LIMIT CONTINUITY (TWO VARIABLES)

Ex: $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{2xy+3}{1+x^2+y^2} = \begin{cases} \lim\limits_{y\to 0} \dfrac{3}{1+y^2} = 3 \\ \lim\limits_{x\to 0} \dfrac{3}{1+x^2} = 3 \end{cases}$$

As both limits are same (taking $x\to 0$ first or taking $y\to 0$ first). Hence the limit exists and equal to 3

Ex: Test the continuity of $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ at $(0,0)$, given $f(0,0)=0$

Sol^n $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2-y^2}{x^2+y^2} = \begin{cases} \lim\limits_{y\to 0} \lim\limits_{x\to 0} \dfrac{x^2-y^2}{x^2+y^2} \\ \lim\limits_{x\to 0} \lim\limits_{y\to 0} \dfrac{x^2-y^2}{x^2+y^2} \end{cases}$$

$$= \begin{cases} \lim\limits_{y\to 0} \dfrac{-y^2}{y^2} \\ \lim\limits_{x\to 0} \dfrac{x^2}{x^2} \end{cases} = \begin{cases} \lim\limits_{y\to 0} -1 = -1 \\ \lim\limits_{x\to 0} 1 = 1 \end{cases}$$

As both limits are different, hence limit does not exist

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) \neq f(0,0) = 0$$

Hence not continuous at $(0,0)$

Partial Derivatives

Ex Find 1st & 2nd partial derivatives of
$$z = x^3+y^3-3axy$$

Sol $\dfrac{\partial z}{\partial x} = 3x^2-3ay \qquad \dfrac{\partial z}{\partial y} = 3y^2-3ax$

$\dfrac{\partial^2 z}{\partial x^2} = 6x \qquad \dfrac{\partial^2 z}{\partial y^2} = 6y$ and $\dfrac{\partial^2 z}{\partial x\,\partial y} = -3a$

Ex If $u = x^y$ prove that $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$

Sol $u = x^y \Rightarrow \log u = \log x^y = y \log x$

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{y}{x} \Rightarrow \frac{\partial u}{\partial x} = u\frac{y}{x} = x^y \frac{y}{x} = y x^{y-1}$$

Also

$$\frac{1}{u}\frac{\partial u}{\partial y} = \log x \Rightarrow \frac{\partial u}{\partial y} = u \log x = x^y \log x$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = x^y \frac{\partial}{\partial x}(\log x) + \log x \frac{\partial}{\partial x}(x^y)$$

$$= x^y \cdot \frac{1}{x} + y x^{y-1} \log x = x^{y-1}(y \log x + 1)$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\left[x^{y-1}(y \log x + 1)\right]$$

$$\Rightarrow \frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial x}\right) = \frac{\partial}{\partial x}\left[x^{y-1}(y \log x + 1)\right]$$

$$= \frac{\partial^3 u}{\partial x^2 \partial y}$$

MAXIMA MINIMA (TWO VARIABLES)

Procedure

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equate each to zero. Solve these equations in x and y. Let $(a, b), (c, d), \ldots$ be the pair of values.
2. Calculate the values of $A = \frac{\partial^2 f}{\partial x^2}$, $B = \frac{\partial^2 f}{\partial x \partial y}$ $C = \frac{\partial^2 f}{\partial y^2}$ for each pair of values
3. If $AC - B^2 > 0$ and $A < 0$ at (a, b) then there is maxima at (a, b)

   If $AC - B^2 > 0$ and $A > 0$ at (a, b) minima at (a, b)

   If $AC - B^2 < 0$ at (a, b) then $f(a, b)$ is not an extreme value i.e. (a, b) is a saddle point

If $AC - B^2 = 0$ at (a,b) then it is doubtful and need further investigation

EX- $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y$ $\quad \frac{\partial f}{\partial y} = 4y^3 + 4x - 4y$

$A = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$, $B = \frac{\partial^2 f}{\partial x \partial y} = 4$ $\quad C = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$

Now $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0 \Rightarrow x^3 - x + y = 0$ — (1)

$y^3 + x - y = 0$ — (2)

Adding these eq's $4(x^3 + y^3) = 0$

$(x+y)(x^2 - xy + y^2) = 0 \Rightarrow y = -x$

Put $y = -x$ in (1) we get $x^3 - 2x = 0$

$x(x^2 - 2) = 0 \Rightarrow x = 0, \pm\sqrt{2}$

$\Rightarrow y = -x = 0, \mp\sqrt{2}$

At $(\sqrt{2}, -\sqrt{2})$ $\quad AC - B^2 = 20 \times 20 - 4^2 > 0$ positive

and A is positive hence minima at $(\sqrt{2}, -\sqrt{2})$

At $(-\sqrt{2}, \sqrt{2})$ $\quad AC - B^2 > 0$, $A > 0$ minima

At $(0,0)$ $\quad AC - B^2 = 0$ hence further investigation is needed

Now $f(0,0) = 0$ and for points along the x-axis where $y = 0$, $f(x,y) = x^4 - 2x^2 = x^2(x^2 - 2)$ which is negative for points in the neighbourhood of origin

Again for points along line $y = x$, $f(x,y) = 2x^4 > 0$

Thus in neighbourhood of $(0,0)$ there are points where $f(x,y) < f(0,0)$ and also $f(x,y) > f(0,0)$

for some other points.

Hence $f(0,0)$ is not an extreme value

Lagrange's method of Undetermined Multipliers

Procedure Let $u = f(x, y, z)$ be function of x, y, z which are connected by the relation $\phi(x, y, z) = 0$.

1. Write $F = f(x, y, z) + \lambda \phi(x, y, z)$
2. Obtain the equations $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$
3. Solve the above equations together with $\phi(x, y, z) = 0$. The values of x, y, z obtained will give the stationary value of $f(x, y, z)$

Ex- Find the maximum and minimum distances of the point $(3, 4, 12)$ from the sphere $x^2 + y^2 + z^2 = 1$

Sol^n Let $P(x, y, z)$ be any point on the sphere and $A(3, 4, 12)$ the given point so that

$$AP^2 = (x-3)^2 + (y-4)^2 + (z-12)^2 = f(x, y, z) \text{ (say)} \quad (1)$$

We have to find the maximum and minimum values of $f(x, y, z)$ subject to the condition

$$\phi(x, y, z) = x^2 + y^2 + z^2 - 1 = 0 \quad (2)$$

Let $F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$

$$= (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

Then $\frac{\partial F}{\partial x} = 2(x-3) + 2\lambda x$, $\frac{\partial F}{\partial y} = 2(y-4) + 2\lambda y$

$$\frac{\partial F}{\partial z} = 2(z-12) + 2\lambda z$$

Now $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial z} = 0$ gives

$$x - 3 + \lambda x = 0, \quad y - 4 + \lambda y = 0 \quad \text{and} \quad z - 12 + \lambda z = 0 \quad (3)$$

which gives $\lambda = -\frac{x-3}{x} = -\frac{y-4}{y} = -\frac{z-12}{z}$

$$= \pm \frac{\sqrt{(x-3)^2+(y-4)^2+(z-12)^2}}{\sqrt{x^2+y^2+z^2}} = \frac{\sqrt{f}}{1}$$

Substituting for λ in eq^n (3) we get

$$x = \frac{3}{1+\lambda} = \frac{3}{1 \pm \sqrt{f}}$$

$$y = \frac{4}{1+\lambda} = \frac{4}{1 \pm \sqrt{f}} \quad \text{and} \quad z = \frac{12}{1+z} = \frac{12}{1 \pm \sqrt{f}}$$

Hence $x^2+y^2+z^2 = \frac{9+16+144}{(1 \pm \sqrt{f})^2} = \frac{169}{(1 \pm \sqrt{f})^2}$

Using eq^n (2) we get

$$1 = \frac{169}{(1 \pm \sqrt{f})^2} \Rightarrow (1 \pm \sqrt{f})^2 = 169$$

$$\Rightarrow 1 \pm \sqrt{f} = 13 \Rightarrow \sqrt{f} = 12, 14$$

We neglect negative values of $\sqrt{f}$ because $\sqrt{f} = AP$ is positive by eq^n (1)

Hence Maximum $AP = 14$ and minimum $AP = 12$

<u>Vector Function</u> A ~~Vector~~ vector whose components are functions of x, y, z

$$\vec{V} = v_1(x,y,z)\,\hat{i} + v_2(x,y,z)\,\hat{j} + v_3(x,y,z)\,\hat{k}$$

A vector function defines a <u>vector field</u> in the region in space in which V is defined with every point in that region. <u>Ex</u>- Velocity field

<u>Scalar point Function</u> A scalar which is a function of x, y, z Ex: $f = e^x \sin yz$

A scalar point function defined a scalar field.

<u>Ex</u>: ~~pressure field~~ Temperature field
potential field

GRADIENT OF A SCALAR FIELD

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Gradient f = grad f $= \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

Physically, grad f represents a vector normal to surface S in the space given by $f(x, y, z) = c$

(4) $f = x^2 + 9y^2$ Find grad f at $(-2, 2)$

Sol^n grad $f = \hat{i}\frac{\partial}{\partial x}(x^2+9y^2) + \hat{j}\frac{\partial}{\partial y}(x^2+9y^2) + \hat{k}\frac{\partial}{\partial z}(x^2+9y^2)$

$= 2x\hat{i} + 18y\hat{j} = 2(-2)\hat{i} + 18(2)\hat{j}$ at $(-2, 2)$

$= -4\hat{i} + 36\hat{j}$

(8) Find a vector normal to $f = \sin(x+z)$ at $(\frac{\pi}{8}, 1, \frac{\pi}{8})$

Sol^n Required normal vector is $\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$= \cos(x+z)\hat{i} + 0 + \cos(x+z)\hat{k}$

$= \cos\frac{\pi}{4}\hat{i} + \cos(\frac{\pi}{4})\hat{k}$ at $(\frac{\pi}{8}, 1, \frac{\pi}{8}) = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

(20) Using gradient find the unit normal vector for $x^2 + y^2 + 2z^2 = 26$ at $P(2, 2, 3)$

Sol^n $f = x^2 + y^2 + 2z^2 - 26$

grad $f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = 2x\hat{i} + 2y\hat{j} + 4z\hat{k}$

$= 4\hat{i} + 4\hat{j} + 12\hat{k}$ at $(2, 2, 3)$

Required unit normal vector

$$= \frac{\text{grad } f}{|\text{grad } f|} = \frac{4\hat{i} + 4\hat{j} + 12\hat{k}}{\sqrt{4^2 + 4^2 + 12^2}}$$

(21) $V = $ grad $f = 2x\hat{i} + 4y\hat{j} + 8z\hat{k}$, Find f ?

Sol^n $\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = 2x\hat{i} + 4y\hat{j} + 8z\hat{k}$

$\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 4y$ and $\frac{\partial f}{\partial z} = 8z$

$\partial f = 2x\,\partial x$, $\partial f = 4y\,\partial y$ and $\partial f = 8z\,\partial z$

$f = x^2 + c$, $f = y^2 + c$ and $f = z^2 + c$

$f = x^2 + y^2 + z^2 + c$

22 $v = \text{grad } f = yz\hat{i} + xz\hat{j} + xy\hat{k}$

$\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = yz\hat{i} + xz\hat{j} + xy\hat{k}$

$\frac{\partial f}{\partial x} = yz$, $\frac{\partial f}{\partial y} = xz$ and $\frac{\partial f}{\partial z} = xy$

$\partial f = yz\,\partial x$, $\partial f = xz\,\partial y$ and $\partial f = xy\,\partial z$

$f = xyz + c$ $\quad f = xyz + c \quad f = xyz + c$

28 Prove that $\nabla(fg) = f\nabla g + g\nabla f$

<u>Proof</u> $\nabla(fg) = \hat{i}\frac{\partial}{\partial x}(fg) + \hat{j}\frac{\partial}{\partial y}(fg) + \hat{k}\frac{\partial}{\partial z}(fg)$

$= \hat{i}\left[f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right] + \hat{j}\left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right) + \hat{k}\left[f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right]$

$= f\left[\hat{i}\frac{\partial g}{\partial x} + \hat{j}\frac{\partial g}{\partial y} + \hat{k}\frac{\partial g}{\partial z}\right] + g\left[\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right]$

$= f\nabla g + g\nabla f$

DIRECTIONAL DERIVATIVE

31 Find the directional derivative of $f = \frac{1}{\sqrt{x^2+y^2+z^2}}$ at $P(3,0,4)$ along direction of vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

<u>Sol</u> $\frac{\partial f}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\,2x = -x(x^2+y^2+z^2)^{-3/2}$

$\frac{\partial f}{\partial y} = -y(x^2+y^2+z^2)^{-3/2}$ and $\frac{\partial f}{\partial z} = -z(x^2+y^2+z^2)^{-3/2}$

$\text{grad } f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$= -x(x^2+y^2+z^2)^{-3/2}\hat{i} - y(x^2+y^2+z^2)^{-3/2}\hat{j} - z(x^2+y^2+z^2)^{-3/2}\hat{k}$

$= -\frac{3}{(9+0+16)^{3/2}}\hat{i} - 0 - \frac{4}{(9+0+16)^{3/2}}\hat{k} = -\frac{3}{125}\hat{i} - \frac{4}{125}\hat{k}$

Required directional derivative at $P(3,0,4)$

$= (\text{grad } f)\cdot\frac{\vec{a}}{|a|} = \left(-\frac{3}{125}\hat{i} - \frac{4}{125}\hat{k}\right)\cdot\frac{(\hat{i}+\hat{j}+\hat{k})}{\sqrt{1^2+1^2+1^2}}$

$= \frac{-\frac{3}{125} + 0 - \frac{4}{125}}{\sqrt{3}} = -\frac{7}{125\sqrt{3}}$

Teacher's Signature

(34) Find directional derivative of $f = x^2 + 3y^2 + 4z^2$ at $(1,0,1)$ along direction of vector $\vec{a} = -\hat{i} - \hat{j} + \hat{k}$

Sol

$$\text{grad } f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= 2x\,\hat{i} + 6y\,\hat{j} + 8z\,\hat{k}$$

$$= 2\hat{i} + 0 + 8\hat{k} \quad \text{at } (1,0,1)$$

Required directional derivative

$$= \text{grad } f \cdot \frac{\vec{a}}{|a|}$$

$$= (2\hat{i} + 8\hat{k}) \cdot \frac{(-\hat{i} - \hat{j} + \hat{k})}{|-\hat{i} - \hat{j} + \hat{k}|}$$

$$= \frac{-2 + 0 + 8}{\sqrt{1+1+1}} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

DIVERGENCE OF A VECTOR FIELD

Let $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

Then $\operatorname{div}\vec{V} = \vec{\nabla}\cdot\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$

(8) Find divergence of $\vec{V} = xyz(x\hat{i} + y\hat{j} + z\hat{k})$

Sol^n $\vec{V} = x^2yz\,\hat{i} + xy^2z\,\hat{j} + xyz^2\,\hat{k}$

$\operatorname{div}\vec{V} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xyz^2) = 6xyz$

(6) Find divergence of $\vec{V} = (x^2+y^2+z^2)^{-3/2}(x\hat{i} + y\hat{j} + z\hat{k})$

Sol^n $\operatorname{div}\vec{V} = \frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{-3/2}\right] + \frac{\partial}{\partial y}\left[y(x^2+y^2+z^2)^{-3/2}\right]$
$+ \frac{\partial}{\partial z}\left[z(x^2+y^2+z^2)^{-3/2}\right]$

$= (x^2+y^2+z^2)^{-3/2} - \frac{3}{2}x(x^2+y^2+z^2)^{-5/2}\,2x$
$+ (x^2+y^2+z^2)^{-3/2} - \frac{3}{2}y(x^2+y^2+z^2)^{-5/2}\,2y$
$+ (x^2+y^2+z^2)^{-3/2} - \frac{3}{2}z(x^2+y^2+z^2)^{-5/2}\,2z$

$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2)$

$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2} = 0$

(11) Check the flow with velocity $\vec{V} = y\hat{i}$ for incompressible

Sol^n $\operatorname{div}\vec{V} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0) = 0$

Hence the flow with velocity $y\hat{i}$ is incompressible

(13) Prove that $\operatorname{div}(fV) = f\operatorname{div}V + V\cdot\nabla f$

RHS $= f\operatorname{div}V + V\cdot\nabla f$

$= f\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right) + (V_1\hat{i} + V_2\hat{j} + V_3\hat{k})\cdot\left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\,\partial f/\partial z\right)$

$= f\frac{\partial V_1}{\partial x} + f\frac{\partial V_2}{\partial y} + f\frac{\partial V_3}{\partial z} + V_1\frac{\partial f}{\partial x} + V_2\frac{\partial f}{\partial y} + V_3\frac{\partial f}{\partial z}$

$= f\frac{\partial V_1}{\partial x} + V_1\frac{\partial f}{\partial x} + f\frac{\partial V_2}{\partial y} + V_2\frac{\partial f}{\partial y} + f\frac{\partial V_3}{\partial z} + V_3\frac{\partial f}{\partial z}$

$$= \frac{\partial}{\partial x}(f v_1) + \frac{\partial}{\partial y}(f v_2) + \frac{\partial}{\partial z}(f v_3) = div(f v)$$

13(c) Prove that $div(f \nabla g) = f \nabla^2 g + \nabla f \cdot \nabla g$

$$LHS = div(f \nabla g) = div\left[i f \frac{\partial g}{\partial x} + j f \frac{\partial g}{\partial y} + k f \frac{\partial g}{\partial z} \right]$$

$$= \frac{\partial}{\partial x}\left(f \frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f \frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f \frac{\partial g}{\partial z}\right)$$

$$= f \frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f \frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f \frac{\partial^2 g}{\partial z^2} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}$$

$$RHS = f \nabla^2 g + \nabla f \cdot \nabla g$$

$$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + \left(i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}\right) \cdot \left(i \frac{\partial g}{\partial x} + j \frac{\partial g}{\partial y} + k \frac{\partial g}{\partial z}\right)$$

$$= f \frac{\partial^2 g}{\partial x^2} + f \frac{\partial^2 g}{\partial y^2} + f \frac{\partial^2 g}{\partial z^2} + \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} = LHS$$

(15) $f = 4x^2 + 9y^2 + z^2$ Find $\nabla^2 f$

Soln

$$grad f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = 8x\hat{i} + 18y\hat{j} + 2z\hat{k}$$

$$\nabla^2 f = div(grad f) = div(8x\hat{i} + 18y\hat{j} + 2z\hat{k})$$

$$= \frac{\partial}{\partial x}(8x) + \frac{\partial}{\partial y}(18y) + \frac{\partial}{\partial z}(2z)$$

$$= 8 + 18 + 2 = 28$$

CURL OF A VECTOR FIELD

If $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ then

$$curl \vec{V} = \vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\hat{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\hat{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\hat{k}$$

If $curl \vec{V} = 0$ then the fluid motion with velocity $\vec{V}$ is irrotational.

(3) Find curl $\vec{v}$ where $\vec{v} = \frac{x^2+y^2+z^2}{2}(\hat{i}+\hat{j}+\hat{k})$

Sol^n

$$\text{curl } \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x^2+y^2+z^2}{2} & \frac{x^2+y^2+z^2}{2} & \frac{x^2+y^2+z^2}{2} \end{vmatrix}$$

$$= \hat{i}\,\frac{1}{2}\left[\frac{\partial}{\partial y}(x^2+y^2+z^2) - \frac{\partial}{\partial z}(x^2+y^2+z^2)\right]$$

$$-\hat{j}\,\frac{1}{2}\left[\frac{\partial}{\partial x}(x^2+y^2+z^2) - \frac{\partial}{\partial z}(x^2+y^2+z^2)\right] + \hat{k}\cdot\frac{1}{2}\left[\frac{\partial}{\partial x}(x^2+y^2+z^2) - \frac{\partial}{\partial y}(x^2+y^2+z^2)\right]$$

$$= (y-z)\,\hat{i} - (x-z)\,\hat{j} + (x-y)\,\hat{k}$$

(11) Check $\vec{v} = y\hat{i} - x\hat{j}$ for irrotational ? incompressible ?

Sol^n $\text{div } \vec{v} = \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) = 0 - 0 = 0$

hence incompressible.

$$\text{curl } \vec{v} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} - 2\hat{k} = -2\hat{k} \neq 0$$

Hence $\vec{v}$ is not irrotational.

14(b) Prove that $\text{div}(\text{curl } \vec{v}) = 0$

Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$$\text{curl } \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} - \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

$$\text{div}(\text{curl } \vec{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$= \frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z} - \frac{\partial^2 v_3}{\partial x \partial y} + \frac{\partial^2 v_1}{\partial y \partial z} + \frac{\partial^2 v_2}{\partial x \partial z} - \frac{\partial^2 v_1}{\partial y \partial z}$$

$$= 0$$

14(d) Prove that $\text{curl}(\text{grad } f) = \vec{0}$

Sol^n By definition $\text{grad } f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$$\text{curl}(\text{grad } f) = \nabla \times (\nabla f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial z}\right)\hat{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z}\right)\hat{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y}\right)\hat{k}$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k} = \vec{0}$$

(17) $u = y\hat{i} + z\hat{j} + x\hat{k}$, $v = yz\hat{i} + zx\hat{j} + xy\hat{k}$ Find $\vec{u} \times \text{curl}\,\vec{v}$

Sol^n

$$\text{curl}\,\vec{v} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right] - \hat{j}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right] + \hat{k}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

$$= (x - x)\hat{i} - (y - y)\hat{j} + (z - z)\hat{k} = \vec{0}$$

Hence $\vec{u} \times \text{curl}\,\vec{v} = \vec{0}$

(19) $u = y\hat{i} + z\hat{j} + x\hat{k}$, $v = yz\hat{i} + zx\hat{j} + xy\hat{k}$ Find $\vec{v} \cdot \text{curl}\,\vec{u}$

Sol^n

$$\text{curl}\,\vec{u} = \nabla \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x) - \frac{\partial}{\partial z}(z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial z}(y)\right] + \hat{k}\left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial y}(y)\right]$$

$$= -\hat{i} - \hat{j} - \hat{k}$$

$$\vec{v} \cdot \text{curl}\,\vec{u} = (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot (-\hat{i} - \hat{j} - \hat{k})$$

$$= -yz - zx - xy$$