MEAN VALUE THEOREM (Lagrange's)

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b) then there exist $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

<u>Note:</u> If $f(a) = f(b)$ then $f'(c) = 0$ which is Rolle's Theorem.

<u>Ex</u> Discuss suitability of Rolle's Th^m & verify it.

<u>Sol^n</u> (a) $f(x) = x^2 - 5x + 6$, $[2, 3]$

Since $f(x)$ is continuous & differentiable (as polynomials are continuous & differentiable)

So Rolle's theorem is applicable.

$$f'(x) = 2x - 5 = 0 \Rightarrow x = 2.5 \in (2, 3)$$

(b) $f(x) = 3 + (x - 2)^{2/3}$ on $[1, 3]$

$$f'(x) = \frac{2}{3}(x - 2)^{-1/3} = \frac{2}{3\sqrt[3]{x - 2}}$$

$f'(x)$ does not exist at $x = 2 \in (1, 3)$

Rolle's theorem is not applicable.

(c) $f(x) = \tan x$ on $[0, \pi)$

Sol^n $f(x) = \tan x$ which is not continuous at $x = \pi/2$. Rolle's Th^m is not applicable.

(d) $f(x) = \sin^2 x, [0, \pi]$

Sol^n $f(x)$ is differentiable on $(0, \pi)$ and continuous in $[0, \pi]$

So Rolle's th^m is applicable.

$f'(x) = 2\sin x \cos x = \sin 2x = 0$

$x = n\pi/2, \frac{\pi}{2} \in [0, \pi]$

Ex(e) $f(x) = \sin x - \sin 2x \quad [0, \pi]$

Sol^n $f'(x) = \cos x - 2\cos 2x = 0$

$\cos x - 2(2\cos^2 x - 1) = 0$

$4\cos^2 x - \cos x - 2 = 0$

$\cos x = \frac{1+\sqrt{33}}{8}, \frac{1-\sqrt{33}}{8} = v$

$x = \cos^{-1} v \in [0, \pi]$

(f) Find the point on the curve $y = \cos x - 1$ in $[\frac{\pi}{2}, \frac{3\pi}{2}]$ at which tangent is parallel to x-axis.

Sol^n $f(x) = \cos x - 1$ is continuous in $[\frac{\pi}{2}, \frac{3\pi}{2}]$ differentiable in $(\frac{\pi}{2}, \frac{3\pi}{2})$

$f'(x) = -\sin x = 0$

$\sin x = 0$

$x = \pi \in (\frac{\pi}{2}, \frac{3\pi}{2})$

Problems on Mean Value Theorem

Ex-1 Verify Mean value theorem for
$f(x) = (x-3)(x-6)(x-9)$ in $[3,5]$

Sol^n As $f(x)$ is a polynomial, it satisfies the conditions of Mean value theorem.

$$f'(c) = \frac{f(5) - f(3)}{5-3} = \frac{8-0}{2} = 4$$

$$f'(x) = 3x^2 - 36x + 99$$

$$f'(c) = 0 \Rightarrow 3c^2 - 36c + 99 = 0 \Rightarrow c = 8.8, 4.8$$

Ex.2 Find a point on the curve $y = \sqrt{x-2}$ on $[2,3]$ where tangent is parallel to the chord joining end points.

Sol^n Mean value theorem is applicable

$$f'(c) = \frac{f(3) - f(2)}{3-2} \quad \text{(slope same)}$$
$$= (1-0)/1 = 1$$

$$f'(x) = \frac{1}{2\sqrt{x-2}} = 1 \Rightarrow x = \frac{9}{4}$$

No. 3 $f(x) = x - 2\sin x$ on $[-\pi, \pi]$

Sol^n As $f(x)$ is combination of polynomial with sine function, so mean value theorem condition is satisfied

$$f'(c) = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)} = \frac{\pi - (-\pi)}{2\pi} = 1$$

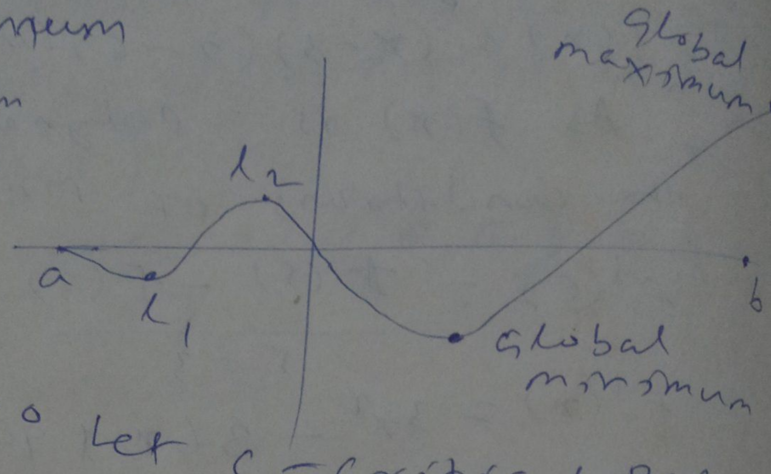
$$1 - 2\cos c = 1 \Rightarrow \cos c = 0 \Rightarrow c = \pm\frac{\pi}{2}$$

FIRST DERIVATIVE TEST FOR LOCAL EXTREMAM

1. If $f'(x)$ changes from positive to negative at c then $f(c)$ is a local maximum.
2. If $f'(x)$ changes from negative to positive at c then $f(c)$ is a local minimum.

3. If $f'(x)$ does not change sign at c then $f(c)$ is neither a local maximum nor local minimum

l_1 = local Minimum
l_2 = local maximum



Step-I Equate $f'(x)$ to zero.

Step II Solve $f'(x) = 0$ Let c = Critical Point

Step III Apply above conditions (1), (2), (3) to c to find maximum or minimum

Examples

1. Find all the points of local maximum & minimum of $f(x) = x^3 - 6x^2 + 9x - 8$

Sol^n $\frac{df}{dx} = 3(x^2 - 4x + 3) = 0$

$3(x-1)(x-3) = 0 \Rightarrow x = 1, 3$

(+) 1 (−) 3 (+)

As $\frac{dy}{dx} = \frac{df}{dx}$ changes sign from positive to negative as increased through $x = 1$ hence $x = 1$ is a local maximum

$\frac{dy}{dx}$ changes sign from negative to positive at $x = 3$ hence $x = 3$ is a local minimum.

2. $f(x) = (x-1)^3(x+1)^2 = y$

$\frac{dy}{dx} = (x-1)^2(x+1)(5x+1)$

$\frac{dy}{dx} = 0 \Rightarrow x = 1, -1, -\frac{1}{5}$

$$\overset{+}{\longleftarrow}\underset{-1}{\circ}\overset{-}{}\underset{-\frac{1}{5}}{\circ}\overset{+}{}\underset{1}{\circ}\overset{+}{\longrightarrow}$$

At $x=1$ Neither local max nor local mini
(point of inflexion)

At $x=-1$ Local Maximum

At $x=-1/5$ Local Minimum

(3) $y=f(x)=x^3-6x^2+12x-8$

$\frac{dy}{dx}=3(x-2)^2=0 \Rightarrow x=2$

$$\overset{+}{\longleftarrow}\underset{2}{\circ}\overset{+}{\longrightarrow}$$

Neither local maxima nor local minima at $x=2$ (point of inflexion)

(4) $f(x)=\sin x+\cos x=y$

Sol^n $\frac{dy}{dx}=\cos x-\sin x=0$

$\tan x=1 \quad x=\frac{\pi}{4}$

If $x<\frac{\pi}{4}$ then $\cos x>\sin x$

If $x>\frac{\pi}{4}$ then $\cos x<\sin x$

dy/dx changes sign from positive to negative so local maximum

(5) $y=\sin^4 x+\cos^4 x, \quad 0<x<\pi/2$

Sol $\frac{dy}{dx}=-2\sin 2x\cos 2x=-\sin 4x$

$dy/dx=0 \Rightarrow \sin 4x=0 \Rightarrow x=\pi/4$

For $x<\frac{\pi}{4} \Rightarrow \sin 4x>0 \Rightarrow -\sin 4x<0$

$dy/dx=-\sin 4x<0$

For $x>\frac{\pi}{4} \quad \frac{dy}{dx}=-\sin 4x>0$

At $x=\pi/4$ point of local minimum

(6) $y=f(x)=(x-2)^4(x+1)^3$

$\frac{dy}{dx}=(x-2)^2(x+1)^2(x-2)(7x-2)$

$dy/dx=0 \Rightarrow x=2,-1,\frac{2}{7}$ (critical points)

$$\overset{+}{\longleftarrow}\underset{-1}{\circ}\overset{+}{}\underset{\frac{2}{7}}{\circ}\overset{-}{}\underset{2}{\circ}\overset{+}{\longrightarrow}$$

$(x-2)^2$ & $(x+1)^2$ are always positive

When $x < \frac{2}{7}$ both $x-2$ & $x - \frac{2}{7}$ negative
hence $(x-2)(x-\frac{2}{7})$ positive ~~negative~~

When $x > \frac{2}{7}$ one is positive other negative
product $(x-2)(x-\frac{2}{7})$ negative

Hence $\frac{2}{7}$ is a local maximum point

$x = -1$ is a point of inflexion

$x = 2$ is local minimum point

TAYLOR SERIES at $x = a$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n$$

R_n = Remainder term

$$= \frac{f^{(n+1)}a}{n!}(x-a)^{n+1} + \cdots\cdots$$

Ex. 1 Find Taylor series of

$f(x) = e^{-x}$ about point $x = -4$

$f(a) = f(-4) = e^4$

$f'(x) = -e^{-x} \Rightarrow f'(a) = f'(-4) = -e^4$

$f''(x) = e^{-x} \Rightarrow f''(a) = f''(-4) = e^4$

$f''' = -e^{x} \Rightarrow f'''(a) = f'''(-4) = -e^4$

Taylor Series at $x = a$ is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots$$

$$= e^4 + (x+4)(-e^4) + \frac{(x+4)^2}{2!}e^4 + \frac{(x+4)^3}{3!}e^{-4} + \cdots$$

$$f(x) = e^{-x} = e^4\left[1 - (x+4) + \frac{(x+4)^2}{2!} - \frac{(x+4)^3}{3!} + \cdots\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} e^4 (x+4)^n$$ Taylor series at $x=-4$

Ex.2 Find Taylor series of $f(x) = \frac{1}{x^2}$ about $x=-$

Sol^n

$f(x) = \frac{1}{x^2} \Rightarrow f(a) = f(-1) = 1$

$f'(x) = -2x^{-3} \Rightarrow f'(a) = f'(-1) = -\frac{2}{-1} = 2$

$f''(x) = 2\times 3x^{-4} = 3!\,x^{-4} \Rightarrow f''(a) = f''(-1) = 3!$

$f'''(x) = -2\times 3\times 4x^{-5} = -4!\,x^{-5} = \frac{-4!}{x^5} \Rightarrow f'''(-1) = 4!$

$\vdots$

$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+2}} \Rightarrow f^{(n)}(a) = f^{(n)}(-1) = \frac{(-1)^n n!}{(-1)^{n+2}} = n!$

$$f(x) = \frac{1}{x^2} = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots$$

$$= 1 + (x+1)2! + \frac{(x+a)^2}{2!}3! + \frac{(x+a)^3}{3!}4! + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(n+1)!}{n!}(x+1)^n = \sum_{n=0}^{\infty}(n+1)(x+1)^n$$

Finite Number of Terms

Ex-3 Find Taylor series of $f(x) = x^3 - 10x^2 + 6$ about $x=3$

Sol^n

$f(x) = x^3 - 10x^2 + 6 \Rightarrow f(a) = f(3) = -57$

$f'(x) = 3x^2 - 20x \Rightarrow f'(a) = f'(3) = -33$

$f''(x) = 6x - 20 \Rightarrow f''(a) = f''(3) = -2$

$f'''(x) = 6 \Rightarrow f'''(a) = f'''(3) = 6$

$f^{(iv)}(x) = 0$

$\vdots$

$f^{(n)}(x) = 0$

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a)$$

$$= f(3) + (x-3) f'(3) + \frac{(x-3)^2}{2!} f''(3) + \frac{(x-3)^3}{3!} f'''(3)$$

$$= -57 - 33(x-3) - (x-3)^2 + (x-3)^3.$$

MACLAURIN SERIES

If we expand Taylor series of $f(x)$ about point $x = 0$ then the resulting series is called Maclaurin series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

Ex Find Maclaurin series of $\frac{1}{1-x}$

Sol $f(x) = \frac{1}{1-x} \Rightarrow f(0) = 1$

$$f'(x) = \frac{-1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2} \Rightarrow f'(0) = 1$$

$$f''(x) = \frac{2}{(1-x)^3} \Rightarrow f''(0) = 2$$

$$f'''(x) = \frac{6}{(1-x)^4} \Rightarrow f'''(0) = 6$$

$$\vdots$$

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \Rightarrow f^{(n)}(0) = n!$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$= 1 + x + \frac{x^2}{2} \times 2 + \frac{x^3}{6} \times 6 + \cdots$$

$$= 1 + x + x^2 + x^3 + \cdots$$

Some Important Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$(1-x)^n = \sum_{r=0}^{\infty} {}^nC_r (-1)^r x^r = 1 - nx + \frac{n(n-1)}{2!} x^2 - \cdots$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Ex: Find Maclaurin series of $\cos 2x$

Sol^n

$f(x) = \cos 2x \Rightarrow f(0) = 1$

$f'(x) = -2\sin 2x \Rightarrow f'(0) = 0$

$f''(x) = -4\cos 2x \Rightarrow f''(0) = -4$

$f'''(x) = 8\sin 2x \Rightarrow f'''(0) = 0$

$f^{iv}(x) = 16\cos 2x \Rightarrow f^{iv}(0) = 16$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$$

$$= 1 + 0 + \frac{x^2}{2!}(-4) + 0 + \frac{x^4}{4!}(16) + \cdots$$

$$= 1 - 2^2 \frac{x^2}{2!} + 2^4 \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}$$

POWER SERIES

A power series about $x=0$ is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

A power series about $x=a$ is $\sum_{n=0}^{\infty} c_n (x-a)^n$
$= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$

Here $x=a$ is center of series and $c_0, c_1, c_2, \ldots c_n \ldots$ are constants called coefficients of series

Note $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots$ is an example of a series but not a power series since power series correspond to a function about a point $x=a$. Hence $(x-a)^n$ terms must be present in each term of the power series, $n = 0, 1, 2, \ldots$

All the series found out in Taylor and Maclaurin series are power series.

Radius of Convergence of Power series

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{\lim_{n \to \infty} |a_n|^{\frac{1}{n}}}$$

Ex. 1 Find the radius of convergence of power series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (x+3)^n$

Sol^n

$a_n = \frac{(-1)^n n}{4^n}$ = Coefficient of $(x+3)^n$

$a_{n+1} = \frac{(-1)^{n+1} (n+1)}{4^{n+1}}$ = Coeff of $(x+3)^{n+1}$

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \left| \frac{(-1)^n n}{4^n} \times \frac{4^{n+1}}{(-1)^{n+1} (n+1)} \right.$$

$$= \lim_{n\to\infty} \left| 4 \frac{n}{n+1} \cdot \frac{1}{(-1)} \right| = \lim_{n\to\infty} \frac{4n}{n+1}$$

$$= \lim_{n\to\infty} \frac{4}{1+\frac{1}{n}} = \frac{4}{1+0} = 4$$

Center = 3, Radius of Convergence $R = 4$

The given power series converges for

$|x-a| < R$ that is $|x+3| < 4$

Ex-2 ~~Test~~ Find radius of convergence of $\sum_{n=0}^{\infty} n!\, x^n$

Soln $a_n = n! =$ Coeff of x^n

$a_{n+1} = (n+1)! =$ Coeff of x^{n+1}

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \Rightarrow \lim_{n\to\infty} \left| \frac{n!}{(n+1)\,n!} \right|$$

$$= \lim_{n\to\infty} \frac{1}{n+1} = 0$$

The power series converges for $|x-a| \leq R$

$|x-0| \leq 0$ that is converges only at $x = 0$

Ex-3 Find radius of convergence of $\sum_{n=1}^{\infty} n^{2n} x^n$

Soln $a_n = n^{2n} =$ Coeff of x^n

$$R = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}} = \frac{1}{\lim_{n\to\infty} (n^{2n})^{1/n}}$$

$$= \frac{1}{\lim_{n\to\infty} n^2} = \frac{1}{\infty} = 0$$

The given power series converges only at $x = 0$.

PARTIAL DERIVATIVE

Ex.1 Find partial derivative of f

(a) $f(x, y, z, w) = x^2 e^{2y+3z} \cos 4w$

(b) $f(r, \theta, z) = \dfrac{r(2 - \cos 2\theta)}{r^2 + z^2}$

Sol^n (a) $f_x = \dfrac{\partial f}{\partial x} = 2x\, e^{2y+3z} \cos 4w = \dfrac{2}{x} f$

$f_y = \dfrac{\partial f}{\partial y} = 2x^2 e^{2y+3z} \cos 4w = 2f$

$f_z = \dfrac{\partial f}{\partial z} = 3x^2 e^{2y+3z} \cos 4w = 3f$

$f_w = \dfrac{\partial f}{\partial w} = -4x^2 e^{2y+3z} \sin 4w$

Ex-2 $V(x, y, z) = \cos 3x \cos 4y \sinh 5z$

Show that $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$

Sol^n $\dfrac{\partial V}{\partial x} = -3 \sin 3x \cos 4y \sinh 5z$

$\dfrac{\partial^2 V}{\partial x^2} = -9 \cos 3x \cos 4y \sinh 5z = -9$ — (1)

$\dfrac{\partial^2 V}{\partial y^2} = -16 \cos 3x \cos 4y \sinh 5z = -16V$ — (2)

$\dfrac{\partial^2 V}{\partial z^2} = 25 \cos 3x \cos 4y \sinh 5z = 25V$ — (3)

Adding eq (1), (2), (3)

$\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$

Ex. 3 If $u = e^{a\theta}\cos(a\ln r)$ then show that $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ —— (1)

Sol $u_r = \frac{\partial u}{\partial r} = -e^{a\theta}\frac{a}{r}\sin(a\ln r)$ —— (2)

$u_{rr} = \frac{\partial^2 u}{\partial r^2} = -\frac{ae^{a\theta}}{r^2}\left\{-\sin(a\ln r) + a\cos(a\ln r)\right\}$ —— (3)

$u_\theta = \frac{\partial u}{\partial \theta} = ae^{a\theta}\cos(a\ln r)$

$u_{\theta\theta} = \frac{\partial^2 u}{\partial \theta^2} = a^2 e^{a\theta}\cos(a\ln r) = a^2 u$ —— (4)

$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ (Putting (2), (3), (4) in eqn (1))

No. 4 If $u = e^{x^2+y^2+z^2}$ then find $\frac{\partial^3 u}{\partial x\partial y\partial z}$

Sol^n $\frac{\partial u}{\partial z} = 2z\,e^{x^2+y^2+z^2} \Rightarrow \frac{\partial^2 u}{\partial y\partial z} = 4yz\,e^{x^2+y^2+z^2}$

$\frac{\partial^3 u}{\partial x\partial y\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y\partial z}\right) = 8xyz\,e^{x^2+y^2+z^2}$
$= 8xyzu$

HESSIAN MATRIX

Let $f: R^n \to R$ $\quad x = (x_1, x_2, \ldots, x_n) \in R^n$

If all second order partial derivatives of f exists then Hessian matrix H of f is a $n\times n$ square matrix

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1\partial x_n} \\ \frac{\partial^2 f}{\partial x_2\partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n\partial x_1} & \frac{\partial^2 f}{\partial x_n\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

The first row is
$$\frac{\partial}{\partial x_1}\left[\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \cdots \quad \frac{\partial f}{\partial x_n}\right]$$

The second row is $\frac{\partial}{\partial x_2}\left[\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \cdots \quad \frac{\partial f}{\partial x_n}\right]$

and the n^{th} row is $\frac{\partial}{\partial x_n}\left[\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \cdots \quad \frac{\partial f}{\partial x_n}\right]$

OR precisely $(H_f)_{ij} = \frac{\partial^2 f}{\partial x_i \, \partial x_j}$

If we need $(2, 3)$ entry of H_f

then $(H_f)_{23} = \frac{\partial^2 f}{\partial x_2 \, \partial x_3}$

If $f : R^2 \to R$ and

if $f(x_1, x_2) = 2x_1^2 + x_2$ then

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \, \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \, \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

JACOBIAN MATRIX

- Let $f: R^n \to R$ be defined as $f = (f_1, f_2, \cdots, f_n)$
  each $f_i = f_i(x_1, x_2, \cdots, x_n)$

Then Jacobian matrix of f is

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

If $n = 2$ then $f(x,y) = (u(x,y), v(x,y))$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Ex-1 $x = r\cos\theta \qquad y = r\sin\theta$

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

Ex. 2 $\quad x = u^2 - v^2 \qquad y = 2uv$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix}$$

$$= 4u^2 + 4v^2$$

Ex-3 $\quad f_1 = x_1^2 + x_2^3 + 3x_3 \qquad f_2 = x_4^3 + 3x_3^2$

$f_3 = x_1^2 + 2x_2^2 + x_3^3$

Find the Jacobian $J(x_1, x_2, x_3)$

Sol

$$J(x_1, x_2, x_3) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} 2x_1 & 3x_2^2 & 3 \\ 3x_1^2 & 0 & 6x_3 \\ 2x_1 & 4x_2 & 3x_3^2 \end{vmatrix}$$