(36) $(y+1)\,dx - (x+1)\,dy = 0$ ——— (1)

$M = y+1 \qquad N = -(x+1)$

$\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = -1$ (Not exact)

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{1-(-1)}{y+1} = \frac{2}{y+1} = f(y)$$

Int factor $F = e^{-\int f(y)\,dy} = e^{-\int \frac{2}{y+1}\,dy}$

$$= e^{-2\ln(y+1)} = e^{\ln(y+1)^{-2}} = (y+1)^{-2} = \frac{1}{(y+1)^2}$$

Multiplying $\frac{1}{(y+1)^2}$ on given eq^n (1) we get

$$\frac{1}{y+1}\,dx - \frac{x+1}{(y+1)^2}\,dy = 0$$

$N_1 = -\frac{x}{(y+1)^2} - \frac{1}{(y+1)^2}$

$\int M_1\,dx$ (y const) $= \frac{1}{y+1}\int dx = \frac{x}{y+1}$

$\int N_1\,dy$ (Terms not containing x) $= -\int \frac{dy}{(y+1)^2} = \frac{1}{y+1}$

Sol^n is $\int M_1\,dx + \int N_1\,dy = C$

$$\frac{x+1}{y+1} = C$$

# EXACT DIFFERENTIAL EQUATIONS

A differential eqⁿ of the form

$M(x,y)\,dx + N(x,y)\,dy = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

To solve exact differential eqⁿ

$$\underset{y \text{ constant}}{\int M\,dx} + \underset{\substack{\text{Terms not} \\ \text{containing } x}}{\int N\,dy} = C = \text{Constant}$$

**No. 16** Test for exact and solve the diffl. equation

$$y e^x\,dx + (2y + e^x)\,dy = 0, \quad y(0) = -1 \quad \text{---} (1)$$

**Solⁿ**

$M = y e^x$ = Coeff of $dx$

$N = 2y + e^x$ = Coeff. of $dy$

$\frac{\partial M}{\partial y} = e^x$ and $\frac{\partial N}{\partial x} = e^x \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, hence eqⁿ (1) is exact

$\underset{y \text{ const}}{\int M\,dx} = y\int e^x\,dx = y e^x$

$\underset{\substack{\text{terms not} \\ \text{containing } x}}{\int N\,dy} = \int 2y\,dy = y^2$ (Terms not containing $x$ means terms with pure $y$ terms or constants)

Solution is $\int M\,dx + \int N\,dy = C$

$y e^x + y^2 = C$ ——— (2)

Given $y(0) = -1 \Rightarrow y = -1$ when $x = 0 \Rightarrow C = 0$

Required solution is $y e^x + y^2 = 0$

**Integrating Factor** It is a function which is multiplied with non-exact diffl. eqⁿ to make it exact.

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ or constant then $F = e^{\int f(x)\,dx}$ is integrating factor

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = f(y)$ or const then $F = e^{-\int f(y)\,dy}$ is integrating factor

**No. 33** Find an integrating factor and solve the diffl eqⁿ $(2\cos y + 4x^2)\,dx = x \sin y\,dy$

Sol$^n$ $M = 2\cos y + 4x^2$ . $N = -x\sin y$

$\frac{\partial M}{\partial y} = -2\sin y \neq \frac{\partial N}{\partial x} = -\sin y$ (Not exact)

Now $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2\sin y + \sin y}{-x\sin y} = \frac{-\sin y}{-x\sin y} = \frac{1}{x} = f(x)$

Integrating factor $F = e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

Multiplying integ. factor $x$ in given eq$^n$ we get

$(2x\cos y + 4x^3)dx - x^2 \cdot \sin y \, dy = 0$

Here $M_1 = 2x\cos y + 4x^3$ = coeff of $dx$

$N_1 = -x^2 \cdot \sin y$ = coeff. of $dy$

$\int M_1 dx$ ($y$ const) $= \cos y \int 2x\,dx + 4\int x^3 dx = x^2\cos y + x^4$

$\int N_1 dy$ (Terms not containing $x$) $= \int 0\,dy = 0$

Solution is $\int M_1 dx + \int N_1 dy = c$

$x^2\cos y + x^4 = c$

LINEAR DIFFERENTAL EQUATION

General Form $\frac{dy}{dx} + p(x)y = Q(x)$

Procedure to solve Int. factor $F = e^{\int p\,dx}$

Sol$^n$ is $y = \frac{1}{F}\left[\int F Q\,dx + c\right]$

No. 9 $xy' = 2y + x^3 e^x$

$y' = \frac{2}{x}y + x^2 e^x \Rightarrow \frac{dy}{dx} - \frac{2}{x}y = x^2 e^x$

Linear eq$^n$ with $p = -\frac{2}{x}$ and $Q = x^2 e^x$

Integrating factor $F = e^{\int p\,dx} = e^{\int -\frac{2}{x}dx} = e^{-2\ln x}$

$= e^{\ln x^{-2}} = x^{-2} = \frac{1}{x^2}$

Solution is $y = \frac{1}{F}\left[\int F\cdot Q\,dx + c\right]$

$= x^2\left[\int \frac{1}{x^2} x^2 e^x dx + c\right] = x^2\left[\int e^x dx + c\right]$

$y = x^2(e^x + c)$

Bernoulli's eqn $\frac{dy}{dx} + p(x)y = Q(x)y^a$

a is any real number

No. 38 $2xyy' + (x-1)y^2 = x^2 e^x$

Soln ~~$y' + \frac{x-1}{2xy}y^2 = \frac{x^2}{2xy}e^x$~~

~~$\frac{dy}{dx} + \frac{1}{2}\left(1-\frac{1}{x}\right)y = \frac{x}{2y}e^x$~~

put $y^2 = t$

$2y\frac{dy}{dx} = \frac{dt}{dx}$

Divide both sides by $x$

$2yy' + \left(1-\frac{1}{x}\right)y^2 = xe^x$

$\frac{dt}{dx} + \left(1-\frac{1}{x}\right)t = xe^x$

It is linear with $P = 1-\frac{1}{x}$ $Q = xe^x$

$F = e^{\int P dx} = e^{\int\left(1-\frac{1}{x}\right)dx} = e^{x-\ln x}$

$= e^x \times e^{-\ln x} = e^x \cdot e^{\ln\frac{1}{x}} = \frac{1}{x}e^x$

Soln is $t = \frac{1}{F}\left[\int F Q\,dx + c\right]$

$= \frac{x}{e^x}\left[\int \frac{e^x}{x} xe^x dx + c\right] = \frac{x}{e^x}\left[\int e^{2x} dx + c\right]$

$t = y^2 = xe^{-x}\left[\frac{e^{2x}}{2} + c\right]$

(34) $\frac{dy}{dx} = \frac{\tan y}{x-1}$

$\frac{dx}{dy} = \frac{x-1}{\tan y} = \frac{x}{\tan y} - \frac{1}{\tan y}$

$= x\cot y - \cot y$

$$\frac{dx}{dy} - (\cot y)\, x = -\cot y$$

It is linear & of the form $\frac{dx}{dy} + P(y)\, x = Q(y)$

$P(y) = -\cot y \qquad Q(y) = -\cot y$

Int factor $F = e^{-\int \cot y\, dy} = e^{-\ln \sin y}$

$= e^{\ln \operatorname{cosec} y} = \operatorname{cosec} y$

Solution is $x = \frac{1}{F}\left[\int F Q\, dy + c\right]$

$$x = \frac{1}{\operatorname{cosec} y}\left[\int \operatorname{cosec} y\,(-\cot y)\, dy + c\right]$$

$$= \frac{1}{\operatorname{cosec} y}\left[\operatorname{cosec} y + c\right] = 1 + c \sin y$$

(33) $y' + \frac{1}{2} y = \frac{1}{3}(1-2x)\, y^4$

$$y^{-4} y' + \frac{1}{2y^3} = \frac{1-2x}{3}$$

Let $\frac{1}{2y^3} = t$ then $-\frac{3}{2y^4}\frac{dy}{dx} = \frac{dt}{dx}$

$-\frac{3}{2} y^{-4} y' = \frac{dt}{dx} \Rightarrow y^{-4} y' = -\frac{2}{3}\frac{dt}{dx}$

$$-\frac{2}{3}\frac{dt}{dx} + t = \frac{1-2x}{3}$$

$$\frac{dt}{dx} - \frac{3t}{2} = x - \frac{1}{2}$$

Linear with $P = -\frac{3}{2} \qquad Q = x - \frac{1}{2}$

$$F = e^{\int P\, dx} = e^{-\frac{3}{2}\int dx} = e^{-3x/2}$$

Sol$^n$ is $t = \frac{1}{F}\left[\int F Q\, dx + c\right]$

$$= e^{3x/2}\left[\int \left(x - \tfrac{1}{2}\right) e^{-3x/2} + c\right]$$

$$= e^{3x/2}\left[\left(-\frac{2x}{3} + \frac{1}{3} - \frac{4}{9}\right) e^{-3x/2} + c\right]$$

$z = \frac{\ldots}{y^2} = \ldots + C$

## Homogeneous Linear ODE of 2nd order

General form: $y'' + p(x)y' + q(x)y = r(x)$

If $r(x) = 0$ → it is homogeneous

if $r(x) \neq 0$ non-homogeneous

Ex: Reduce $2xy'' = 3y'$ to first order and solve

Soln: Let $y' = p$ then $y'' = \frac{dp}{dx}$

Given eqn will be $2x\frac{dp}{dx} = 3p$

$$\int \frac{dp}{p} = \frac{3}{2}\int \frac{dx}{x} \Rightarrow \ln p = \frac{3}{2}\ln x + \ln c$$

$$\ln p = \ln c x^{3/2}$$

$$p = \frac{dy}{dx} = c x^{3/2}$$

$$\int dy = c\int x^{3/2}\,dx$$

$$y = \frac{c\,x^{5/2}}{5/2} + d = c_1 x^{5/2} + d$$

Ex: $x^2y'' - 5xy' + 9y = 0$ — (1)

one soln $y_1$ is given by $y_1 = x^3$.

Soln:

$$y'' - \frac{5}{x}y' + \frac{9}{x^2}y = 0$$ (Divide both sides of eq (1) by $x^2$)

Here $p = -\frac{5}{x}$ = coeff of $y'$

$$U = \frac{1}{y_1^2}e^{-\int p\,dx} = \frac{1}{x^6}e^{\int \frac{5}{x}dx} = \frac{1}{x^6}e^{5\ln x}$$

$= \frac{1}{x^6} e^{\ln x^5} = \frac{1}{x^6} \cdot x^5 = \frac{1}{x}$

Then another solution $y_2 = u y_1 = y_1 \int U \, dx$

$= x^3 \int \frac{1}{x} dx = x^3 \ln x$

General solution is $y = c_1 y_1 + c_2 y_2$

$y = c_1 x^3 + c_2 x^3 \ln x = x^3 (c_1 + c_2 \ln x)$

~~Second Order~~

Homogeneous Linear ODES with Constant Coefficients

Gen Form $y'' + ay' + by = 0$ where $a, b$ are constants

Let $y = e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$, $y' = \lambda e^{\lambda x}$ then

$(\lambda^2 + a\lambda + b) e^{\lambda x} = 0$

$\lambda^2 + a\lambda + b = 0$ characteristic equation

$\lambda = \lambda_1, \lambda_2$

Case-I $\lambda_1$ & $\lambda_2$ are real distinct

General solution $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

Case II $\lambda_1 = \lambda_2$ (real and equal) $= \lambda$ (say)

General solⁿ $y = (c_1 + c_2 x) e^{\lambda x}$

Case III $\lambda = p \pm iq$ complex conjugates

Gen solution $y = e^{px}(A \cos qx + B \sin qx)$

No. 6 $y'' + y' - 12y = 0$

put $y' = \lambda y$

$y'' = \lambda^2 y$

$(\lambda^2 + \lambda - 12) y = 0$

Solⁿ Characteristic eqⁿ

$\lambda^2 + \lambda - 12 = 0$

$(\lambda - 3)(\lambda + 4) = 0 \Rightarrow \lambda = 3, -4$ real & distinct

General solution is $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

$y = c_1 e^{3x} + c_2 e^{-4x}$

(13) $9y'' + 6y' + y = 0$, $y(0) = 4$, $y'(0) = -\frac{13}{3}$

Sol^n Characteristic eq^n $9\lambda^2 + 6\lambda + 1 = 0$

$(3\lambda+1)(3\lambda+1) = 0 \Rightarrow \lambda = -\frac{1}{3}, -\frac{1}{3}$

real & equal

General sol^n $y = (c_1 + c_2 x)e^{\lambda x}$

$y = (c_1 + c_2 x) e^{-x/3}$ —— (1)

$y' = \left(-\frac{c_1}{3} + c_2 - \frac{c_2 x}{3}\right) e^{-x/3}$ —— (2)

$y(0) = 4 \Rightarrow y = 4$ when $x = 0$

Eq^n (1) gives $c_1 = 4$

$y'(0) = -\frac{13}{3} \Rightarrow y' = -\frac{13}{3}$ when $x = 0$

Eq^n (2) gives $-\frac{c_1}{3} + c_2 = -\frac{13}{3} \Rightarrow c_2 = \frac{c_1}{3} - \frac{13}{3} = -3$

putting values of $c_1 = 4$, $c_2 = -3$ in eq^n (1)

we get $y = (4 - 3x) e^{-x/3}$

No. 7 $16y'' - 8y' + 5y = 0$ —— (1)

Sol^n Putting $y' = \lambda y$ $y'' = \lambda^2 y$ we get

$16\lambda^2 y - 8\lambda y + 5y = 0 \Rightarrow 16\lambda^2 - 8\lambda + 5 = 0$

characteristic eq^n

$\lambda = \frac{1}{4} \pm \frac{i}{2} = p \pm iq \Rightarrow p = \frac{1}{4}$ real part

$q = \frac{1}{2} =$ Imag. part

General Solution is $y = e^{px}[A \cos qx + B \sin qx]$

$y = e^{x/4}\left[A \cos \frac{x}{2} + B \sin \frac{x}{2}\right]$

# EULER CAUCHY EQUATION

General Form $x^2 y'' + axy' + by = 0$ — (1)

Put $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

Eq^n (1) becomes $[m^2 + (a-1)m + b]\, x^m = 0$

Auxillary eq^n is $m^2 + (a-1)m + b = 0$

$m = m_1, m_2$

| Types of roots of m | General solution |
|---|---|
| $m_1 \neq m_2$ Real Distinct | $y = c_1 x^{m_1} + c_2 x^{m_2}$ |
| $m_1 = m_2 = m$ (say) Real equal | $y = (c_1 + c_2 \ln x)\, x^m$ |
| $m = \mu \pm i\nu$ Complex Conjugates | $y = x^{\mu}[A\cos(\nu \ln x) + B\sin(\nu \ln x)]$ |

No. 4 $xy'' + 2y' = 0$

$x^2 y'' + 2xy' = 0$

Here $a = 2$ = coeff of $xy'$
$b = 0$ = coeff of $y$

Auxillary eq^n $m^2 + (a-1)m + b = 0$

$m^2 + m = 0$

$m(m+1) = 0 \Rightarrow m = 0, -1$

General sol^n $y = c_1 x^{m_1} + c_2 x^{m_2} = c_1 + c_2 x^{-1}$

No. 7 $x^2 y'' + xy' + y = 0$

Here $a = 1$ $b = 1$ = coeff of $y$

$a$ = coeff of $xy' = 1$

$m^2 + (a-1)m + b = 0 \Rightarrow m^2 + 1 = 0$

$m = \pm i = \cancel{\mu} \pm i\nu$

$\mu$ = real part $= 0$

$\nu$ = imaginary part $= 1$

$$y = x^{\alpha}\left[A\cos(\beta \ln x) + B\sin(\beta \ln x)\right]$$
$$= x^{0}\left[A\cos(\ln x) + B\sin(\ln x)\right]$$

No. 13 $(x^2D^2 + 7xD + 9)y = 0$

$$x^2y'' + 7xy' + 9y = 0$$

Here $a =$ coeff of $xy' = 7$

$b = 9 =$ coeff of $y$

Auxillary eqn $m^2 + (a-1)m + b = 0$

$m^2 + 6m + 9 = 0$ $\qquad (m+3)(m+3) = 0$

$m = -3, -3$ (real & equal)

General soln $y = (c_1 + c_2 \ln x)x^m$

$$y = (c_1 + c_2 \ln x)x^{-3}$$

## Non-Homogeneous ODES

General Form $y'' + p(x)y' + q(x)y = r(x) \neq 0$

| Term in $r(x)$ | Choice for $x_p$ |
|---|---|
| $ke^{ax}$ | $ce^{ax}$ |
| $kx^n$ | $a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ |
| $k\cos wx$, $k\sin wx$ | $a\cos wx + b\sin wx$ |
| $ke^{cx}\cos wx$, $ke^{cx}\sin wx$ | $e^{cx}(a\cos wx + b\sin wx)$ |

No. 7 $y'' + y' - 6y = -6x^3 + 3x^2 + 6x$ ——— ①

Sol$^n$ Homogeneous part $y'' + y' - 6y = 0$

C.E. $\to$ $\lambda^2 + \lambda - 6 = 0 \Rightarrow \lambda = 2, -3$

Homogeneous sol$^n$ $y_h = c_1 e^{2x} + c_2 e^{-3x}$

Non-homogeneous part

Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ ——— ③

$y' = a_1 + 2a_2 x + 3a_3 x^2$ and $y'' = 2a_2 + 6a_3 x$

putting in eq$^n$ ①

$2a_2 + 6a_3 x + a_1 + 2a_2 x + 3a_3 x^2$

$-6a_0 - 6a_1 x - 6a_2 x^2 - 6a_3 x^3 = -6x^3 + 3x^2 + 6x$

Equating coeff. of $x^3$, $x^2$, $x$ & const. terms both sides we get

$-6a_3 = -6 \Rightarrow a_3 = 1$

$3a_3 - 6a_2 = 3 \Rightarrow a_2 = \dfrac{3a_3 - 3}{6} = 0$

$6a_3 + 2a_2 - 6a_1 = 6 \Rightarrow a_1 = \dfrac{6a_3 + 2a_2 - 6}{6} = 0$

$2a_2 + a_1 - 6a_0 = 0 \Rightarrow a_0 = \dfrac{2a_2 + a_1}{6} = 0$

putting in eq$^n$ ② $y = x^3 = y_p$

General solution is $y = y_h + y_p = c_1 e^{2x} + c_2 e^{-3x} + x^3$

No. 16 Find general sol$^n$ of ODE $3y'' + 10y' + 3y = 9x + 5\cos x$

Sol$^n$ Homogeneous part $3y'' + 10y' + 3y = 0$ ——— ①

C.E. $\to$ $3\lambda^2 + 10\lambda + 3 = 0 \Rightarrow \lambda = -3, -1/3$

Homog. sol$^n$ $y_h = c_1 e^{-3x} + c_2 e^{-x/3}$

Non-homogeneous part Let $y = a_0 + a_1 x + A\cos x + B\sin x$ ——— ②

$y' = a_1 - A\sin x + B\cos x$

$y'' = -A\cos x - B\sin x$

Putting on eqⁿ ① we get

$-3A\cos x - 3B\sin x + 10a_1 - 10A\sin x + 10B\cos x$
$+ 3a_0 + 3a_1 x + 3A\cos x + 3B\sin x = 9x + 5\cos x$

Coeff of $x$ $\quad 3a_1 = 9 \Rightarrow a_1 = 3$

Constant term $\quad 10a_1 + 3a_0 = 0 \Rightarrow a_0 = \frac{-10a_1}{3} = -10$

Coeff. of $\cos x$ $\quad -3A + 10B + 3A = 5$

$10B = 5 \Rightarrow B = 0.5$

Coeff of $\sin x$ $\quad -3B - 10A + 3B = 0 \Rightarrow A = 0$

putting value of these constants on eqⁿ ②

$y = -10 + 3x + 0.5\sin x = y_p$

General solⁿ $y = y_h + y_p$

$= C_1 e^{-3x} + C_2 e^{-x/3} - 10 + 3x + 0.5\sin x$

No. 9 $\quad y'' + 2y' - 35y = 12e^{5x} + 37\sin 5x$

solⁿ Homogeneous part $\quad y'' + 2y' - 35y = 0$ ——①

C.E. → $\lambda^2 + 2\lambda - 35 = 0 \Rightarrow \lambda = 5, -7$

Homogeneous solution $y_h = C_1 e^{-7x} + C_2 e^{5x}$

Non-homogeneous part

Let $y = (cx+d)e^{5x} + A\cos 5x + B\sin 5x$ ——②

$y' = (5cx + c + 5d)e^{5x} - 5A\sin 5x + 5B\cos 5x$

~~$y'' = 25ce^{5x} - 25A\cos 5x - 25B\sin 5x$~~

~~Putting on eqⁿ ① we get~~

$y'' = (25cx + 10c + 25d)e^{5x} - 25A\cos 5x$
$- 25B\sin 5x$ $\cancel{+5B\cos 5x}$

$25ce^{5x} - 25A\cos 5x - 25B\sin 5x + 10ce^{5x}$
$-10A\sin 5x + 10B\cos 5x - 35ce^{5x} - 35A\cos 5x - 35B\sin 5x$
$= 12e^{5x} + 37\sin 5x$

Coefficient of $e^{5x}$ $\quad 25c + 10c - 35c = 12$

$0(c) = 12$ ~~c can take any value~~
~~let c = 1~~

coeff of $\cos 5x$ $\quad -25A + 10B - 35A = 0$
$\Rightarrow B = 6A$

coeff of $\sin 5x$ $\quad -25B - 10A - 35B = 37$
$-10A - 60B = 37 \Rightarrow -370A = 37 \Rightarrow A = -0.1$
$B = 6A = -0.6$

putting in eqn ① we get

$(25cx + 10c + 25d)e^{5x} - 25A\cos 5x - 25B\sin 5x$
$+ (10cx + 2c + 10d)e^{5x} - 10A\sin 5x + 10B\cos 5x$
$- 35(cx + d)e^{5x} - 35A\cos 5x - 35B\sin 5x$
$= 12e^{5x} + 37\sin 5x$

Coeff of $\cos 5x$ $\quad -25A + 10B - 35A = 0 \Rightarrow B = 6A$

Coeff of $\sin 5x$ $\quad -25B - 10A - 35B = 37$
$-10A - 60B = 37 \Rightarrow -370A = 37$
$A = -0.1$ and $B = 6A = -0.6$

Coeff of $xe^{5x}$ $\quad 25c + 10c - 35c = 0$
$0 \times c = 0 \quad$ let $c = 1$

Coeff of $e^{5x}$ $\quad 10c + \cancel{25d} + 2c + \cancel{10d} - \cancel{35d} = 12$
$12c = 12 \Rightarrow c = 1 \quad$ let $d = 0$

putting in eqn ②
$y_p = xe^{5x} - 0.1\cos 5x - 0.6\sin 5x$
General solⁿ $y = y_h + y_p = c_1 e^{-7x} + c_2 e^{5x}$
$+ xe^{5x} - 0.1\cos 5x - 0.6\sin 5x$

No. 8 $y'' + 6y' + 9y = 50 e^{-x} \cos x$ —— ①

Sol^n Homogeneous part $y'' + 6y' + 9y = 0$

C.E. $r^2 + 6r + 9 = 0 \Rightarrow r = -3, -3$

Homogeneous sol^n $y_h = (C_1 + C_2 x) e^{-3x}$

Non homogeneous part

Let $y = e^{-x} (A \cos x + B \sin x)$ —— ②

$y' = -e^{-x} (A \cos x + B \sin x) + e^{-x}(-A \sin x + B \cos x)$

$= e^{-x} [(B - A) \cos x - (B + A) \sin x]$

$y'' = e^{-x} [2A \sin x - 2B \cos x]$

putting in eq^n ① we get

$[2A \sin x - 2B \cos x + (6B - 6A) \cos x + (-6B - 6A) \sin x$
$+ 9A \cos x + 9B \sin x] e^{-x} = 50 e^{-x} \cos x$

Coeff of $e^{-x} \sin x$ $\quad 2A - 6B - 6A + 9B = 0$

$B = 4A/3$

Coeff of $e^{-x} \cos x$ $\quad -2B + 6B - 6A + 9A = 50$

$4B + 3A = 50 \Rightarrow \frac{16A}{3} + 3A = 50 \Rightarrow A = 6$

$B = \frac{4A}{3} = 8$

Putting in eq^n ② $y = e^{-x} (6 \cos x + 8 \sin x)$

General sol^n $y = y_h + y_p$

$= (C_1 + C_2 x) e^{-3x} + e^{-x} (6 \cos x + 8 \sin x)$

# Solution by variation of parameter

$y'' + p(x)y' + q(x)y = r(x)$

Let $y_1$ & $y_2$ be two solutions of homogeneous eqn $y'' + p(x)y' + q(x)y = 0$. Then non-homog. solution $y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$

where $W = y_1 y_2' - y_2 y_1'$ = Wronskian.

**No. 2** $y'' + 9y = \sec 3x$ ——— (1)

**Sol^n** Homogeneous part is $y'' + 9y = 0$ ——— (2)

C.E. $\lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i$

Homogeneous sol^n $y_h = A\cos 3x + B\sin 3x$

Here $y_1 = \cos 3x$, $y_2 = \sin 3x$ are two sol^ns of homog. eq^n (2)

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$

$= 3(\cos^2 3x + \sin^2 3x) = 3 \times 1 = 3$

Non-homog sol^n $y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$

$= -\cos 3x \frac{1}{3} \int \sin 3x \sec 3x \, dx + \frac{\sin 3x}{3} \int \cos 3x \sec 3x \, dx$

$= -\frac{\cos 3x}{3} \int \tan 3x \, dx + \frac{\sin 3x}{3} \int dx$

$= \frac{\cos 3x}{9} \ln \cos 3x + \frac{x}{9} \sin 3x$

General sol^n $y = y_h + y_p = A\cos 3x + B\sin 3x + \frac{x}{9}\sin 3x + \frac{1}{9}\cos 3x \ln \cos 3x$

**No. 15** $x^2 y'' - 2xy' + 2y = x^3 \cos x$ ——— (1)

**Sol^n** Homog. part $x^2 y'' - 2xy' + 2y = 0$

Euler Cauchy eq^n $a = -2$ $b = 2$ ——— (2)

A.E. $\rightarrow$ $m^2 + (a-1)m + b = 0$

$m^2 - 3m + 2 = 0 \Rightarrow m = 2, 1$

Homogeneous solution

$y = c_1 x^{m_1} + c_2 x^{m_2}$

$y_h = c_1 x + c_2 x^2$

Here $y_1 = x$, $y_2 = x^2$ are solutions of homogeneous eq (2)

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$

$= 2x^2 - x^2 = x^2$

Non-homogeneous solution

$y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$

$= -x \int \frac{x^2 x^3 \cos x}{x^2} dx + x^2 \int \frac{x\, x^3 \cos x}{x^2} dx$

$= -x \int x^3 \cos x\, dx + x^2 \int x^2 \cos x\, dx$

$= -x \left[ x^3 \sin x - \int 3x^2 \sin x\, dx \right]$
$+ x^2 \left[ x^2 \sin x - \int 2x \sin x\, dx \right]$

$= -x \left[ x^3 \sin x - 3\left(-x^2 \cos x + \int 2x \cos x\, dx\right) \right]$
$+ x^2 \left[ x^2 \sin x - 2\left(-x \cos x + \int \cos x\, dx\right) \right]$

$= \cancel{-x^4 \sin x} - 3x^3 \cos x + 6x(x \sin x + \cos x)$
$+ \cancel{x^4 \sin x} + 2x^3 \cos x - 2x^2 \sin x$

$= -x^3 \cos x - 4x^2 \sin x + 6x \cos x$

# Modeling of Electric Circuit

**RLC Circuit** $LI' + RI + \frac{1}{C}\int I\,dt = E(t)$

$$LI'' + RI' + \frac{1}{C}I = \frac{dE}{dt}$$

**No. 4** Given $R = 2$ ohms, $L = 1$ henry, $C = 0.5$ farad
$E = 50 \sin t$ volts

$$LI'' + RI' + \frac{1}{C}I = \frac{dE}{dt} = 50\cos t \quad \text{—— (1)}$$

$$I'' + 2I' + 2I = 50\cos t \quad \text{—— (1)}$$

**Homog. part** $I'' + 2I' + 2I = 0$

$\lambda^2 + 2\lambda + 2 = 0 \qquad \lambda = -1 \pm i = p \pm iq$

$I_h = e^{pt}(A\cos qx + B\sin qt) = e^{-t}[A\cos t + B\sin t]$

**Non-homog. part**

Let $I = A_1\cos t + B_1\sin t$

$I' = -A_1\sin t + B_1\cos t \qquad I'' = -A_1\cos t - B_1\sin t$

Putting in eqn (1) we get

$-A_1\cos t - B_1\sin t + 2B_1\cos t - 2A_1\sin t + 2A_1\cos t$
$+ 2B_1\sin t = 50\cos t$

Equating coeff of $\sin t$ & $\cos t$ we get

$-B_1 - 2A_1 + 2B_1 = 0 \Rightarrow B_1 - 2A_1 = 0 \Rightarrow B_1 = 2A_1$

and
$-A_1 + 2B_1 + 2A_1 = 50 \Rightarrow A_1 + 2B_1 = 50$

$\Rightarrow 5A_1 = 50 \Rightarrow A_1 = 10$

$\Rightarrow B_1 = 2A_1 = 20$

Non-homogeneous soln $I_p = 10\cos t + 20\sin t$

General solution $I = I_h + I_p$

$= e^{-t}[A\cos t + B\sin t] + 10\cos t + 20\sin t$

POWER SERIES SOL^n OF DIFFERENTIAL EQUATION MODULE - II

Procedure Step-I Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$ be sol^n of given eq^n — (1)

II We are putting these series in $y$ and its derivatives $y'$ & $y''$ in ~~eq~~ given eq^n.

III Compare the coefficients of like powers of $x$ both sides

IV The higher coefficients $c_2, c_3, c_4, \ldots$ will be expressed in terms of $c_0, c_1$ or both $c_0, c_1$

V Putting these coefficient values in ~~given~~ eq^n (1) we get solution.

No. 7 $y'' + 4y = 0$ ——— (1)

Sol^n Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$ be power series sol^n of eq^n (1) ——— (2)

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots$

$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + \ldots$

Putting in eq^n (1)

$2a_2 + 6a_3 x + 12a_4 x^2 + \ldots$
$+ 4a_0 + 4a_1 x + 4a_2 x^2 + \ldots = 0$

Coeff of $x^0$ (const terms)

$2a_2 + 4a_0 = 0 \Rightarrow a_2 = -2a_0$

Coeff of $x$ $6a_3 + 4a_1 = 0 \Rightarrow a_3 = -\frac{2}{3}a_1$

Coeff of $x^2$ $12a_4 + 4a_2 = 0 \Rightarrow a_4 = -\frac{a_2}{3} = \frac{2a_0}{3}$

Sol^n is $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$

$= a_0 + a_1 x - 2a_0 x^2 - \frac{2}{3} a_1 x^3 + \frac{2a_0}{3} x^4 + \ldots$

$= a_0 \left[1 - 2x^2 + \frac{2x^4}{3} - \ldots\right] + a_1 \left[x - \frac{2}{3} x^3 + \ldots\right]$

No. 5 Solve: $y'' - 3y' + 2y = 0$ by power series method. —— (1)

Sol$^n$ Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$ be solution of eq$^n$ (1). Then —— (2)

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots$

$y'' = 2a_2 + 6a_3 x + 12 a_4 x^2 + \ldots$

Putting in eq$^n$ (1) we get

$2a_2 + 6a_3 x + 12a_4 x^2 + \ldots$

$-3a_1 - 6a_2 x - 9a_3 x^2 - \ldots$

$+2a_0 + 2a_1 x + 2a_2 x^2 + \ldots = 0$

Equating coeff of like powers of $x$ both sides

$2a_2 - 3a_1 + 2a_0 = 0 \Rightarrow a_2 = \frac{3}{2}a_1 - a_0$ (Const terms)

$6a_3 - 6a_2 + 2a_1 = 0 \Rightarrow a_3 = a_2 - \frac{a_1}{3} = \frac{3}{2}a_1 - a_0 - \frac{a_1}{3}$ (Coefficient of $x$)

$= \frac{7}{6}a_1 - a_0$

$12a_4 - 9a_3 + 2a_2 = 0$ (Coeff. of $x^2$)

$\Rightarrow a_4 = \frac{3}{4}a_3 - \frac{a_2}{6} = \frac{3}{4}\left(\frac{7}{6}a_1 - a_0\right) - \frac{a_1}{4} + \frac{a_0}{6}$

Putting in eq$^n$ (2) these values of $a_2, a_3, a_4$ we get

$y = a_0 + a_1 x + \left(\frac{3}{2}a_1 - a_0\right)x^2 + \left(\frac{7}{6}a_1 - a_0\right)x^3$
$+ \left(\frac{5a_1}{8} - \frac{7}{12}a_0\right)x^4 + \ldots$

Radius of Convergence

$\frac{1}{R} = \lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right|$ and $\frac{1}{R} = \lim_{m\to\infty} |a_m|^{\frac{1}{m}}$

where $a_m$ = coeff of $x^m$

No. 14 Find radius of convergence of $\sum_{m=0}^{\infty} m(m+1)x^m$

Sol$^n$ $a_m = m(m+1)$ = coeff of $x^m$

$a_{m+1} = (m+1)(m+2)$ = coeff of $x^{m+1}$

$\frac{1}{R} = \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty}\left|\frac{m+2}{m}\right| = \lim_{m\to\infty} 1 + \frac{2}{m}$

$= 1 + 0 = 1$

$\frac{1}{R} = 1 \Rightarrow R = 1$

No. 20 Find radius of convergence of $\sum_{m=0}^{\infty} m^m x^m$

Sol^n $a_m = m^m$ = coeff of $x^m$

$$\frac{1}{R} = \lim_{m\to\infty} |a_m|^{\frac{1}{m}} = \lim_{m\to\infty} (m^m)^{\frac{1}{m}}$$

$$= \lim_{m\to\infty} m = \infty \quad \text{Hence } R = 0$$

No. 18 Find radius of convergence of $\sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!}$

Sol^n $a_{2m+1} = \frac{1}{(2m+1)!}$ = coeff of $x^{2m+1}$

$a_{2m+3} = \frac{1}{(2m+3)!}$ = coeff of $x^{2m+3}$

$$\frac{1}{R^2} = \lim_{m\to\infty} \left| \frac{a_{2m+3}}{a_{2m+1}} \right| = \lim_{m\to\infty} \frac{(2m+1)!}{(2m+3)!}$$

$$= \lim_{m\to\infty} \frac{(2m+1)!}{(2m+3)(2m+2)(2m+1)!}$$

$$= \lim_{m\to\infty} \frac{1}{(2m+3)(2m+2)}$$

$$= 0$$

$$R = \infty$$

# Legendre's Equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Its solution is called Legendre's polynomial

$$P_n(x) = \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n (n-1)!\,(n-2)!} x^{n-2} + \cdots$$

$P_0 = 1$, $P_1(x) = x$, $P_2(x) = \dfrac{3x^2-1}{2}$

$P_3(x) = \dfrac{5x^3-3x}{2}$, $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

Ex: Express $5x^3 + x^2 - x + 7$ in terms of Legendre polynomial.

Sol$^n$ $5x^3 + x^2 - x + 7$

$= (5x^3 - 3x) + 3x + x^2 - x + 7$

$= 2P_3(x) + \cancel{\frac{3x^2}{2}}\ \frac{2}{3}\left(\frac{3x^2}{2} - \frac{1}{2} + \frac{1}{2}\right) + 2x + 7$

$= 2P_3(x) + \frac{2}{3}\left(\frac{3x^2-1}{2}\right) + 2x + \frac{1}{3} + 7$

$= 2P_3(x) + \frac{2}{3}P_2(x) + 2P_1(x) + \frac{22}{3}P_0$

No. 6 Prove the Rodrigues Formule

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$

or $2^n n!\, P_n(x) = \dfrac{d^n}{dx^n}(x^2-1)^n$

Proof $(x^2-1)^n = x^{2n} - n x^{2n-2} + \dfrac{n(n-1)}{2} x^{2n-4} - \cdots$

RHS $=$

$\dfrac{d^n}{dx^n}(x^2-1)^n = 2n(2n-1)\cdots\cdots(1+n)x^n$

$- n\left[(2n-2)(2n-3)\cdots(n-1)x^{n-2}\right]$

$+ \cdots$

$$= \frac{2n(2n-1)\cdots(n+1)n(n-1)\cdots 3\cdot 2\cdot 1}{n(n-1)\cdots 3\cdot 2\cdot 1}x^n$$

$$-n\left[\frac{(2n-2)(2n-3)\cdots(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n-2)\cdots 3\cdot 2\cdot 1}\right]x^{n-2}$$

$$+ \cdots$$

$$= \frac{(2n)!}{n!}x^n - n\frac{(2n-2)!}{(n-2)!}x^{n-2} + \cdots$$

$$LHS = 2^n n!\, P_n(x)$$

$$= 2^n n!\left[\frac{(2n)!}{2^n (n!)^2}x^n - \frac{(2n-2)!}{2^n (n-1)!\,(n-2)!}x^{n-2} + \cdots\right]$$

$$= \frac{(2n)!}{n!}x^n - n\frac{(2n-2)!}{(n-2)!}x^{n-2} + \cdots = RHS$$

**No. 10(a)** Prove that the generating function for Legendre polynomial is

$$G(u, x) = \frac{1}{\sqrt{1-2xu+u^2}} = \sum_{n=0}^{\infty} P_n(x)u^n$$

**Proof**

$$LHS = (1-2xu+u^2)^{-1/2} = (1-v)^{-1/2}$$ where $v = 2xu - u^2$

$$= 1 + \tfrac{1}{2}v + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}v^2 + \cdots$$

$$= 1 + \frac{v}{2} + \frac{3}{8}v^2 + \cdots$$

$$= 1 + \frac{2xu - u^2}{2} + \frac{3}{8}(4x^2u^2 + u^4 - 4xu^3) + \cdots$$

$$= 1 + ux + u^2\left(\frac{3x^2}{2} - \frac{1}{2}\right) + \cdots$$

$$= P_0 + uP_1(x) + u^2P_2(x) + \cdots$$

$$= \sum_{n=0}^{\infty} P_n(x)u^n = RHS$$

Put $1 = P_0$

$x = P_1(x)$

$\frac{3x^2-1}{2} = P_2(x)$

Ex. Express in terms of Legendre polynomial

$$f(x) = x^4 + 2x^3 + 2x^2 - x - 3$$

Solution

$$x^4 + 2x^3 + 2x^2 - x - 3$$

$$= \frac{8}{35}\left(\frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}\right) + \frac{20}{7}x^2 + 2x^3 - x - \frac{108}{35}$$

$$= \frac{8}{35}P_4(x) + \frac{4}{5}\left(\frac{5}{2}x^3 - \frac{3x}{2}\right) + \frac{20}{7}x^2 + \frac{x}{5} - \frac{108}{35}$$

$$= \frac{8}{35}P_4(x) + \frac{4}{5}P_3(x) + \frac{40}{21}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$$

$$+ \frac{x}{5} - \frac{108}{35} + \frac{20}{21}$$

$$= \frac{8}{35}P_4(x) + \frac{4}{5}P_3(x) + \frac{40}{21}P_2(x)$$

$$+ \frac{1}{5}P_1(x) - \frac{32}{15}P_0$$