$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] R_1 \to R_1 + R_2$$

$$\qquad I \qquad\qquad\qquad A^{-1}$$

Required inverse is $A^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$

EIGEN VALUES, EIGEN VECTORS

Let $A = [a_{jk}]$ be a given $n \times n$ matrix and consider the vector equation $AX = \lambda X$ ——— (1)

A value λ for which eq^n (1) has a solution $X \neq 0$ is called eigen value or characteristic value of the matrix A. The corresponding solutions $X \neq 0$ of eq^n (1) are called eigen vectors or characteristic vectors of A.

For eigen values $|A - \lambda I| = 0$

For eigen vectors we solve the equations $[A - \lambda I] X = 0$

No. 5 Find the eigen values and eigen vectors of $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

Sol^n Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ unit matrix, order 2

For eigen values $|A - \lambda I| = 0$

$$\left| \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-3-\lambda) - 16 = 0$$

$$\lambda^2 - 25 = 0 \Rightarrow \boxed{\lambda = \pm 5}$$

Eigen values

For Eigen vector

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} (3-\lambda) & 4 \\ 4 & (-3-\lambda) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (3-\lambda)x_1 + 4x_2 \\ 4x_1 - (3+\lambda)x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(3-\lambda)x_1 + 4x_2 = 0 \quad \text{——— (1)}$$

$$4x_1 - (3+\lambda)x_2 = 0 \quad \text{——— (2)}$$

Case - I $\quad \lambda = 5$

Eqⁿ (1) & (2) become $\quad -2x_1 + 4x_2 = 0$ ——— (3)

$4x_1 - 8x_2 = 0$ ——— (4)

From eqⁿ (3) & (4) $\quad x_1 = 2x_2$

Let $x_2 = 1$ then $x_1 = 2$

One eigen value is $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Case - II $\quad \lambda = -5$

Eqⁿ (1) & (2) becomes $\quad 8x_1 + 4x_2 = 0$ ——— (5)

$4x_1 + 2x_2 = 0$

Eq (5) & (6) $\Rightarrow$ $x_2 = -2x_1$

Let $x_1 = 1$ then $x_2 = -2$

Another eigen value is $X_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

No. 10 Find eigen values and eigen vectors of $A = \begin{bmatrix} -10 & 10 & -15 \\ 10 & 5 & -30 \\ -5 & -10 & 0 \end{bmatrix}$

Sol^n. Let $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, an 3x3 unit matrix

For eigen values

$|A - \lambda I| = 0$

$$\begin{vmatrix} -10-\lambda & 10 & -15 \\ 10 & 5-\lambda & -30 \\ -5 & -10 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -10-\lambda & 10 & -15 \\ 0 & -15-\lambda & -30-2\lambda \\ -15-\lambda & 0 & -15-\lambda \end{vmatrix} = 0 \quad \begin{matrix} R_2 \to R_2 + 2R_3 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\Rightarrow (-15-\lambda)(-15-\lambda) \begin{vmatrix} -10-\lambda & 10 & -15 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (-15-\lambda)(-15-\lambda)(25-\lambda) = 0$$

$$\Rightarrow \boxed{\lambda = -15, -15, 25} \text{ (Eigen values)}$$

For Eigen vectors

$[A - \lambda I] X = 0$

$$\begin{bmatrix} -10-\lambda & 10 & -15 \\ 10 & 5-\lambda & -30 \\ -5 & -10 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow (-10-\lambda)x_1 + 10x_2 - 15x_3 = 0$ —— (1)

$10x_1 + (5-\lambda)x_2 - 30x_3 = 0$ —— (2)

$-5x_1 - 10x_2 - \lambda x_3 = 0$ —— (3)

Case I $\lambda = -15$

Eqn (1), (2), (3) become $5x_1 + 10x_2 - 15x_3 = 0$ —— (4)

$10x_1 + 20x_2 - 30x_3 = 0$ —— (5)

$-5x_1 - 10x_2 + 15x_3 = 0$ —— (6)

we observe that

Eqn (5) = 2 × eqn (4) and

Eqn (6) = −1 × eqn (4).

From eqn (4) $x_3 = \dfrac{5x_1 + 10x_2}{15}$

Let $x_1 = 0$ and $x_2 = 1$ then $x_3 = \frac{2}{3}$

One eigen value is $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2/3 \end{bmatrix}$

Case II $\lambda = -15$

Eqn (1), (2) (3) becomes

$5x_1 + 10x_2 - 15x_3 = 0$ —— (4)

$10x_1 + 20x_2 - 30x_3 = 0$ —— (5)

$-5x_1 - 10x_2 + 15x_3 = 0$ —— (6)

Let $x_1 = 1$ and $x_2 = 0$

then $x_3 = \frac{1}{3}$.

Another eigen vector $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1/3 \end{bmatrix}$

<u>Case - III</u> $\lambda = 25$

Eqⁿ (1), (2), (3) become

$$-35x_1 + 10x_2 - 15x_3 = 0$$
$$10x_1 - 20x_2 - 30x_3 = 0$$
$$-5x_1 - 10x_2 - 25x_3 = 0$$

OR
$$7x_1 - 2x_2 + 3x_3 = 0 \quad \text{—— (7)}$$
$$x_1 - 2x_2 - 3x_3 = 0 \quad \text{—— (8)}$$
$$x_1 + 2x_2 + 5x_3 = 0 \quad \text{—— (9)}$$

Eqⁿ (8) + Eqⁿ (9) $\Rightarrow 2x_1 + 2x_3 = 0$

$$x_3 = -x_1 \quad \text{—— (10)}$$

Putting $x_3 = -x_1$ in eqⁿ (7), we get

$4x_1 - 2x_2 = 0 \quad \Rightarrow \quad x_2 = 2x_1$ —— (11)

Let $x_1 = 1$ then $x_2 = 2$ $x_3 = -1$

Another eigen vector is $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

SOME APPLICATIONS OF EIGEN VALUE PROBLEM

<u>No. 3</u> Find the principal direction and corresponding factors of extension or contraction of the elastic deformation $Y = AX$ with given $A = \begin{bmatrix} 3 & 1.5 \\ 1.5 & 3 \end{bmatrix}$

<u>Solⁿ</u> For eigen values $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 1.5 \\ 1.5 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 6\lambda + 6.75 = 0 \quad \Rightarrow \quad \boxed{\lambda = 1.5, 4.5}$$

Eigen values are $\lambda = 1.5, 4.5$ are corresponding factors of extension.

<u>For eigen vectors</u>

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 3-\lambda & 1.5 \\ 1.5 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(3-\lambda)x_1 + 1.5x_2 = 0 \quad \text{---} \quad (1)$$

$$1.5x_1 + (3-\lambda)x_2 = 0 \quad \text{---} \quad (2)$$

<u>Case - I</u> <u>$\lambda = 1.5$</u> Eqn (1) & (2) becomes

$$\left.\begin{aligned} 1.5x_1 + 1.5x_2 = 0 \\ 1.5x_1 + 1.5x_2 = 0 \end{aligned}\right\} \Rightarrow x_2 = -x_1$$

Let $x_1 = 1$ then $x_2 = -1$

$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is one eigen vector for eigen value 1.5

<u>Case - II</u> $\lambda = 4.5$ Eqn (1) & (2) becomes

$$\left.\begin{aligned} -1.5x_1 + 1.5x_2 = 0 \\ 1.5x_1 - 1.5x_2 = 0 \end{aligned}\right\} \Rightarrow x_1 = x_2$$

Let $x_1 = 1$ then $x_2 = 1$

$X_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is another eigen vector for eigen value 4.5

These vectors X_1 and X_2 make -45° and 45° angle with the positive x_1 direction

$$\theta_1 = \tan^{-1}\frac{x_2}{x_1} = \tan^{-1}(-1) = -45^\circ$$

$$\theta_2 = \tan^{-1}\frac{x_2}{x_1} = \tan^{-1} 1 = 45^\circ$$

They give the principal directions.

Symmetric, Skew-Symmetric & Orthogonal Matrices

Def: **Symmetric matrix**: A real square matrix $A = [a_{jk}]$ is symmetric if $A^T = A$
(transpose makes unchanged)

Skew Symmetric If $A^T = -A$
(transpose A gives negative of A)

Orthogonal If $A^T = A^{-1}$ (transpose of A gives inverse of A)

No. 1 Is the matrix $A = \begin{bmatrix} 0.96 & -0.28 \\ 0.28 & 0.96 \end{bmatrix}$

Symmetric, skew symmetric or orthogonal?
Find eigen values

Sol:

$$A^T = \begin{bmatrix} 0.96 & 0.28 \\ -0.28 & 0.96 \end{bmatrix} \neq A$$
$$\neq -A$$

Hence A is neither symmetric nor skew-symmetric

$$|A| = 0.96^2 + 0.28^2 = 1$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 0.96 & 0.28 \\ -0.28 & 0.96 \end{bmatrix} = \begin{bmatrix} 0.96 & 0.28 \\ -0.28 & 0.96 \end{bmatrix} = A^T$$

$A^{-1} = A^T$ Hence matrix A is orthogonal

For eigen values $|A - \lambda I| = 0$

$$\begin{vmatrix} 0.96-\lambda & -0.28 \\ 0.28 & 0.96-\lambda \end{vmatrix} = 0$$

$$0.9216 + \lambda^2 - 1.92\lambda + 0.0784 = 0$$

$$\lambda^2 - 1.92\lambda + 1 = 0$$

$$\lambda = 0.96 \pm 0.28i$$

Hence the eigen values are complex conjugates $0.96 + 0.28i$ and $0.96 - 0.28i$

No. 5 Test the matrix $A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$

for symmetric, skew symmetric, orthogonal & find eigen values

Sol^n

$$A^T = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix} = -A$$

Hence A is skew symmetric

For eigen values $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 9 & -12 \\ -9 & -\lambda & 20 \\ 12 & 20 & -\lambda \end{vmatrix} =$$

$$= \begin{vmatrix} -\lambda & 9 & -12 \\ 0 & 60-4\lambda & 80-3\lambda \\ -20\lambda-108 & 0 & 9\lambda-240 \end{vmatrix} \quad \begin{matrix} R_2 \to 4R_2 + 3R_3 \\ R_3 \to 20R_3 - 9R_1 \end{matrix}$$

$$= -\lambda(60-4\lambda)(9\lambda-240) + 9(80-3\lambda)(-20\lambda-108)$$
$$-12(60-4\lambda)(20\lambda+108) = 0$$

$$\Rightarrow \lambda(\lambda^2 + 625) = 0$$

$$\Rightarrow \lambda = 0, \quad \lambda^2 = -625$$

$$\Rightarrow \lambda = 0, \; 25i, \; -25i$$

No. 7 Show that main diagonal entries of a skew symmetric matrix must be zero

Sol^n Let matrix $A = [a_{jk}]$ be skew symmetric. Then by definition

$$A^T = -A$$

and $a_{jk} = -a_{kj}$

for diagonal elements $j=k$
like a_{11}, a_{22} and a_{33}

Hence $a_{kk} = -a_{kk}$ (for $k=1,2,3$)

$a_{kk} + a_{kk} = 0$

$2a_{kk} = 0 \Rightarrow a_{kk} = 0$ for $k=1,2,3$

Hence $a_{11} = 0$ $a_{22} = 0$ $a_{33} = 0$

$\Rightarrow$ All diagonal elements of a skew symmetric matrices are zero

Complex Matrices

Hermitian, Skew Hermitian, Unitary

A matrix $A = [a_{jk}]$ is called

Hermitian if $\bar{A}^T = A \Rightarrow \bar{a}_{kj} = a_{jk}$

Skew Hermitian if $\bar{A}^T = -A \Rightarrow \bar{a}_{kj} = -a_{jk}$

Unitary if $\bar{A}^T = A^{-1}$

No-5 Check the matrix $A = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$
for Hermitian, Skew-Hermitian, Unitary
Find its eigen values and eigen vectors ?

Sol $\bar{A} = \begin{bmatrix} 2 & 3-4i \\ 3+4i & 2 \end{bmatrix}$

$$\bar{A}^T = \begin{pmatrix} 2 & 3+4i \\ 3-4i & 2 \end{pmatrix} = A$$

Hence A is Hermitian

Eigen values $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 3+4i \\ 3-4i & 2-\lambda \end{vmatrix} = 0$$

$$4 + \lambda^2 - 4\lambda - (3+4i)(3-4i) = 0$$

$$\lambda^2 - 4\lambda - 21 = 0$$

$$(\lambda+3)(\lambda-7) = 0$$

$$\lambda = -3, 7 \text{ (Eigen values)}$$

For Eigen vector $[A - \lambda I]X = 0$

$$\begin{bmatrix} 2-\lambda & 3+4i \\ 3-4i & 2-\lambda \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(2-\lambda)x_1 + (3+4i)x_2 = 0 \quad \text{——— (1)}$$

$$(3-4i)x_1 + (2-\lambda)x_2 = 0 \quad \text{——— (2)}$$

Case - I $\lambda = -3$ Eq^n (1) & (2) become

$$5x_1 + (3+4i)x_2 = 0 \quad \text{——— (3)}$$

$$(3-4i)x_1 + 5x_2 = 0 \quad \text{——— (4)}$$

From eq^n (3)

$$x_1 = -\frac{(3+4i)x_2}{5} = \frac{-(3+4i)(3-4i)}{5(3-4i)}x_2$$

$$= -\frac{25}{5(3-4i)}x_2 = -\frac{5}{3-4i}x_2$$

$$-5x_2 = (3-4i)x_1 \quad \text{which is}$$

same as eq^n (4)

Let $x_2 = 1$ then $x_1 = -\frac{(3+4i)}{5}$

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} - \frac{4}{5}i \\ 1 \end{pmatrix}$$

is one eigen vector

Case - II $\quad \lambda = 7 \quad$ Eq^n (1) & (2) become

$$-5x_1 + (3+4i)x_2 = 0 \quad \text{--- (5)}$$

$$(3-4i)x_1 - 5x_2 = 0 \quad \text{--- (6)}$$

From (5) & (6) $x_2 = \frac{5}{3+4i}x_1 = \frac{3-4i}{5}x_1$

Let $x_1 = 1$ then $x_2 = \frac{5}{3+4i}$

$$X_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{5}{3+4i} \end{pmatrix}$$ is another

eigen vector

No. 7 $\quad A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2}i \\ \frac{\sqrt{3}}{2}i & \frac{1}{2} \end{bmatrix}$

$$|A| = 1 - \frac{3}{4}i^2 = 1 + \frac{3}{4} = 1$$

$$A^{-1} = \frac{1}{|A|}\operatorname{Adj} A = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2}i \\ -\frac{\sqrt{3}}{2}i & \frac{1}{2} \end{bmatrix}$$

$$\overline{A}^T = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2}i \\ -\frac{\sqrt{3}}{2}i & \frac{1}{2} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2}i \\ -\frac{\sqrt{3}}{2}i & \frac{1}{2} \end{bmatrix}$$

$= A^{-1}$

$\bar{A}^T = A^{-1}$ hence A is unitary matrix

For eigen values $|A - \lambda I| = 0$

$$\begin{vmatrix} \frac{1}{2} - \lambda & \frac{\sqrt{3}}{2} i \\ \frac{\sqrt{3}}{2} i & \frac{1}{2} - \lambda \end{vmatrix} = 0$$

$$\lambda^2 + \frac{1}{4} - \lambda - \frac{3}{4} i^2 = 0$$

$$\lambda^2 - \lambda + 1 = 0$$

Eigen values are $\lambda = \frac{1}{2} - \frac{\sqrt{3}}{2} i$ and $\frac{1}{2} + \frac{\sqrt{3}}{2} i$

<u>For Eigen vectors</u> $[A - \lambda I] X = 0$

$$\left(\frac{1}{2} - \lambda\right) x_1 + \frac{\sqrt{3}}{2} i \, x_2 = 0 \quad \text{——— (1)}$$

$$\frac{\sqrt{3}}{2} i x_1 + \left(\frac{1}{2} - \lambda\right) x_2 = 0 \quad \text{——— (2)}$$

<u>Case I</u> $\lambda = \frac{1}{2} - \frac{\sqrt{3}}{2} i$ Eqn (1) & (2) become $\frac{\sqrt{3}}{2} i x_1 + \frac{\sqrt{3}}{2} i x_2 = 0$

$$\frac{\sqrt{3}}{2} i x_1 + \frac{\sqrt{3}}{2} i x_2 = 0$$

$\Rightarrow x_2 = -x_1$

Let $x_1 = 1$ then $x_2 = -1$

$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is one eigen vector.

<u>Case - II</u> $\lambda = \frac{1}{2} + \frac{\sqrt{3}}{2} i$

From eqn (1) & (2) we get $x_2 = x_1$

Let $x_1 = 1$, then $x_2 = 1$ and $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is another eigen vector

NO. (14) Find the Symmetric Coefficient matrix C of the quadratic form $Q = X^T C X$ given by

$$-2x_1^2 + 2x_1 x_2 + 4x_2 x_3 - 9x_3^2$$

Soln

$$-2x_1^2 + 2x_1x_3 + 4x_2x_3 - 9x_3^2$$

$$= -2x_1x_1 + 0(x_1x_2) + 2x_1x_3$$
$$+ 0(x_2x_1) + 0(x_2x_2) + 4x_2x_3$$
$$+ 0(x_3x_1) + 0(x_3x_2) - 9x_3x_3$$

The coefficient matrix is

$$A = \begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & -9 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & \text{[illegible]} \\ 2 & 4 & -9 \end{bmatrix}$$

Required symmetric coefficient matrix is $C = \frac{1}{2}(A + A^T)$

$$= \frac{1}{2}\begin{bmatrix} -4 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & -18 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & -9 \end{bmatrix}$$

NO. 21 Is $A = \begin{bmatrix} 4 & 3-2i \\ 3+2i & -4 \end{bmatrix}$ Hermitian or Skew Hermitian? Find $\bar{X}^T A X$?

Sol^n $\bar{A}^T = \begin{bmatrix} 4 & 3+2i \\ 3-2i & -4 \end{bmatrix}^T$

$= \begin{bmatrix} 4 & 3-2i \\ 3+2i & -4 \end{bmatrix} = A$ Since $\bar{A}^T = A$,

Hence A is Hermitian.

NOW

$\bar{X}^T A X = \begin{pmatrix} 2i & 1-i \end{pmatrix} \begin{bmatrix} 4 & 3-2i \\ 3+2i & -4 \end{bmatrix} \begin{bmatrix} -2i \\ 1+i \end{bmatrix}$

$= \begin{bmatrix} 5+7i & 10i \end{bmatrix} \begin{bmatrix} -2i \\ 1+i \end{bmatrix} = 4$

DIAGONALIZATION

NO. 13
Find a basis of eigen vectors and diagonalize $A = \begin{bmatrix} 16 & 0 & 0 \\ 48 & -8 & 0 \\ 84 & -24 & 4 \end{bmatrix}$

Sol^n For eigen values $|A - \lambda I| = 0$

$$\begin{vmatrix} 16-\lambda & 0 & 0 \\ 48 & -8-\lambda & 0 \\ 84 & -24 & 4-\lambda \end{vmatrix} = 0$$

$(16-\lambda)(-8-\lambda)(4-\lambda) = 0$

$\lambda = 16, -8, 4$ (eigen values)

For eigen vectors $[A - \lambda I]X = 0$

$$\begin{bmatrix} 16-\lambda & 0 & 0 \\ 48 & -8-\lambda & 0 \\ 84 & -24 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(16-\lambda)x_1 = 0$ —— (1)

$48x_1 - (8+\lambda)x_2 = 0$ —— (2)

$84x_1 - 24x_2 + (4-\lambda)x_3 = 0$ —— (3)

Case I $\lambda = 16$ Eq^n (1), (2), (3) become

$0 \cdot x_1 = 0$

$48x_1 - 24x_2 = 0 \Rightarrow x_2 = 2x_1$

$84x_1 - 24x_2 - 12x_3 = 0$

$\Rightarrow x_3 = 7x_1 - 2x_2$

Let $x_1 = 1$ then $x_2 = 2$, $x_3 = 3$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$ is one eigen vector

Case II $\lambda = -8$ Eq^n (1), (2), (3) become

$24x_1 = 0 \Rightarrow x_1 = 0$

$48x_1 = 0$

$84x_1 - 24x_2 + 12x_3 = 0$

$x_3 = 2x_2 - 7x_1$

Let $x_2 = 1$ then $x_1 = 0$ $x_3 = 2$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$ is another eigen vector

<u>Case - III</u> $\lambda = 4$ Eq (1), (2), (3) become

$12 x_1 = 0 \Rightarrow x_1 = 0$ —— (4)

$48 x_1 - 12 x_2 = 0 \Rightarrow x_2 = 4 x_1 = 0$ —— (5)

$84 x_1 - 24 x_2 = 0 \Rightarrow x_2 = 3.5 x_1 = 0$

~~Het~~ $x_3 = 1$

then $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is another eigen vector

Eigen vector matrix is

$$X = [X_1 \; X_2 \; X_3] = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$[X \vdots I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) R_3 \to R_3 - 2R_2$$

$\qquad I \qquad\qquad\qquad X^{-1}$

Required diagonal matrix is

$$D = X^{-1}AX = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 \\ 48 & -8 & 0 \\ 84 & -24 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 0 & 0 \\ 16 & -8 & 0 \\ 4 & -8 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The eigen vectors X_1, X_2, X_3 form a basis of eigen vectors.

EX-6 Find out what type of conic section the following quadratic form & transform it to principal axes.

Sol $Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 + 128$

The coefficient matrix is $B = \begin{pmatrix} 17 & -30 \\ 0 & 17 \end{pmatrix}$

as we can write

$Q = 17x_1x_1 - 30x_1x_2 + 0(x_2x_1) + 17x_2x_2$

$$B^T = \begin{bmatrix} 17 & 0 \\ -30 & 17 \end{bmatrix}$$

$$A = \frac{1}{2}(B + B^T) = \frac{1}{2}\begin{bmatrix} 34 & -30 \\ -30 & 34 \end{bmatrix} = \begin{pmatrix} 17 & -15 \\ -15 & 17 \end{pmatrix}$$

is the symmetric coefficient matrix

For eigen values of A $\quad |A - \lambda I| = 0$

$$\begin{vmatrix} 17-\lambda & -15 \\ -15 & 17-\lambda \end{vmatrix} = 0$$

$(17-\lambda)^2 - 15^2 = 0$

$\lambda^2 - 34\lambda + 64 = 0 \Rightarrow \boxed{\lambda = 2, 32}$

$\Rightarrow$ Eigen values are $\lambda = 2, 32$

$\lambda_1 = 2$ and $\lambda_2 = 32$.

Hence $Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 = 2y_1^2 + 32y_2^2$

We observe that $Q = 128$ means

$$2y_1^2 + 32y_2^2 = 128$$

$$\frac{y_1^2}{64} + \frac{y_2^2}{4} = 1$$

$$\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$$ which represents an ellipse.

<u>For eigen vectors</u> $[A - \lambda I]X = 0$.

$$\begin{bmatrix} 17-\lambda & -15 \\ -15 & 17-\lambda \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(17-\lambda)x_1 - 15x_2 = 0$ ——— (1)

$-15x_1 + (17-\lambda)x_2 = 0$ ——— (2)

<u>Case I</u> $\lambda = 2$ Eq (1) & (2) become

$$\left.\begin{matrix} 15x_1 - 15x_2 = 0 \\ -15x_1 + 15x_2 = 0 \end{matrix}\right] \Rightarrow x_1 = x_2.$$

Let $x_1 = \frac{1}{\sqrt{2}}$ then $x_2 = \frac{1}{\sqrt{2}}$

Normalized eigen vector $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Case - II $\lambda = 32$ Eq^n (1) & (2) become

$$\left.\begin{aligned} -15x_1 - 15x_2 = 0 \\ -15x_1 - 15x_2 = 0 \end{aligned}\right\} \Rightarrow x_1 = -x_2$$

Let $x_2 = \frac{1}{\sqrt{2}}$ then $x_1 = -\frac{1}{\sqrt{2}}$

Another normalized eigen vector $X_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Eigen vector matrix is

$$X = \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Hence $x = Xy$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\cancel{\Rightarrow x_1 = y_1\sqrt{2} - y_2}$$

$$= \begin{bmatrix} \frac{y_1}{\sqrt{2}} - \frac{y_2}{\sqrt{2}} \\ \frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{2}} \end{bmatrix}$$

Hence

$$x_1 = \frac{y_1}{\sqrt{2}} - \frac{y_2}{\sqrt{2}}$$

$$x_2 = \frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{2}}$$

Thus it makes a 45° rotation.