4·12. (1) MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

Def. *A function $f(x, y)$ is said to have a* **maximum** *or* **minimum** *at $x = a, y = b$, according as*

$$f(a + h, b + k) < \text{or} > f(a, b),$$

for all positive or negative small values of h and k.

In other words, if $\Delta = f(a + h, b + k) - f(a, b)$ *is of the same sign for all small values of h, k, and if this sign is negative, then $f(a, b)$ is a maximum. If this sign is positive, $f(a, b)$ is a minimum.*

Considering $z = f(x, y)$ as a surface, maximum value of z occurs at the top of an elevation (*e.g.* a dome) from which the surface descends in every direction and a minimum value occurs at the bottom of a depression (*e.g.* a bowl) from which the surface ascends in every direction. Sometimes the maximum or minimum value may form a *ridge* such that the surface descends or ascends in all directions except that of the ridge. Besides these, we have such a point of the surface, where the tangent plane is horizontal and the surface falls for displacement in certain directions and rises for displacements in other directions. Such a point is called a **saddle point.**

Note. *A maximum or minimum value of a function is called its* **extreme value.**

(2) Conditions for f(x, y) to be maximum or minimum.

Using Taylor's theorem page 176, we have $\Delta = f(a + h, b + k) - f(a, b)$

$$= \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{a,\,b} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots\ldots \qquad \ldots(i)$$

For small values of h and k, the second and higher order terms are still smaller and hence may be neglected. Thus

$$\text{sign of } \Delta = \text{sign of } [hf_x(a, b) + kf_y(a, b)].$$

Taking $h = 0$ we see that the right hand side changes sign when k changes sign. Hence $f(x, y)$ cannot have a maximum or a minimum at (a, b) unless $f_y(a, b) = 0$.

Similarly taking $k = 0$, we find that $f(x, y)$ cannot have a maximum or minimum at (a, b) unless $f_x(a, b) = 0$.

Hence *the necessary conditions for f (x, y) to have a maximum or a minimum at (a, b) are that*

$$\mathbf{f_x(a, b) = 0,\ f_y(a, b) = 0.}$$

If these conditions are satisfied, then for small value of h and k, (i) gives

$$\text{sign of } \Delta = \text{sign of} \left[\frac{1}{2!}(h^2r + 2hks + k^2t)\right] \text{ where } r = f_{xx}(a, b),\ s = f_{xy}(a, b) \text{ and } t = f_{yy}(a, b).$$

Now $\quad h^2r + 2hks + k^2t = \frac{1}{r}\left[h^2r^2 + 2hkrs + k^2rt\right] = \frac{1}{r}\left[(hr + ks)^2 + k^2(rt - s^2)\right]$

Thus sign of Δ = sign of $\frac{1}{2r}\{(hr + ks)^2 + k^2(rt - s^2)\}$ $\qquad \ldots(ii)$

In (ii), $(hr + ks)^2$ is always positive and $k^2(rt - s^2)$ will be positive if $rt - s^2 > 0$. In this case, Δ will have the same sign as that of r for all values of h and k.

Hence if $rt - s^2 > 0$, then $f(x, y)$ *has a maximum or a minimum at (a, b) according as* $r <$ or > 0.

If $rt - s^2 < 0$, then Δ will change with h and k and hence there is no maximum or minimum at (a, b) *i.e.* it is a *saddle point.*

If $rt - s^2 = 0$, further investigation is required to find whether there is a maximum or minimum at (a, b) or not.

Note. Stationary value. *$f(a, b)$ is said to be a stationary value of $f(x, y)$, if $f_x(a, b) = 0$ and $f_y(a, b) = 0$ i.e.* the function is stationary at (a, b).

Thus every extreme value is a stationary value but the converse may not be true.

(3) Working rule to find the maximum and minimum values of f(x, y).

1. *Find $\partial f/\partial x$ and $\partial f/\partial y$ and equate each to zero. Solve these as simultaneous equations in x and y.* Let (a, b), (c, d),..... be the pairs of values.

2. *Calculate the value of $r = \partial^2 f/\partial x^2$, $s = \partial^2 f/\partial x \partial y$, $t = \partial^2 f/\partial y^2$ for each pair of values.*

3. (i) *If $rt - s^2 > 0$ and $r < 0$ at (a, b), $f(a, b)$ is a max. value.*

(ii) *If $rt - s^2 > 0$* and $r > 0$ *at (a, b), $f(a, b)$ is a min. value.*

(iii) *If $rt - s^2 < 0$ at (a, b), $f(a, b)$ is not an extreme value, i.e. (a, b) is a saddle point.*

(iv) *If $rt - s^2 = 0$ at (a, b), the case is doubtful and needs further investigation.*

Similarly examine the other pairs of values one by one.

Example 4·24. *Examine the following function for extreme values :*

$$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2. \qquad \textit{(Delhi, 1997)}$$

We have $\qquad f_x = 4x^3 - 4x + 4y\ ;\ f_y = 4y^3 + 4x - 4y$

and $\qquad r = f_{xx} = 12x^2 - 4,\ s = f_{xy} = 4,\ t = f_{yy} = 12y^2 - 4 \qquad \ldots(i)$

New $f_x = 0,\ f_y = 0$ give $x^3 - x + y = 0$, ...(*i*) $y^3 + x - y = 0$ ...(*ii*)

Adding these, we get $4(x^3 + y^3) = 0$ or $y = -x$.

Putting $y = -x$ in (*i*), we obtain $x^3 - 2x = 0$, *i.e.* $x = \sqrt{2}, -\sqrt{2}, 0$.

$\therefore$ Corresponding values of y are $-\sqrt{2}, \sqrt{2}, 0$.

At $(\sqrt{2}, -\sqrt{2})$, $rt - s^2 = 20 \times 20 - 4^2 = +$ve and r is also +ve. Hence $f(\sqrt{2}, -\sqrt{2})$ is a minimum value.

At $(-\sqrt{2}, \sqrt{2})$ also both $rt - s^2$ and r are + ve.

Hence $f(-\sqrt{2}, \sqrt{2})$ is also a minimum value.

At (0, 0), $rt - s^2 = 0$ and, therefore, further investigation is needed.

Now $f(0, 0) = 0$ and for points along the x-axis, where $y = 0$, $f(x, y) = x^4 - 2x^2 = x^2(x^2 - 2)$, which is negative for points in the neighbourhood of the origin.

Again for points along the line $y = x$, $f(x, y) = 2x^4$ which is positive.

Thus in the neighbourhood of (0, 0) there are points where $f(x, y) < f(0, 0)$ and there are points where $f(x, y) > f(0, 0)$.

Hence $f(0, 0)$ is not an extreme value.

Example 4·25. *Discuss the maxima and minima of* $f(x, y) = x^3y^2(1 - x - y)$.

(*Gauhati, 1999; Ranchi, 1998*)

We have $f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$; $f_y = 2x^3y - 2x^4y - 3x^3y^2$

and $r = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$; $s = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$; $t = f_{yy} = 2x^3 - 2x^4 - 6x^3y$.

When $f_x = 0,\ f_y = 0$, we have $x^2y^2(3 - 4x - 3y) = 0,\ x^3y(2 - 2x - 3y) = 0$

Solving these, the stationary points are (1/2, 1/3), (0, 0).

Now $rt - s^2 = x^4y^2[12(1 - 2x - y)(1 - x - 3y) - (6 - 8x - 9y)^2]$

At (1/2, 1/3), $rt - s^2 = \frac{1}{16} \cdot \frac{1}{9}\left[12\left(1 - 1 - \frac{1}{3}\right)\left(1 - \frac{1}{2} - 1\right) - (6 - 4 - 3)^2\right] = \frac{1}{14} > 0$

Also $r = 6\left(\frac{1}{2} \cdot \frac{1}{9} - \frac{2}{4} \cdot \frac{1}{9} - \frac{1}{2} \cdot \frac{1}{27}\right) = -\frac{1}{9} < 0$

Hence $f(x, y)$ has a maximum at (1/2, 1/3) and maximum value $= \frac{1}{8} \cdot \frac{1}{9}\left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}$.

At (0, 0), $rt - s^2 = 0$ and therefore further investigation is needed.

For points along the line $y = x$, $f(x, y) = x^5(1 - 2x)$ which is positive for $x = 0.1$ and negative for $x = -0.1$ *i.e.* in the neighbourhood of (0, 0), there are points where $f(x, y) > f(0, 0)$ and there are points where $f(x, y) < f(0, 0)$. Hence $f(0, 0)$ is not an extreme value.

Example 4·26. *In a plane triangle, find the maximum value of* cos *A* cos *B* cos *C*.

(*Raipur, 1998*)

We have $A + B + C = \pi$ so that $C = \pi - (A + B)$.

$$\therefore \cos A \cos B \cos C = \cos A \cos B \cos[\pi - (A + B)]$$
$$= -\cos A \cos B \cos(A + B) = f(A, B), \text{ say.}$$

We get $\frac{\partial f}{\partial A} = \cos B\,[\sin A \cos(A + B) + \cos A \sin(A + B)]$

$$= \cos B \sin(2A + B)$$

and $\frac{\partial f}{\partial B} = \cos A \sin(A + 2B)$

$$\frac{\partial f}{\partial A}=0,\ \frac{\partial f}{\partial B}=0 \text{ only when } A=B=\pi/3.$$

Also $r=\dfrac{\partial^2 f}{\partial A^2}=2\cos B\cos(2A+B)$, $t=\dfrac{\partial^2 f}{\partial B^2}=2\cos A\cos(A+2B)$

$$s=\frac{\partial^2 f}{\partial A\partial B}=-\sin B\sin(2A+B)+\cos B\cos(2A+B)=\cos(2A+2B)$$

When $A=B=\pi/3$, $r=-1$, $s=-\dfrac{1}{2}$, $t=-1$ so that $rt-s^2=3/4$.

These show that $f(A,B)$ is maximum for $A=B=\pi/3$.

Then $C=\pi-(A+B)=\pi/3$.

Hence $\cos A\cos B\cos C$ is maximum when each of the angles is $\pi/3$ *i.e.* triangle is equilateral and its maximum value = 1/8.

4·13. LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

Sometimes it is required to find the stationary values of a function of several variables which are not all independent but are connected by some given relations. Ordinarily, we try to convert the given function to the one, having least number of independent variables with the help of given relations. Then solve it by the above method. When such a procedure becomes impracticable, Lagrange's method* proves very convenient. Now we explain this method.

Let $\qquad u=f(x,y,z) \qquad$ ...(1)

be a function of three variables x, y, z which are connected by the relation.

$$\phi(x,y,z)=0 \qquad \text{...(2)}$$

For u to have stationary values, it is necessary that

$$\partial u/\partial x=0,\ \partial u/\partial y=0,\ \partial u/\partial z=0.$$

$$\therefore \quad \frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=du=0 \qquad \text{...(3)}$$

Also differentiating (2), we get $\dfrac{\partial\phi}{\partial x}dx+\dfrac{\partial\phi}{\partial y}dy+\dfrac{\partial\phi}{\partial z}dz=d\phi=0$ ...(4)

Multiply (4) by a parameter λ and add to (3). Then

$$\left(\frac{\partial u}{\partial x}+\lambda\frac{\partial\phi}{\partial x}\right)dx+\left(\frac{\partial u}{\partial y}+\lambda\frac{\partial\phi}{\partial y}\right)dy+\left(\frac{\partial u}{\partial z}+\lambda\frac{\partial\phi}{\partial z}\right)dz=0$$

This equation will be satisfied if $\dfrac{\partial u}{\partial x}+\lambda\dfrac{\partial\phi}{\partial x}=0$, $\dfrac{\partial u}{\partial y}+\lambda\dfrac{\partial\phi}{\partial y}=0$, $\dfrac{\partial u}{\partial z}+\lambda\dfrac{\partial\phi}{\partial z}=0$.

These three equations together with (2) will determine the values of x, y, z and λ for which u is stationary.

Working rule : 1. *Write* $F=f(x,y,z)+\lambda\phi(x,y,z)$

2. *Obtain the equations* $\dfrac{\partial F}{\partial x}=0,\ \dfrac{\partial F}{\partial y}=0,\ \dfrac{\partial F}{\partial z}=0.$

3. *Solve the above equations together with* $\phi(x,y,z)=0$.

The values of x, y, z so obtained will give the stationary value of $f(x,y,z)$.

Obs. Although the Lagrange's method is often very useful in application yet the drawback is that we cannot determine the nature of the stationary point. This can sometimes, be determined from physical considerations of the problem.

* See footnote page 122.

Example 4·27. *A rectangular box open at the top is to have volume of 32 cubic ft. Find the dimensions of the box requiring least material for its construction.* (*J.N.T.U., 1998*)

Let x, y and z ft. be the edges of the box and S be its surface.

Then $$S = xy + 2yz + 2zx \quad ...(i)$$

and $$xyz = 32 \quad ...(ii)$$

Eliminating z from (*i*) with the help of (*ii*), we get $S = xy + 2(y + x)\dfrac{32}{xy} = xy + 64\left(\dfrac{1}{x} + \dfrac{1}{y}\right)$

$\therefore\ \partial S/\partial x = y - 64/x^2 = 0$ and $\partial S/\partial y = x - 64/y^2 = 0$.

Solving these, we get $x = y = 4$.

Now $r = \partial^2 S/\partial x^2 = 128/x^3$, $s = \partial^2 S/\partial x\partial y = 1$, $t = \partial^2 S/\partial y^2 = 128/y^3$.

At $x = y = 4$, $rt - s^2 = 2 \times 2 - 1 = +$ ve and r is also + ve.

Hence S is minimum for $x = y = 4$. Then from (*ii*), $z = 2$.

Otherwise (*by Lagrange's method*) :

Write $$F = xy + 2yz + 2zx + \lambda\,(xyz - 32)$$

Then $$\frac{\partial F}{\partial x} = y + 2z + \lambda yz = 0 \quad ...(iii)$$

$$\frac{\partial F}{\partial y} = x + 2z + \lambda zx = 0 \quad ...(iv)$$

$$\frac{\partial F}{\partial z} = 2y + 2x + \lambda xy = 0 \quad ...(v)$$

Multiplying (*iii*) by x and (*iv*) by y and subtracting, we get $2zx - 2zy = 0$ or $x = y$.

[The value $z = 0$ is neglected, as it will not satisfy (*ii*)]

Again multiplying (*iv*) be y and (*v*) be z and subtracting, we get $y = 2z$.

Hence the dimensions of the box are $x = y = 2z = 4$ ...(*vi*)

Now let us see what happens as z increases from a small value to a large one. When z is small, the box is flat with a large base showing that S is large. As z increases, the base of the box decreases rapidly and S also decreases. After a certain stage, S again starts increasing as z increases. Thus S must be a minimum at some intermediate stage which is given by (*vi*). Hence S is minimum when $x = y = 4$ ft and $z = 2$ ft.

Example 4·28. *Find the maximum and minimum distances of the point (3, 4, 12) from the sphere* $x^2 + y^2 + z^2 = 1$. (*Kanpur, 1996*)

Let $P(x, y, z)$ be any point on the sphere and $A(3, 4, 12)$ the given point so that

$$AP^2 = (x - 3)^2 + (y - 4)^2 + (z - 12)^2 = f(x, y, z), \text{ say} \quad ...(i)$$

We have to find the maximum and minimum values of $f(x, y, z)$ subject to the condition

$$\phi(x, y, z) = x^2 + y^2 + z^2 - 1 = 0 \quad ...(ii)$$

Let $F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$

$$= (x - 3)^2 + (y - 4)^2 + (z - 12)^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

Then $$\frac{\partial F}{\partial x} = 2(x - 3) + 2\lambda x,\quad \frac{\partial F}{\partial y} = 2(y - 4) + 2\lambda y,\quad \frac{\partial F}{\partial z} = 2(z - 12) + 2\lambda z$$

$\therefore\ \dfrac{\partial F}{\partial x} = 0,\ \dfrac{\partial F}{\partial y} = 0$ and $\dfrac{\partial F}{\partial z} = 0$ give

$$x - 3 + \lambda x = 0,\ y - 4 + \lambda y = 0,\ z - 12 + \lambda z = 0 \quad ...(iii)$$

which give $\lambda = -\frac{x-3}{x} = -\frac{y-4}{y} = -\frac{z-12}{z}$

$$= \pm \frac{\sqrt{[(x-3)^2 + (y-4)^2 + (z-12)^2]}}{\sqrt{(x^2 + y^2 + z^2)}} = \pm \frac{\sqrt{f}}{1}$$

Substituting for λ in (*iii*), we get

$$x = \frac{3}{1+\lambda} = \frac{3}{1 \pm \sqrt{f}}, y = \frac{4}{1 \pm \sqrt{f}}, z = \frac{12}{1 + \pm \sqrt{f}}$$

$$\therefore \qquad x^2 + y^2 + z^2 = \frac{9 + 16 + 144}{(1 \pm \sqrt{f})^2} = \frac{169}{(1 \pm \sqrt{f})^2}$$

Using (*ii*), $1 = \frac{169}{(1 \pm \sqrt{g})^2}$ or $1 \pm \sqrt{f} = \pm 13, \sqrt{f} = 12, 14.$

[We have left out the negative values of $\sqrt{f}$, because $\sqrt{f} = AP$ is +ve by (*i*)]

Hence maximum $AP = 14$ and minimum $AP = 12$.

Example 4·29. *Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \qquad \textit{(Andhra, 1998; A.M.I.E., 1997 ; Rewa, 1994)}$$

Let the edges of the parallelopiped be $2x$, $2y$ and $2z$ which are parallel to the axes.

Then its volume $V = 8xyz$.

Now we have to find the maximum value of V subject to the condition that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \qquad ...(i)$$

Write $\qquad F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$

Then $\qquad \frac{\partial F}{\partial x} = 8yz + \lambda\left(\frac{2x}{a^2}\right) = 0 \qquad ...(ii)$

$$\frac{\partial F}{\partial y} = 8zx + \lambda\left(\frac{2y}{b^2}\right) = 0 \qquad ...(iii)$$

$$\frac{\partial F}{\partial z} = 8xy + \lambda\left(\frac{2z}{c^2}\right) = 0 \qquad ...(iv)$$

Equating the values of λ from (*ii*) and (*iii*), we get $x^2/a^2 = y^2/b^2$

Similarly from (*iii*) and (*iv*), we obtain $y^2/b^2 = z^2/c^2 \quad \therefore \; x^2/a^2 = y^2/b^2 = z^2/c^2$

Substituting these in (*i*), we get $x^2/a^2 = \frac{1}{3}$ *i.e.* $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3}$

These give $x = a/\sqrt{3}, y = b/\sqrt{3}, z = c/\sqrt{3}$ ...(*v*)

When $x = 0$, the parallelopiped is just a rectangular sheet and as such its volume $V = 0$.

As x increases, V also increases continuously.

Thus V must be greatest at the stage given by (*v*).

Hence the greatest volume $= \frac{8abc}{3\sqrt{3}}$.