16 Special Functions

16.1 Introduction

A class of mathematical functions owes it established names and notations to the importance of these functions in mathematical analysis, functional analysis, physics, and statistics. Although there is no general formal definition of *special functions*, a list of mathematical functions are commonly accepted as special, and therefore, are known as special functions. Most of the special functions are named after the mathematicians who first introduced them or contributed much to their theory.

Many special functions appear as integrals of elementary functions or solutions of differential equations. Here the scope of the discussion is limited to only four functions, namely, beta, gamma, Bessel, and Legendre, although there are several others in this class.

Beta and gamma functions fall into the first category, i.e., they are defined as integrals of some elementary functions. It is convenient to express many other integrals in terms of beta and gamma functions. Gamma function is one of the most commonly used non-elementary function; it has applications in diverse areas such as quantum physics, astrophysics, and fluid dynamics.

Bessel and Legendre functions are generally used to express solutions of linear differential equations. Bessel function expresses the solution of differential equations in a system with cylindrical symmetry, whereas Legendre function expresses that in a system with spherical symmetry. For these reasons Bessel functions are also called *cylindrical harmonic functions* and Legendre functions are called *spherical harmonic functions*.

16.2 Gamma Function

The gamma function may be thought of as a factorial function with its domain extended to include non-integers also. The problem of extending the factorial to non-integer arguments was apparently first considered by Daniel Bernoulli and Christian Goldbach in the 1720s. However, it was solved by Leonhard Euler (1707–1783), a Swiss mathematician and physicist in 1729. Around 1811, Adrien-Marie Legendre (1752–1833), an important French mathematician (also credited for *Legendre polynomials*) introduced the name *gamma function* and the symbol Γ for it; he also rewrote Euler's integral definition in its modern form. Johann Carl Friedrich Gauss, a German mathematician and scientist, and Karl Weierstrass also made significant contributions to further the studies of the function.

Gamma Function The *gamma function*, denoted by Γ, is defined for any positive number x by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{16.1}$$

16.2.1 Convergence of Gamma Function

We can write

$$\Gamma(x) = \int_0^1 t^{x-1} e^{-t} dt + \int_1^\infty t^{x-1} e^{-t} dt = I_1(x) + I_2(x)$$

where

$$I_1(x) = \int_0^1 t^{x-1} e^{-t} dt \text{ and } I_2(x) = \int_1^\infty t^{x-1} e^{-t} dt$$

Now, for $0 < x < 1$, $I_1(x)$ is an improper integral but is absolutely convergent by the μ-test, because $t^{1-x}\left[t^{x-1}e^{-t}\right] = e^{-t} \to 1$ as $t \to 0+$.

By the μ-test, the second integral

$$I_2(x) = \int_1^\infty t^{x-1} e^{-t} dt$$

is also absolutely convergent for all values of x, because we have $t^2\left[t^{x-1}e^{-t}\right] = t^{x+1}e^{-t} \to 0$ as $t \to \infty$.

Thus, the gamma function is well defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

16.2.2 Properties of Gamma Function

We can evaluate $\Gamma(1)$ by direct integration, i.e.

$$\Gamma(1) = \int_0^\infty e^{-t} dt = \left[-e^{-t}\right]_0^\infty = 1 \tag{16.2}$$

Integrating by parts, we have

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = \left[t^x\left(-e^{-t}\right)\right]_0^\infty - \int_0^\infty \frac{d}{dt}(t^x)\left(-e^{-t}\right)dt$$

$$= 0 + \int_0^\infty x\, t^{x-1} e^{-t}\, dt = x \int_0^\infty t^{x-1} e^{-t}\, dt$$

Hence, we get

$$\Gamma(x+1) = x\,\Gamma(x) \tag{16.3}$$

Thus, Eqns (16.2) and (16.3) describe the functional properties of the gamma function.

Combining these properties, we have $\Gamma(n+1) = n\Gamma(n) = \cdots = n!\Gamma(1) = n!$ for all natural numbers n.

Thus, the functional property in Eqn (16.3) generalizes the relation $n! = n(n-1)!$ of the factorial function.

It is also possible to extend the definition of gamma function for negative values of x, by inverting the functional equation as

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} \quad \text{for } -1 < x < 0$$

16.2.3 Another Definition by Euler and Gauss

For $x > 0$, define

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)\cdots(x+p)} = \frac{p^x}{x(1+x/1)\cdots(1+x/p)} \tag{16.4}$$

then

$$\Gamma(x) = \lim_{p\to\infty} \Gamma_p(x) \tag{16.5}$$

Clearly

$$\Gamma_p(1) = \frac{p!p}{1(1+1)\cdots(1+p)} = \frac{p}{1+p}$$

and

$$\Gamma_p(1+x) = \frac{p!\,p^{x+1}}{(x+1)(x+1+1)\cdots(x+1+p)}$$

$$= \frac{p}{x+1+p} x\Gamma_p(x)$$

Hence making $p \to \infty$, we see that $\Gamma(1) = 1$ and $\Gamma(x+1) = x\Gamma(x)$.

Thus, this definition satisfies the functional equations given by the basic definition of the gamma function.

It is interesting to observe that the definition is valid also for negative values of x, except for the values $0, -1, -2, \ldots$. The graph of gamma function (corresponding to this definition) is shown in Fig. 16.1. This definition is useful to establish the complement or reflection property of the gamma function as shown in Section 16.2.4.

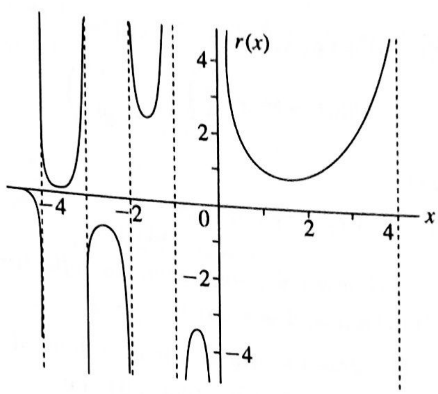

Fig. 16.1 Gamma function

16.2.4 Weierstrass Formula

We can express

$$p^x = e^{x\log p} = e^{x\left(\log p - 1 - \frac{1}{2} - \cdots - \frac{1}{p}\right)} e^{\left(x + \frac{x}{2} + \cdots + \frac{x}{p}\right)}$$

Thus, Eqn (16.4) can be rewritten as

$$\Gamma_p(x) = \frac{1}{x}\cdot\frac{1}{x+1}\cdot\frac{2}{x+2}\cdots\frac{p}{x+p}p^x$$

$$= \frac{e^{x\left(\log p - 1 - \frac{1}{2} - \cdots - \frac{1}{p}\right)} e^{\left(x + \frac{x}{2} + \cdots + \frac{x}{p}\right)}}{x\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\cdots\left(1+\frac{x}{p}\right)}$$

$$= e^{x\left(\log p - 1 - \frac{1}{2} - \cdots - \frac{1}{p}\right)} \frac{1}{x}\cdot\frac{e^x}{1+x}\cdot\frac{e^{\frac{x}{2}}}{1+\frac{x}{2}}\cdots\frac{e^{\frac{x}{p}}}{1+\frac{x}{p}}$$

Now

$$\lim_{p\to\infty}\left(-\log p + 1 + \frac{1}{2} + \cdots + \frac{1}{p}\right) = 0.5772156649$$

is a constant, known as *Euler's constant* and is denoted by γ.

For any real number x, except for the values $0, -1, -2, \ldots$, gamma function may be expressed in terms of Euler's constant as the infinite product

$$\frac{1}{\Gamma(x)} = xe^{\gamma x}\prod_{p=1}^{\infty}\left[\left(1+\frac{x}{p}\right)e^{-\frac{x}{p}}\right]$$

This is known as the *Weierstrass form of the gamma function.*

There is an important identity connecting the gamma function at the complementary values x and $1-x$. To obtain this identity, start with Weierstrass formula, which gives

$$\frac{1}{\Gamma(x)}\cdot\frac{1}{\Gamma(-x)} = -x^2 e^{\gamma x} e^{-\gamma x}\prod_{p=1}^{\infty}\left[\left(1+\frac{x}{p}\right)e^{-\frac{x}{p}}\left(1-\frac{x}{p}\right)e^{\frac{x}{p}}\right]$$

But the functional equation gives $\Gamma(-x) = -x\sqrt{(1-x)}$ and the equality simplifies to

$$\frac{1}{\Gamma(x)}\cdot\frac{1}{\Gamma(1-x)} = x\prod_{p=1}^{\infty}\left[\left(1-\frac{x^2}{p^2}\right)\right]$$

Now, use the well-known infinite product:

$$\sin(\pi x) = \pi x \prod_{p=1}^{\infty}\left(1 - \frac{x^2}{p^2}\right)$$

Thus, we get

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \tag{16.6}$$

This identity is known as *complement* or *reflection property* of gamma function and is valid for $0 < x < 1$.

Note The gamma function is also defined for a complex number z with a positive real part, i.e.

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt \text{ for } Re(z) > 0$$

and thus, it is an extension of the factorial function to complex numbers with positive real parts.

Example 16.1 Evaluate

$$\int_0^1 \frac{1}{\sqrt{-\log x}}dx$$

Solution Substitute $-\log x = y$, i.e., $x = e^{-y}$ so that $dx = -e^{-y}dy$.

When $x = 0$, we have $y = \infty$ and when $x = 1$, we have $y = 0$.

Thus, we have

$$\int_0^1 \frac{1}{\sqrt{-\log x}}dx = \int_{\infty}^0 \frac{1}{\sqrt{y}}(-e^{-y}dy)$$

$$= \int_0^{\infty} y^{-\frac{1}{2}}e^{-y}dy = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$ □

Example 16.2 Prove that

$$\int_0^{\infty} e^{-m^2x^2}dx = \frac{\sqrt{\pi}}{2m}$$

Solution Substitute $t = m^2x^2$, i.e., $x = \frac{\sqrt{t}}{m}$ so that $dx = \frac{1}{2m}t^{-\frac{1}{2}}dt$.

When $x = 0$, we have $t = 0$ and when $x = \infty$, t is also ∞.

Thus, we have

$$\int_0^{\infty} e^{-m^2x^2}dx = \frac{1}{2m}\int_0^{\infty} e^{-t}t^{-\frac{1}{2}}dt$$

$$= \frac{1}{2m}\int_0^{\infty} e^{-t}t^{\frac{1}{2}-1}dt = \frac{1}{2m}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2m}$$

Example 16.3 Prove that

$$\Gamma\left(\frac{1}{9}\right)\cdot\Gamma\left(\frac{2}{9}\right)\cdots\Gamma\left(\frac{8}{9}\right) = \frac{16}{3}\pi^4$$

Solution We write

$$\Gamma\left(\frac{1}{9}\right)\cdot\Gamma\left(\frac{2}{9}\right)\cdots\Gamma\left(\frac{8}{9}\right)$$

$$= \Gamma\left(\frac{1}{9}\right)\cdot\Gamma\left(\frac{8}{9}\right)\cdot\Gamma\left(\frac{2}{9}\right)\cdot\Gamma\left(\frac{7}{9}\right)\cdot\Gamma\left(\frac{3}{9}\right)\cdot\Gamma\left(\frac{6}{9}\right)\cdot \Gamma\left(\frac{4}{9}\right)\cdot\Gamma\left(\frac{5}{9}\right)$$

$$= \frac{\pi}{\sin\left(\frac{\pi}{9}\right)}\cdot\frac{\pi}{\sin\left(\frac{2\pi}{9}\right)}\cdot\frac{\pi}{\sin\left(\frac{3\pi}{9}\right)}\cdot\frac{\pi}{\sin\left(\frac{4\pi}{9}\right)}$$

[Using complement property]

$$= \frac{2\pi^4}{\sqrt{3}}\frac{1}{\sin 20^\circ \sin 40^\circ \sin 80^\circ}$$

$$= \frac{2\pi^4}{\sqrt{3}}\frac{2}{\sin 20^\circ(\cos 40^\circ - \cos 120^\circ)}$$

$$= \frac{2\pi^4}{\sqrt{3}}\frac{4}{2\sin 20^\circ\cos 40^\circ + 2\sin 20^\circ\cos 60^\circ}$$

$$= \frac{8\pi^4}{\sqrt{3}}\frac{1}{\sin 60^\circ - \sin 20^\circ + \sin 20^\circ} = \frac{16}{3}\pi^4$$

Section Review 16.1

1. Show that (a) $\Gamma(7) = 720$, (b) $\Gamma\left(\frac{7}{2}\right) = \frac{3}{2}\sqrt{\pi}$.
2. Show that
   (a) $\Gamma\left(\frac{1}{3}\right)\cdot\Gamma\left(\frac{2}{3}\right) = \frac{2}{\sqrt{3}}\pi$
   (b) $2^{2p-1}\Gamma(p)\cdot\Gamma\left(p+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2p)$
3. Show that
   (a) $\int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$
   (b) $\int_0^{\infty} 3^{-x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{\log 3}}$
4. (a) Show that
   (i) $\int_0^{\infty} e^{-x}x^{\frac{5}{2}}dx = \frac{15}{8}\sqrt{\pi}$
   (ii) $\int_0^{\infty} e^{-a^2x}x^{\frac{3}{2}}dx = \frac{3}{4a^5}\sqrt{\pi}$
   (b) Evaluate
   $$\int_0^{\infty} e^{-kx}x^{m-1}dx, k > 0$$
   (c) Show that
   $$\int_0^1 x^m(\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$$
   where n is a positive integer and $m > -1$
5. Prove that
   (a) $\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2m+1)}{2^{2m}\Gamma(m+1)}$
   (b) $\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$

16.3 Beta Function

Leonhard Euler and Adrien-Marie Legendre studied and introduced the *beta function*, however, the name *beta* was given by Jacques Philippe Marie Binet (1786–1856), a French mathematician, physicist, and astronomer. The *beta function*, which is also called *the Euler integral of the first kind*, is another special function (of two independent variables) defined by

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \tag{16.7}$$

for $x, y > 0$.

It may be defined for complex arguments x and y, provided both $Re(x)$ and $Re(y)$, the real parts of x and y, respectively, are positive.

16.3.1 Property and other Forms of Beta Function

The beta function is symmetric, i.e., $\beta(x, y) = \beta(y, x)$. It has many other forms including

$$\beta(x, y) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta \cos^{2y-1}\theta d\theta \tag{16.8}$$

$$\beta(x, y) = 2\int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}}dt \tag{16.9}$$

16.3.2 Relationship between Gamma and Beta Functions

To derive the integral representation of the beta function, write the product of two gamma functions as

$$\Gamma(x)\Gamma(y) = \int_0^{\infty} e^{-u}u^{x-1}du \int_0^{\infty} e^{-v}v^{y-1}dv$$

Now, substituting $u = t^2$ and $v = s^2$, we have

$$\Gamma(x)\Gamma(y) = \left\{2\int_0^{\infty} e^{-t^2}t^{2x-1}dt\right\}\left\{2\int_0^{\infty} e^{-s^2}s^{2y-1}ds\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(t^2+s^2)}|t|^{2x-1}|s|^{2y-1}dtds$$

Transforming to polar co-ordinates with the substitutions $t = r\cos\theta$ and $s = r\sin\theta$, we have

$$\Gamma(x)\Gamma(y) = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}|r\cos\theta|^{2x-1}|r\sin\theta|^{2y-1}rdrd\theta$$

$$= \int_0^{\infty} e^{-r^2}r^{2x+2y-1}dr \int_0^{2\pi}\left|\cos^{2x-1}\theta\sin^{2y-1}\theta\right| d\theta$$

$$= \left\{\frac{1}{2}\int_0^{\infty} e^{-r^2}r^{2(x+y-1)}d(r^2)\right\}\left\{4\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta\sin^{2y-1}\theta d\theta\right\}$$

$$= \Gamma(x+y) \times \left\{2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta\sin^{2y-1}\theta d\theta\right\}$$

$$= \Gamma(x+y) \times \beta(x, y)$$

Thus, the relation between beta function and gamma functions is given by

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{16.10}$$

Example 16.4 From the relationship between gamma and beta functions, find the value of $\Gamma\left(\frac{1}{2}\right)$.

Solution In the relation

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

take $x = y = \frac{1}{2}$.

Then we have

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) \quad [\text{as } \Gamma(1) = 1]$$

or

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \beta\left(\frac{1}{2}, \frac{1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^0\theta\cos^0\theta d\theta = 2 \times \frac{\pi}{2} = \pi$$

Thus, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. □

Example 16.5 Assuming the result

$$\int_0^{\infty} \frac{x^{p-1}}{1+x}dx = \frac{\pi}{\sin(\pi p)}$$

derive the reflection formula of gamma function, i.e.

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(\pi p)}$$

Solution In the result

$$\beta(m, m) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

put $m = p$ and $n = 1 - p$.

Then, we have

$$\frac{\Gamma(p)\Gamma(1-p)}{\Gamma(p+1-p)} = \beta(p, 1-p)$$

or

$$\Gamma(p)\,\Gamma(1-p) = \int_0^{\infty} \frac{x^{p-1}}{1+x}dx = \frac{\pi}{\sin(\pi p)}$$

Thus

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(\pi p)}$$

□

Example 16.6 Prove that

$$\beta(m, m) = \frac{\sqrt{\pi}\Gamma(m)}{2^{2m-1}\Gamma\left(m+\frac{1}{2}\right)} = \frac{\beta\left(m, \frac{1}{2}\right)}{2^{2m-1}}.$$

Solution We know

$$\beta(m,m) = \frac{\Gamma(m)\Gamma(m)}{\Gamma(m+m)} = \frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)}$$

$$= \frac{\Gamma(m)\Gamma(m)}{(2m-1)\cdot(2m-2)\cdot(2m-3)\cdot(2m-4)\cdots4\cdot3\cdot2\cdot1}$$

$$= \frac{\Gamma(m)\Gamma(m)}{2^{2m-1}\left(m-\frac{1}{2}\right)\cdot(m-1)\cdot\left(m-\frac{3}{2}\right)\cdot(m-2)\cdots2\cdot\frac{3}{2}\cdot1\cdot\frac{1}{2}}$$

$$= \frac{\Gamma(m)\Gamma(m)}{2^{2m-1}\left\{\left(m-\frac{1}{2}\right)\cdot\left(m-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\right\}\cdot\{(m-1)\cdot(m-2)\cdots2\cdot1\}}$$

$$= \frac{\Gamma(m)}{2^{2m-1}\left\{\left(m-\frac{1}{2}\right)\cdot\left(m-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\right\}}$$

$$= \frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{2^{2m-1}\left\{\left(m-\frac{1}{2}\right)\cdot\left(m-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\right\}\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{2^{2m-1}\Gamma\left(m+\frac{1}{2}\right)} = \frac{\sqrt{\pi}\Gamma(m)}{2^{2m-1}\Gamma\left(m+\frac{1}{2}\right)} \quad \left[\text{as } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

Again,

$$\frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} = \beta\left(m,\frac{1}{2}\right)$$

and hence, the second result follows. □

Example 16.7 Prove that

$$\int_0^1 x^{m-1}(1-x^2)^{n-1}dx = \frac{1}{2}\beta\left(\frac{m}{2},n\right)$$

Solution Substitute $x = \sin\theta$ so that $dx = \cos\theta d\theta$. When $x = 0$, we have $\theta = 0$ and when $x = 1$, $\theta = \frac{\pi}{2}$. Thus, we have

$$\int_0^1 x^{m-1}(1-x^2)^{n-1}dx = \int_0^{\pi/2} \sin^{m-1}\theta(1-\sin^2\theta)^{n-1}\cos\theta d\theta$$

$$= \int_0^{\pi/2} \sin^{m-1}\theta\cos^{2n-1}\theta d\theta$$

$$= \frac{1}{2}\beta\left(\frac{m}{2},n\right)$$

□

Example 16.8 Prove that

$$\int_0^1 \frac{1}{(1-x^6)^{\frac{1}{6}}}dx = \frac{\pi}{3}$$

Solution Substitute $x^3 = \sin\theta$ so that $3x^2dx = \cos\theta d\theta$ or $dx = \frac{\cos\theta}{3\sin^{\frac{2}{3}}\theta}d\theta$.

When $x = 0$, we have $\theta = 0$ and when $x = 1$, $\theta = \frac{\pi}{2}$.

Thus

$$\int_0^1 \frac{1}{(1-x^6)^{\frac{1}{6}}}dx = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{3\sin^{\frac{2}{3}}\theta\cos^{\frac{2}{6}}\theta}d\theta$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{2}} \sin^{-\frac{2}{3}}\theta\cos^{\frac{2}{3}}\theta d\theta = \frac{1}{6}\beta\left(\frac{1}{6},\frac{5}{6}\right)$$

$$= \frac{1}{6}\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1+5}{6}\right)} = \frac{1}{6}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)$$

$$= \frac{1}{6}\frac{\pi}{\sin\left(\frac{\pi}{6}\right)} \quad \text{[Using complement property]}$$

$$= \frac{\pi}{3}$$

Example 16.9 Evaluate

$$\int_0^a x^9\sqrt[3]{a^6-x^6}dx$$

where a is a constant.

Solution Substitute $x^6 = a^6y$, i.e., $x = ay^{\frac{1}{6}}$ so that $dx = ay^{-\frac{5}{6}}\frac{dy}{6}$.

Then $x^9 = a^9y^{\frac{9}{6}}$ and $\sqrt[3]{a^6-x^6} = a(1-y)^{\frac{1}{3}}$.
When $x = 0$, we have $y = 0$ and when $x = a$, we have $y = 1$.
Thus, the given integral

$$= \int_0^1 a^9y^{\frac{9}{6}}a^2(1-y)^{\frac{1}{3}}y^{-\frac{5}{6}}\frac{dy}{6}$$

$$= \frac{a^{12}}{6}\int_0^1 y^{\frac{2}{3}}(1-y)^{\frac{1}{3}}dy$$

$$= \frac{a^{12}}{6}\beta\left(\frac{2}{3}+1,\frac{1}{3}+1\right) = \frac{a^{12}}{6}\beta\left(\frac{5}{3},\frac{4}{3}\right)$$

$$= \frac{a^{12}}{6}\frac{\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{5}{3}+\frac{4}{3}\right)}$$

$$= \frac{a^{12}}{6}\frac{\frac{2}{3}\Gamma\left(\frac{2}{3}\right)\frac{1}{3}\Gamma\left(\frac{1}{3}\right)}{\Gamma(3)}$$

$$= \frac{a^{12}}{18}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)$$

$$= \frac{a^{12}}{18}\frac{\pi}{\sin\frac{\pi}{3}} \quad \text{[Using complement property]}$$

$$= \frac{\pi a^{12}}{9\sqrt{3}}$$

Example 16.10 Show that

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(b+cx)^{m+n}}dx = \frac{1}{b^n(b+c)^{m+n}}\beta(m,n),$$

where b and c are constants.

Solution Substitute

$$x = \frac{by}{b+c-cy}$$

i.e.,

$$y = \frac{(b+c)x}{b+cx}$$

so that

$$dx = \frac{(b+c-cy)b - by(-c)}{(b+c-cy)^2}dy = \frac{b(b+c)}{(b+c-cy)^2}dy$$

Now

$$1-x = 1 - \frac{by}{b+c-cy} = (b+c)\frac{1-y}{b+c-cy}$$

and

$$b+cx = b + c\frac{by}{b+c-cy} = \frac{b+c}{b+c-cy}$$

When $x=0$, we have $y=0$ and when $x=1$, we have $y=1$. Thus, we have

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(b+cx)^{m+n}}dx$$

$$= \int_0^1 \left(\frac{by}{b+c-cy}\right)^{m-1}\left\{(b+c)\frac{1-y}{b+c-cy}\right\}^{n-1} \times \left(\frac{b+c-cy}{b+c}\right)^{m+n}\frac{b(b+c)}{(b+c-cy)^2}dy$$

$$= \int_0^1 b^{m-1}y^{m-1}(b+c)^{n-1}(1-y)^{n-1}\frac{1}{(b+c)^{m+n}}b(b+c)dy$$

$$= \frac{1}{b^n(b+c)^m}\int_0^1 y^{m-1}(1-y)^{n-1}dy = \frac{1}{b^n(b+c)^m}\beta(m,n)$$

□

Example 16.11 Show that

$$\int_0^{\frac{\pi}{2}} \left(\tan\theta + \sqrt{\sec\theta}\right)d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left[\Gamma\left(\frac{3}{4}\right) + \frac{\pi}{\Gamma\left(\frac{1}{4}\right)}\right]$$

Solution

$$\int_0^{\frac{\pi}{2}} \left(\sqrt{\tan\theta} + \sqrt{\sec\theta}\right)d\theta$$

$$= \int_0^{\frac{\pi}{2}} \tan\theta d\theta + \int_0^{\frac{\pi}{2}} \sqrt{\sec\theta}d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta\cos^{-\frac{1}{2}}\theta d\theta + \int_0^{\frac{\pi}{2}} \cos^{-\frac{1}{2}}\theta d\theta$$

$$= \frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2}, \frac{-\frac{1}{2}+1}{2}\right) + \frac{1}{2}\beta\left(\frac{1}{2}, \frac{-\frac{1}{2}+1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{4}\right) + \frac{1}{2}\beta\left(\frac{1}{2},\frac{1}{4}\right)$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)} + \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{4}\right)}$$

$$= \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) + \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) + \frac{1}{2}\sqrt{\pi}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left[\Gamma\left(\frac{3}{4}\right) + \frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}\right]$$

□

16.3.3 Derivatives

The derivative of beta function may be expressed as

$$\frac{\partial}{\partial x}\beta(x,y) = \beta(x,y)\left\{\frac{1}{\Gamma(x)}\frac{\partial\Gamma(x)}{\partial x} - \frac{1}{\Gamma(x+y)}\frac{\partial\Gamma(x+y)}{\partial x}\right\}$$
$$= \beta(x,y)\{\psi(x) - \psi(x+y)\}$$

where $\psi(x) = \dfrac{1}{\Gamma(x)}\dfrac{\partial\Gamma(x)}{\partial x}$, which is known as the *digamma function.*

Section Review 16.2

1. Prove that
   (a) $\dfrac{\beta(m,n+1)}{n} = \dfrac{\beta(m+1,n)}{m} = \dfrac{\beta(m,n)}{m+n}$
   (b) $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$
2. Show that $\beta(p,q)\cdot\beta(p+q,r) = \beta(q,r)\cdot\beta(q+r,m) = \beta(r,p)\cdot\beta(r+p,q)$.
3. Evaluate the following integrals:
   (a) $\displaystyle\int_0^2 x\sqrt[3]{8-x^3}dx$
   (b) $\displaystyle\int_0^a x^3(a^5-x^5)^3dx$
   (c) $\displaystyle\int_0^a x^3(a^3-x^3)^4dx$
   (d) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5 x\cos^7 x dx$
   (e) $\displaystyle\int_{-1}^1 \sqrt{\frac{1-x}{1+x}}dx$

Gamma Function

The gamma function is a factorial function with its domain extended to include non-integers.

The gamma function denoted by Γ is defined for any positive number n as

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} \, dt \quad (1)$$

$$\Gamma(-a) = \int_0^\infty e^{-t} t^{-a-1} \, dt$$

Properties

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2) = \cdots = n!\,\Gamma(1) = n!$$

$$\Gamma(0) = \int_0^\infty t^{-1} e^{-t} \, dt = \infty$$

$$\Gamma(1) = \int_0^\infty e^{-t} \, dt = \left[\frac{-1}{e^t}\right]_0^\infty = -(0-1) = 1$$

$$\Gamma(2) = \int_0^\infty t e^{-t} \, dt = -te^{-t} - e^{-t} \Big]_0^\infty$$

$$= -\frac{(t+1)}{e^t}\Big]_0^\infty = -(0-1) = 1$$

$$\Gamma(2) = 1 \times 1 = 1\,\Gamma(1)$$

$$\Gamma(x+1) = x\Gamma(x)$$

Proof

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} \, dt \quad \left(\text{Putting } n = x+1 \text{ in eq } (1)\right)$$

$$= t^x(-e^{-t})\Big]_0^\infty - \int \frac{d}{dt}(t^x)(-e^{-t}) \, dt$$

(integrating by parts)

$$= 0 + \int_0^\infty x t^{x-1} e^{-t} \, dt$$

$$= x\int_0^\infty t^{x-1} e^{-t} \, dt = x\Gamma(x)$$

$$\Gamma(x+1) = x\,\Gamma(x)$$

Properties

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$\Gamma(n+1) = n!$$

$$* \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Ex- $\Gamma(5) = \Gamma(4+1) = 4!$

$$\Gamma(9) = \Gamma(8+1) = 8!$$

$$\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right)$$

$$= \frac{5}{2}\Gamma\left(\frac{3}{2}+1\right) = \frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}$$

Ex- Evaluate $\int_0^\infty e^{2ax-x^2}\,dx$ using gamma funct

Sol^n $\int_0^\infty e^{-(x^2-2ax)}\,dx$

$$= \int_0^\infty e^{-(x^2-2ax+a^2-a^2)}\,dx$$

$$= \int_0^\infty e^{-(x-a)^2}\,e^{a^2}\,dx$$

$$= e^{a^2}\int_0^\infty e^{-(x-a)^2}\,dx$$

$$= \frac{e^{a^2}}{2} \int_0^{\infty} e^{-t}\, t^{-\frac{1}{2}}\, dt$$

Put $\sqrt{t} = x - a$

$dx = \frac{1}{2} t^{-1/2} dt$

$$= \frac{e^{a^2}}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} e^{a^2}$$

Properties

* ~~Beta Function~~ $\Gamma(n+1) = n\Gamma(n)$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \Rightarrow \Gamma(0) = \frac{\Gamma(1)}{0} \to \infty$$

$$\Gamma(-a) = \frac{\Gamma(-a+1)}{-a}$$ (For negative number)

$$\Gamma(-1) = \frac{\Gamma(0)}{-1} = \infty$$

Ex- Evaluate $I = \int_0^1 \frac{1}{\sqrt{-\log x}}\, dx$ using gamma function

Sol^n put $-\log x = y \Rightarrow x = e^{-y}$

$dx = -e^{-y} dy$

when $x = 0$ $y = \infty$

when $x = 1$ $y = 0$

$$I = \int_{\infty}^{0} \frac{1}{\sqrt{y}} (-e^{-y})\, dy$$

$$= \int_0^{\infty} y^{-\frac{1}{2}} e^{-y}\, dy = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

s

a function

Ex -

Prove that

$$\int_0^\infty e^{-m^2x^2}\,dx = \frac{\sqrt{\pi}}{2m}$$

Sol^n put $t = m^2x^2 \Rightarrow x = \frac{\sqrt{t}}{m}$

$$dx = \frac{1}{2m}\,t^{-1/2}\,dt$$

when $x = 0$ $t = 0$

when $x = \infty$ $t = \infty$

$$\int_0^\infty e^{-m^2x^2}\,dx = \frac{1}{2m}\int_0^\infty e^{-t}\,t^{-\frac{1}{2}}\,dt$$

$$= \frac{1}{2m}\int_0^\infty e^{-t}\,t^{\frac{1}{2}-1}\,dt$$

$$= \frac{1}{2m}\,\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2m}$$

BETA FUNCTION

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt$$

for $x, y > 0$ is called Beta function. It is also called Euler integral of first kind is a special function (two independent variables)

Properties $\beta(x, y) = \beta(y, x)$

It has other forms as

$$\beta(x, y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2y-1}\theta \, d\theta$$

$$\beta(x, y) = 2\int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}} \, dt$$

Relation between Gamma & Beta Function

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Ex – Find value of $\Gamma(\tfrac{1}{2})$ from the relation between gamma and beta function.

Soln $\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

Putting $x = \frac{1}{2}$ and $y = \frac{1}{2}$ we get

$$\beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2})}{\Gamma(1)} = \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2$$

(since $\Gamma(1) = 1$)

Hence

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = 2\int_0^{\pi/2} \sin^0\theta \cos^0\theta \, d\theta$$

OXFORD

$$= 2\int_0^{\pi/2} d\theta = 2[\theta]_0^{\pi/2} = 2\cdot\frac{\pi}{2} = \pi$$

Hence $[\Gamma(\tfrac{1}{2})]^2 = \pi \Rightarrow \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

<u>EX</u> Evaluate $I = \int_0^a x^9 \cdot \sqrt[3]{a^6 - x^6}\, dx$

where a is a constant.

<u>Solⁿ</u> Putting $x^6 = a^6 y$

$\Rightarrow x = a(y)^{1/6} \Rightarrow dx = \frac{a}{6} y^{-5/6}\, dy$

Hence $x^9 = a^9 y^{3/2}$ and $\sqrt[3]{a^6 - x^6} = a(1-y)^{\frac{1}{3}}$

When $x = 0$, $y = 0$ and when $x = a$, $y = 1$

Then $I = \int_0^1 \frac{a^9}{6} y^{\frac{9}{6}} a^2 (1-y)^{\frac{1}{3}} y^{-\frac{5}{6}}\, dy$

$$= \frac{a^{12}}{6} \int_0^1 y^{2/3} (1-y)^{1/3}\, dy$$

$$= \frac{a^{12}}{6} \beta\left(\frac{2}{3}+1, \frac{1}{3}+1\right) = \frac{a^{12}}{6} \beta\left(\frac{5}{3}, \frac{4}{3}\right)$$

$$= \frac{a^{12}}{6} \frac{\Gamma(\frac{5}{3})\Gamma(\frac{4}{3})}{\Gamma(\frac{5}{3}+\frac{4}{3})}$$

$$= \frac{a^{12}}{6} \frac{\frac{2}{3}\Gamma(\frac{2}{3})\,\frac{1}{3}\Gamma(\frac{1}{3})}{\Gamma(3)}$$

$$= \frac{a^{12}}{6} \times \frac{2}{3} \times \frac{1}{2!} \times \frac{1}{3}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)$$

$$= \frac{a^{12}}{18}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{a^{12}}{18} \frac{\pi}{\sin\frac{\pi}{3}}$$

$$= \frac{\pi a^{12}}{9\sqrt{3}}$$

(Using Complement property $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$)

Ex Prove that
$$I = \int_0^1 \frac{1}{(1-x^6)^{1/6}}\,dx = \frac{\pi}{3}$$

Sol^n put $x^3 = \sin\theta$

$$\Rightarrow 3x^2\,dx = \cos\theta\,d\theta \Rightarrow dx = \frac{\cos\theta\,d\theta}{3\sin^{2/3}\theta}$$

when $x = 0$, $\theta = 0$ and when $x = 1$ $\theta = \pi/2$

$$I = \int_0^{\pi/2} \frac{\cos\theta\,d\theta}{3\sin^{2/3}\theta\,\cos^{\frac{1}{3}}\theta} = \frac{1}{3}\int_0^{\pi/2} \sin^{-2/3}\theta\,\cos^{\frac{2}{3}}\theta\,d\theta$$

$$= \frac{1}{6}B\left(\frac{1}{6},\frac{5}{6}\right) = \frac{1}{6}\,\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{6}+\frac{5}{6}\right)}$$

$$= \frac{1}{6}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)$$ (as denom $= \Gamma(1) = 1$)

$$= \frac{1}{6}\,\frac{\pi}{\sin\frac{\pi}{6}} = \frac{\pi}{3}$$ (Using complement property)