LINEAR SYSTEM OF EQUATIONS

A linear system of m equations in n unknowns $x_1, x_2, \ldots, x_n$ is

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n$$

In matrix form $AX = b$

where the coefficient matrix A is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ and } X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

~~Geometric~~

$$\bar{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{bmatrix}$$ is called augmented matrix.

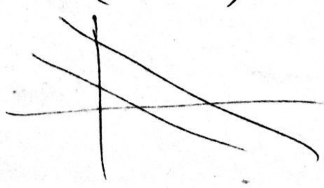
Geometric Interpretation : Existence of solution

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Case-1 If $\dfrac{a_{11}}{a_{21}} = \dfrac{a_{12}}{a_{22}} \neq \dfrac{b_1}{b_2}$

there is no solution. The lines are parallel

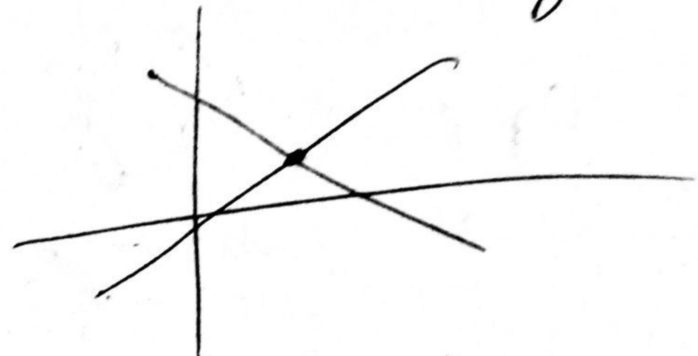
Ex. $x + y = 1$
$2x + 2y = 5$



Case-2 $\dfrac{a_{11}}{a_{21}} \neq \dfrac{a_{12}}{a_{22}} \neq \dfrac{b_1}{b_2}$

Precisely one solution. Intersecting lines

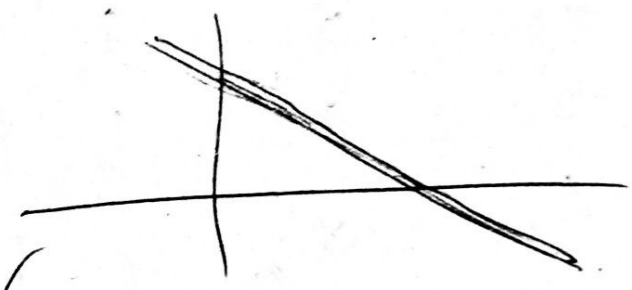
EX
$$x + y = 1$$
$$x - y = 0$$



Case-3 $\dfrac{a_{11}}{a_{21}} = \dfrac{a_{12}}{a_{22}} = \dfrac{b_1}{b_2}$

Infinitely many solutions. Coincident lines.

EX:
$$x + y = 1$$
$$2x + 2y = 2$$



THREE UNKNOWNS

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$$a_1x + a_2y + a_3z = d_1$$
$$b_1x + b_2y + b_3z = d_2$$
$$c_1x + c_2y + c_3z = d_3$$

$$D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad D_1 = \begin{vmatrix} d_1 & a_2 & a_3 \\ d_2 & b_2 & b_3 \\ d_3 & c_2 & c_3 \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_1 & d_1 & a_3 \\ b_1 & d_2 & b_3 \\ c_1 & d_3 & c_3 \end{vmatrix} \quad D_3 = \begin{vmatrix} a_1 & a_2 & d_1 \\ b_1 & b_2 & d_2 \\ c_1 & c_2 & d_3 \end{vmatrix}$$

$$x = \frac{D_1}{D} \quad y = \frac{D_2}{D} \quad z = \frac{D_3}{D}$$

If $D \neq 0$ Unique solution

If $D = 0$, $D_1 = 0$, $D_2 = 0$, $D_3 = 0$ infinitely many solutions

If $D = 0$ and atleast one of D_1, D_2, D_3 is non-zero then no solution.

Gauss Elimination Method

Step-1 Write the given eq^{n}s in matrix form. Take the augmented matrix A.

Step-2 Convert matrix A to upper-triangular form by row operation.

Step-3 Write the triangular matrix in equation form. Find the values of x, y, z by back substitution.

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Ex. 3 $3w + 2x + 2y - 5z = 8$ ——— (1)

$2w + 5x + 5y - 18z = 9$ ——— (2)

$4w - x - y + 8z = 7$ ——— (3)

Sol^n Solve by Gauss Elimination method

The augmented matrix is

$$A = \left[\begin{array}{cccc:c} 3 & 2 & 2 & -5 & 8 \\ 2 & 5 & 5 & -18 & 9 \\ 4 & -1 & -1 & 8 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc:c} 3 & 2 & 2 & -5 & 8 \\ 0 & 11 & 11 & -44 & 11 \\ 0 & -11 & -11 & 44 & -11 \end{array}\right] \begin{array}{l} R_2 \to 3R_2 - 2R_1 \\ R_3 \to 3R_3 - 4R_1 \end{array}$$

$$\sim \left[\begin{array}{ccccc} 3 & 2 & 2 & -5 & 8 \\ 0 & 11 & 11 & -44 & 11 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_3 \to R_3 + R_2$$

$$\sim \left[\begin{array}{ccccc} 3 & 2 & 2 & -5 & 8 \\ 0 & 1 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_2 \to \frac{1}{11} R_2$$

Writing in equation form we get

$3w + 2x + 2y - 5z = 8$ ——— (4)

$x + y - 4z = 1$ ——— (5)

$0 = 0$ ——— (6)

From eq^n (5), $y = 4z - x + 1$

Putting in eq^n (4) $3w + \cancel{2x} + 8z - \cancel{2x} + 2 - 5z = 8$

$3w - 3z = 6$

$w = 2 - z$

Solutions are $x = 4z - y + 1$

$y = 4z - x + 1$, $w = 2 - z$

In particular If $x = 0$, $z = 1$ then $y = 5$, $w = 1$

If $x = 1$, $z = 2$ then $y = 8$, $w = 0$

If $x = 2$, $z = 3$ then $y = 11$, $w = -1$

Infinite many solutions

EX-4 Solve by Gauss elimination

$$-x+y+2z=2$$
$$3x-y+z=6$$
$$-x+3y+4z=4$$

Sol Augmented matrix $A = \left[\begin{array}{ccc:c} -1 & 1 & 2 & 2 \\ 3 & -1 & 1 & 6 \\ -1 & 3 & 4 & 4 \end{array}\right]$

$$\sim \left[\begin{array}{ccc:c} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 2 & 2 & 2 \end{array}\right] \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 0 & 5 & 10 \end{array}\right] R_3 \to R_2 - R_3$$

writing in eqn form

$$-x+y+2z=2 \quad \text{——— (4)}$$
$$2y+7z=12 \quad \text{——— (5)}$$
$$5z=10 \quad \text{——— (6)}$$

Back substitution

From eqn (6) $z=2$

Putting $z=2$ in eqn (5) we get $y=-1$

Putting $z=2$, $y=-1$ in eqn (4) $x=1$

EX-5 Solve by Gauss elimination method

$$3x+2y+z=3 \quad \text{——— (1)}$$
$$2x+y+z=0 \quad \text{——— (2)}$$
$$6x+2y+4z=6 \quad \text{——— (3)}$$

Sol

Augmented matrix

$$A = \left[\begin{array}{cccc} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

$$\sim \left(\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 0 & -1 & 1 & -6 \\ 0 & -2 & 2 & 0 \end{array}\right) \begin{array}{l} R_2 \to 3R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 0 & -1 & 1 & -6 \\ 0 & 0 & 0 & 12 \end{array}\right) R_3 \to R_3 - 2R_2$$

$$3x + 2y + z = 3 \quad \text{———} (4)$$
$$-y + z = -6 \quad \text{———} (5)$$
$$0 = 12 \quad \text{———} (6)$$

Eq^n (6) is impossible, hence no sol^n.

No. (11) $7x - 4y - 2z = -6$ ——— (1)

$16x + 2y + z = 3$ ——— (2)

Sol^n Augmented matrix is

$$A = \begin{bmatrix} 7 & -4 & -2 & -6 \\ 16 & 2 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 7 & -4 & -2 & -6 \\ 39 & 0 & 0 & 0 \end{bmatrix} R_2 \to 2R_2 + R_1$$

$$7x - 4y - 2z = -6 \quad \text{———} (3)$$
$$39x = 0 \Rightarrow x = 0 \quad \text{———} (4)$$

Putting $x = 0$ in eq^n (3)

$4y + 2z = 6 \Rightarrow 2z = 6 - 4y$

Sol^n is $x = 0$, $z = 3 - 2y$ infinite many solution

RANK OF MATRIX

Rank of a matrix is defined as maximum number of linearly independent row vectors of a matrix A

OR Maximum number of linear independent column vectors

OR Number of non-zero rows in echelon form.

Echelon form

1. Matrix should be in upper triangular form.
2. The number of zeros should appear in increasing order
3. The first non-zero element in each row should be 1 (one).

Ex. 6 Find rank of the matrix $A = \begin{pmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 24 \\ 21 & -21 & 0 & -15 \end{pmatrix}$

Sol^n Given $A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 24 \\ 21 & -21 & 0 & -15 \end{bmatrix}$

$$\sim \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 28 \\ 0 & -21 & -14 & -29 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 7R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 28 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to 2R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{2}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \to \frac{1}{3} R_1 \\ R_2 \to \frac{1}{42} R_2 \\ \cancel{R_3 \to} \end{matrix}$$

It is in upper-triangular form

Rank A = 2 = number of non-zero rows in echelon form

(17) Find Rank A, $A = \begin{pmatrix} 9 & 3 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{pmatrix}$ (27)

Sol Given $A = \begin{bmatrix} 9 & 3 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{bmatrix}$

$$\sim \begin{bmatrix} 9 & 3 & 1 & 0 \\ 0 & -3 & 2 & -18 \\ 0 & 6 & 8 & 9 \\ 0 & -6 & 1 & 9 \end{bmatrix} \begin{array}{l} R_2 \to 3R_2 - R_1 \\ R_3 \to 9R_3 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 9 & 3 & 1 & 0 \\ 0 & -3 & 2 & -18 \\ 0 & 0 & 12 & -27 \\ 0 & 0 & 9 & 18 \end{bmatrix} \begin{array}{l} R_3 \to R_3 + 2R_2 \\ R_4 \to R_4 + R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{9} & 0 \\ 0 & 1 & -\frac{2}{3} & 6 \\ 0 & 0 & 1 & \frac{9}{4} \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{array}{l} R_1 \to \frac{1}{9}R_1 \\ R_2 \to -\frac{1}{3}R_2 \\ R_3 \to \frac{1}{12}R_3 \\ R_4 \to \frac{1}{9}R_4 \end{array}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{9} & 0 \\ 0 & 1 & -2/3 & 6 \\ 0 & 0 & 1 & 9/4 \\ 0 & 0 & 0 & 1/4 \end{bmatrix} R_4 \to -R_4 + R_3$$

$$\sim \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{9} & 0 \\ 0 & 1 & -2/3 & 6 \\ 0 & 0 & 1 & 9/4 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_4 \to 4R_4$$

This is in echelon form. ~~and~~

Rank A = 4 = number of non-zero rows in echelon form.

LINEAR INDEPENDENSE (L.I.)

p vectors $x_1, x_2, \cdots, x_p$ with n components each are linearly independent if the matrix with row vectors $x_1, x_2, \cdots, x_p$ has rank p, they are linearly dependent (L.D.) if rank $< p$.

No. 1 Are the following set of vectors linearly independent? or dependent?
$[1 \quad 0 \quad 0], [1 \quad 1 \quad 0], [1 \quad 1 \quad 1]$

Sol^n Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_3$

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \quad \begin{matrix} R_2 \to -R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 \leftrightarrow R_3$

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 \to -R_2$

Rank $A = 3$ = number of rows of A

$\therefore$ Row vectors are linearly independent

No. 6 Are the following row vectors linearly dependent? independent?
$[1 \quad 2 \quad 3], [0 \quad 0 \quad 0], [5 \quad 5 \quad 1]$

Sol Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 5 & 5 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -5 & -14 \end{bmatrix} R_3 \to R_3 - 5R_1$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -14 \\ 0 & 0 & 0 \end{bmatrix} R_2 \leftrightarrow R_3$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 14/5 \\ 0 & 0 & 0 \end{bmatrix} R_2 \to \frac{1}{-5} R_2$

This is in echelon form

Rank $A = 2 < 3 =$ no of rows of A

Hence given row vectors are linearly dependent.

VECTOR SPACE

A vector space is a non-empty set V of vectors such that with any two elements a, b in V

(i) $a + b = b + a$

(ii) $a + (b + c) = (a + b) + c$

(iii) $a + 0 = a$

(iv) $a + (-a) = 0$

(v) $k(a + b) = ka + kb$

(vi) $(k_1 + k_2) a = k_1 a + k_2 a$

(vii) $k_1 (k_2 a) = (k_1 k_2) a$

(viii) $1 \cdot a = a$

Dimension The maximum number of linearly independent vectors in V is called the dimension of vector space V and is denoted by dim V.

Basis A linearly independent set in V consisting of a maximum possible number of vectors in V is called a basis for V.

Thus number of vectors (elements) of a basis for V equals dim V.

The set of all linear combinations of given vectors $a_1, a_2, \ldots, a_p$ with same number of components is called the span of these vectors.

No. 19 Is the set of all vectors in R^3 such that $2V_1 + 3V_3 = 0$ vector space? If yes find the dimension and a basis.

Soln Let V = set of vectors in R^3 such that $2V_1 + 3V_3 = 0$.

Let $a, b, c \in V$, then

$a = (a_1, 0, -\frac{2}{3}a_1)$, $b = (b_1, 0, -\frac{2}{3}b_1)$

$c = (c_1, 0, -\frac{2}{3}c_1)$

(i) $a + b = (a_1 + b_1, 0, -\frac{2}{3}a_1 - \frac{2}{3}b_1) = b + a$

(ii) $a + (b + c) = (a_1, 0, -\frac{2}{3}a_1) + (b_1 + c_1, 0, -\frac{2}{3}b_1 - \frac{2}{3}c_1)$

$= (a_1 + b_1 + c_1, 0, -\frac{2}{3}a_1 - \frac{2}{3}b_1 - \frac{2}{3}c_1)$

$= (a_1 + b_1, 0, -\frac{2}{3}a_1 - \frac{2}{3}b_1) + (c_1, 0, -\frac{2}{3}c_1)$

$= (a + b) + c$

(iii) $a + 0 = (a_1, 0, -\frac{2}{3}a_1) + (0, 0, 0)$
$= (a_1, 0, -\frac{2}{3}a_1) = a$

(iv) $a + (-a) = (a_1, 0, -\frac{2}{3}a_1) + (-a_1, 0, \frac{2}{3}a_1)$
$= (a_1 - a_1, 0, -\frac{2}{3}a_1 + \frac{2}{3}a_1) = (0, 0, 0) = 0$

(v) $k(a+b) = k(a_1 + b_1, 0, -\frac{2}{3}a_1 - \frac{2}{3}b_1)$
$= (ka_1 + kb_1, 0, -\frac{2}{3}ka_1 - \frac{2}{3}kb_1)$
$= (ka_1, 0, -\frac{2}{3}ka_1) + (kb_1, 0, -\frac{2}{3}kb_1)$
$= k(a_1, 0, -\frac{2}{3}a_1) + k(b_1, 0, -\frac{2}{3}b_1)$
$= ka + kb$

(vi) $(k_1 + k_2)a = (k_1 + k_2)(a_1, 0, -\frac{2}{3}a_1)$
$= (k_1a_1 + k_2a_1, 0, -\frac{2}{3}k_1a_1 - \frac{2}{3}k_2a_1)$
$= (k_1a_1, 0, -\frac{2}{3}k_1a_1) + (k_2a_1, 0, -\frac{2}{3}k_2a_1)$
$= k_1(a_1, 0, -\frac{2}{3}a_1) + k_2(a_1, 0, -\frac{2}{3}a_1)$
$= k_1a + k_2a$

(vii) $k_1(k_2a) = k_1(k_2a_1, 0, -\frac{2}{3}k_2a_1)$
$= k_1k_2(a_1, 0, -\frac{2}{3}a_1) = (k_1k_2)a$

(viii) $1 \cdot a = 1(a_1, 0, -\frac{2}{3}a_1)$
$= (a_1, 0, -\frac{2}{3}a_1) = a$

As all the properties are satisfied, hence V is a vector space

Dimension of V = 2

A basis of V is $\{[3 \ \ 0 \ -2], [0 \ \ 1 \ \ 0]\}$