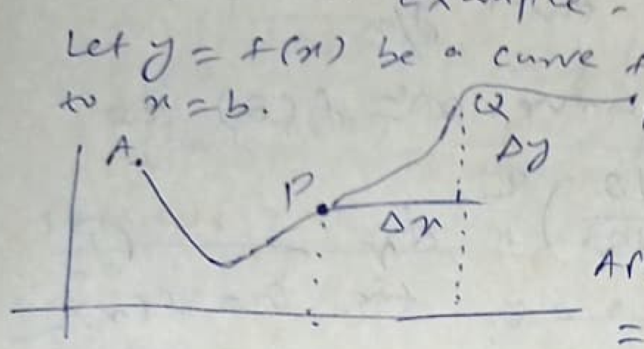
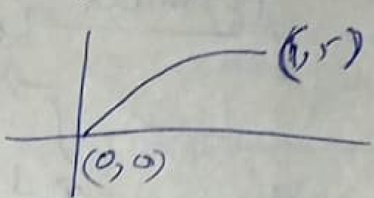
LENGTH OF PLANE CURVE

The curve which lies in a single plane is called plane curve. Example - $y = 5x^{2/3}$, $0 \le x \le 1$

Let $y = f(x)$ be a curve from $x = a$ to $x = b$.





Approx. length of PQ
$= \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$

If AB will be divided into n number of pieces like PQ then length of AB is

$$AB = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

Let c_k be a point between x_{k-1} and x_k (that is P & Q)

From calculus $f'(c_k) = \dfrac{\Delta y_k}{\Delta x_k}$

$\Delta y_k = f'(c_k)\,\Delta x_k$

$\Rightarrow$ Length $AB = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + [f'(x_k)\,\Delta x_k]^2}$ ——— (1)

The function $g(x) = \sqrt{1 + [f'(x)]^2}$ is continuous on $[a, b]$

Length of curve $L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

EX: Find the length of curve from $x = 0$ to $x = \pi/3$ of the curve $y = \log \sec x$

Sol^n $y' = \dfrac{1}{\sec x} \dfrac{d}{dx}(\sec x) = \dfrac{1}{\sec x}(\sec x \tan x)$
$= \tan x$

Length $L = \int_0^{\pi/3} \sqrt{1 + y'^2}\, dx = \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, dx$

$= \int_0^{\pi/3} \sec x\, dx = \left[\ln(\sec x + \tan x)\right]_0^{\pi/3}$

$= \log\left(\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right) = \ln(2 + \sqrt{3})$

POLAR FORM $\quad x = r\cos\theta \quad y = r\sin\theta$

$x^2+y^2 = r^2 \quad \frac{y}{x} = \tan\theta$

$r = \sqrt{x^2+y^2} \quad \theta = \tan^{-1} y/x$

Formula: Length of curve $r = f(\theta)$ is

$$L = \int_{r_1}^{r_2} \sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}\, dr \quad \text{—— (1)}$$

for $\theta = f(r)$

$$= \int_{\theta_1}^{\theta_2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad \text{—— (2)}$$

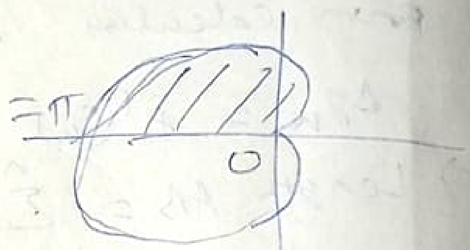
for $r = f(\theta)$

Ex: Find length of cardioid $r = a(1-\cos\theta)$

Sol^n $\frac{dr}{d\theta} = a\sin\theta$

NOW

$r^2 + \left(\frac{dr}{d\theta}\right)^2$



$= a^2(1-\cos\theta)^2 + a^2\sin^2\theta$

$= a^2[1+\cos^2\theta - 2\cos\theta + \sin^2\theta]$

$= a^2[1+1-2\cos\theta] = 2a^2(1-\cos\theta)$

$= 2a^2\left(2\sin^2\frac{\theta}{2}\right) = 4a^2\sin^2\frac{\theta}{2}$

~~Area~~ Length $= \int_0^{\pi} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$= \int_0^{\pi} 2a\sin\frac{\theta}{2}\, d\theta = -2a \left.\frac{\cos\frac{\theta}{2}}{1/2}\right]_0^{\pi}$

$= 4a\left[-\cos\frac{\pi}{2} + \cos 0\right] = 4a$

Total length $= 2\times 4a = 8a$

EX- Find Length of Curve $y = \frac{4\sqrt{2}}{3} x^{3/2} - 1$, $0 \le x \le 1$

Sol^n Length $L = \int_0^1 \sqrt{1 + y'^2}\, dx = \int_0^1 \sqrt{1 + (2\sqrt{2}\sqrt{x})^2}\, dx$

$= \int_0^1 \sqrt{1+8x}\, dx = \frac{1}{8} \int u^{\frac{1}{2}} du = \frac{1}{8} \frac{u^{3/2}}{3/2}$ $\quad 1+8x = u$, $dx = \frac{1}{8} du$

$= \left[\frac{1}{12} (1+8x)^{3/2}\right]_0^1 = 13/6$

parametric form $x = f(t)$ $y = g(t)$, $L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

EX- $x = a\cos^3 t$, $y = a\sin^3 t$, $0 \le t \le \pi/2$

Sol^n Length $L = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int \sqrt{(-3a\cos^2 t \sin t)^2 + (3a\sin^2 t \cos t)^2}\, dt$

$= \int \sqrt{9a^2 \cos^2 t \sin^2 t (\cos^2 t + \sin^2 t)}\, dt = \int_0^{\pi/2} \sqrt{(3a\cos t \sin t)^2}\, dt$

$= \int_0^{\pi/2} 3a \cos t \sin t\, dt = \frac{3a}{2} \int_0^{\pi/2} \sin 2t\, dt = \frac{3a}{2} \left[\frac{-\cos 2t}{2}\right]_0^{\pi/2}$

$= 3a/2$

AREA of SURFACES OF REVOLUTION

Surface area for revolution about x-axis

$S = \int_a^b 2\pi y \sqrt{1 + y'^2}\, dx = \int_a^b 2\pi f(x) \sqrt{1 + f'^2}\, dx$

EX-1 Find area of surface generated by revolving the curve
P402 $y = 2\sqrt{x}$, $1 \le x \le 2$ about x-axis

Sol^n $a = 1$, $b = 2$ $y = 2\sqrt{x}$ $\frac{dy}{dx} = y' = \frac{1}{\sqrt{x}}$

$\sqrt{1 + y'^2} = \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}}$

~~$S = \int_2^4 1 + y'^2\, dx$~~ $\quad S = \int_1^2 2\pi y \sqrt{1 + y'^2}\, dx$

$= \int_1^2 2\pi\, 2\sqrt{x} \frac{\sqrt{x+1}}{\sqrt{x}}\, dx = 4\pi \int_1^2 \sqrt{x+1}\, dx$

$= 4\pi \frac{2}{3} (x+1)^{3/2} \Big]_1^2 = \frac{8\pi}{3} (3\sqrt{3} - 2\sqrt{2})$

Ex-2 The line segment $x = 1-y$, $0 \le y \le 1$ is
P. 402 revolved about y-axis to generate cone.
Find its lateral surface area.

Sol^n $c = 0$ $d = 1$ $x = 1-y$ $dx/dy = -1$

$$S = \int_0^1 2\pi x \sqrt{1 + (dx/dy)^2}\, dy = \int_0^1 2\pi (1-y)\sqrt{2}\, dy$$

$$= 2\pi\sqrt{2}\left[y - y^2/2\right] = \pi\sqrt{2}$$

Short differential form

$S = \int_a^b 2\pi y\, ds$ or $\int_a^b 2\pi x\, ds$ where $ds = \sqrt{dx^2 + dy^2}$

Ex-3 Find area of surface generated by revolving curve $y = x^3$, $0 \le x \le \frac{1}{2}$ about x-axis

Sol^n $S = \int_0^{1/2} 2\pi y \sqrt{dx^2 + dy^2}$ $\left[\begin{matrix} y = x^3 \\ (\text{given}) \end{matrix}\right.$, $dy = 3x^2\, dx$

$$= \int_0^{1/2} 2\pi x^3 \sqrt{(dx)^2 + 9x^4 (dx)^2} = 2\pi \int_0^{1/2} x^3 \sqrt{1 + 9x^4}\, dx$$

$$= 2\pi \frac{1}{36} \int v^{1/2}\, dv = \frac{\pi}{18} \frac{v^{3/2}}{3/2} = \frac{\pi}{27} (1+9x^4)^{3/2} \Big|_0^{1/2}$$ $\left[\begin{matrix} \text{put } 1 + 9x^4 = v \\ 36x^3 dx = dv \end{matrix}\right.$

$$= 61\pi / 1728$$

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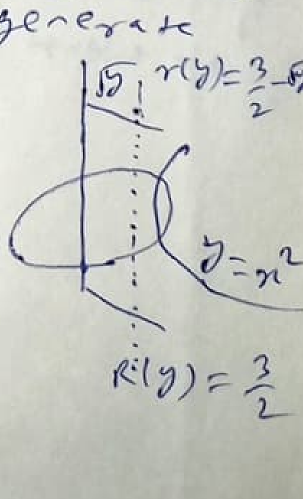
volume $V = \int_a^b \pi \left[[R(x)]^2 - [r(x)]^2 \right] dx$

$R(x)$ = outer radius $r(x)$ = inner radius

Ex. 7 The region in 1st quadrant enclosed by
P. 384 parabola $y = x^3$, the y-axis & line $y = 1$ is
revolved about line $x = 3/2$ to generate
a solid. Find volume of the solid

Sol^n Step-I Draw the region & sketch a line
across it ⊥r to axis of revolution
In this case the line $x = 3/2$



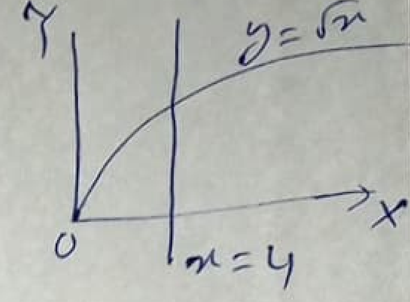
Step II Limits of integration are $y = 0$ to $y = 1$

Step III Radii of washer swept out by line segment $R(y) = 3/2$ $r(y) = \frac{3}{2} - \sqrt{y}$

$$\text{Volume } V = \int_c^d \pi \left[(R(y))^2 - (r(y))^2 \right] dy$$

$$= \int_0^1 \pi \left[\left(\frac{3}{2}\right)^2 - \left(\frac{3}{2} - \sqrt{y}\right)^2 \right] dy = \pi \int_0^1 (3\sqrt{y} - y)\, dy = \frac{3\pi}{2}$$

Ex. 1 p. 388 The region bounded by curve $y = \sqrt{x}$, the x-axis and line $x = 4$ is revolved about the y-axis to generate a solid. Find vol. of the solid.



Sol Step-I Sketch the region and draw a line segment across it parallel to the axis of revolution

Step II Limits of integration x varies from $a = 0$ to $b = 4$

Formula: Volume $V = \int_a^b 2\pi x\, f(x)\, dx = \int_0^4 2\pi x \sqrt{x}\, dx$

$= 2\pi \int_0^4 x^{3/2}\, dx = 2\pi \left[\frac{x^{5/2}}{5/2}\right]_0^4 = 128\frac{\pi}{5}$

Ex. 2 The region bounded by curve $y = \sqrt{x}$, the x-axis and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

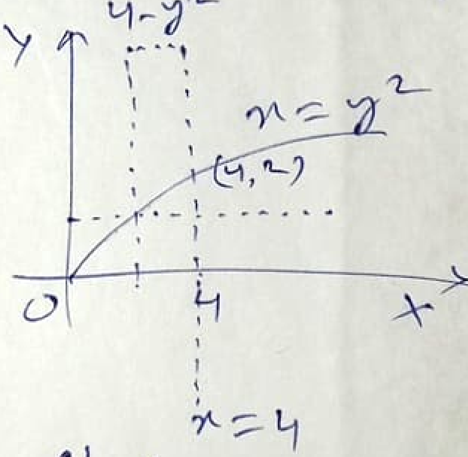
Solution

Formula $V = \int_c^d 2\pi y\, f(y)\, dy$

$= \int_0^2 2\pi y (4 - y^2)\, dy$

$= 2\pi \left[2y^2 - \frac{y^4}{4}\right]_0^2 = 8\pi$

(Since $y^2 = x = 4$, given $y = \sqrt{x}$
$y^2 = 4 \Rightarrow 4 - y^2 = 0$)



Shell radius $= y$
Shell height $= 4 - y^2$